N6

Mathematics

Lecturer Guide

Sparrow Consulting

Additional resource material available for this title includes:

- Exemplar examination paper
- Interactive toys
- Past exam papers with memos
- Posters
- PowerPoint presentation.

Scan the QR code below or visit this link: futman.pub/N6MathematicsLG

SCAN ME

ISBN 978-1-77637-227-0

FutureManagers
SIYAFUNDA • SIYAKHULA

Published by
Future Managers (Pty) Ltd
PO Box 13194, Mowbray, 7705
Tel (021) 462 3572
Fax (021) 462 3681
E-mail: info@futuremanagers.com
Website: www.futuremanagers.com

CONTENTS

1. Subject aims

1.1 General subject aims

Mathematics N6 aims to provide learners with the skills to identify and calculate mathematical problems in N6 and the content forms part of engineering calculation problems from industry.

Furthermore, Mathematics N6 will equip students with the relevant knowledge to enable them to integrate meaningfully into their trade subjects and also serve as the foundation for the Mathematics N6 syllabus in order to achieve a national diploma.

Upon completion of this subject, the student should be able to:

- apply the necessary knowledge of Mathematics to various engineering fields in their respective working environments;
- apply higher cognitive skills pertaining to application, analysis, synthesis and evaluation, and logical and critical thought processes;
- apply their understanding in the interpretation of real world problems;
- promote Mathematics as a tool to be used to trouble shoot in different fields of study; and
- calculate using certain theorems, the proofs of which are not examinable.

1.2 Specific subject aims

The specific aims of Mathematics N6 is to continue with the study of Differential and Integral Calculus.

Mathematics N6 strives to assist students to obtain trade-specific calculation knowledge.

Other specific aims of Mathematics N6 also include:

- Promote correct mathematical terminology.
- Promote and focus on word problems and the problem solving thereof, in order to prepare the students for their relevant careers.
- Use technology in Mathematics and apply Mathematics to further technology.

2. Admission requirements

For admission to N6 Mathematics, a student must have passed N5 Mathematics.

3. Duration of course

The duration of the subject is one trimester on full-time, part-time or distance-learning mode.

4. Evaluation

Candidates must be evaluated continually as follows:

4.1 ICASS Trimester Mark
- Assessment marks are valid for a period of one year and are referred to as ICASS Trimester marks.
- A minimum of 40% is required for a student to qualify for entry to the final examination.
- Two formal class tests for full-time and part-time students (or two assignments for distance-learning students only).

4.2 Calculation of trimester mark will be as follows:
- weight of test or assignment 1 = 30% of the syllabus; and
- weight of test or assignment 2 = 70% of the syllabus.

5. Examination

A final examination will be conducted in April, August and November of each year. The pass requirement is 40%.

The final examination will consist of 100% of the syllabus

The duration of the final examination will be 3 hours.

The final examination will be a closed book examination.

Minimum pass percentage will be 40%.

Assessments will be based on the cognitive domain of Bloom's Taxonomy, that is remember, understand, apply, analyse, evaluate, and create.

The division of these aspects are as follows:

Remember	Understand	Apply	Analyse	Evaluate	Create
20%	20%	20%	10%	20%	10%

6. General information

Problems should be based on real world scenarios allowing students to relate theory to practice.

Emphasis of correct mathematical terminology should be encouraged and promoted at all times.

A systematic approach to problem solving should be adhered to.

Students should be encouraged to understand rather than memorise the basic formulae applicable to N6 Mathematics.

Calculators may be used to do mathematical calculations.

Answers to all calculations must be approximated correctly to three decimal places, unless otherwise stated. Unless otherwise stated, approximations may not be done during calculations. The final answer must be approximated to the stipulated degree of accuracy.

The weight value of a module gives an indication of the time to be spent on teaching the module as well as the relative percentage of the total marks allocated to the module in the final exam examination (1 mark = 1,8 minutes).

7. Subject matter

Mathematics N6 strives to assist students to obtain trade-specific calculation knowledge. Students should be able to acquire in-depth knowledge of the following content:

Module	Weighted value (%)
1. Differentiation	6
2. Integration techniques	18
3. Partial fractions	12
4. Differential equations	12
5. Areas and volumes	15
6. Centroids and centre of gravity	10
7. Second moment of area, moment of inertia and centre of fluid pressure	15
8. Combinations of differentiation and integration	12
Total	100

8. Workschedule

Week	Module	Topic	Activities	Hours
1	Module 1 Differentiation	1.1 Partial differentiation 1.2 Applications of partial differentiation 1.3 Differentiation of parametric equations	Activity 1.1 Activity 1.2 Activity 1.3 Activity 1.4 Summative assessment: Module 1	6 hours
1–3	Module 2 Integration techniques	2.1 Integration by parts 2.2 Integration of trigonometric functions 2.3 Integration by completing the square	Activity 2.1 Activity 2.2 Activity 2.3 Activity 2.4 Summative assessment: Module 2	18 hours
3–4	Module 3 Partial fractions	3.1 Partial fraction decomposition (Revision) 3.2 Single recursive factor 3.3 Two recursive factors 3.4 One trinomial factor with recursive factors 3.5 Improper rational factors	Activity 3.1 Activity 3.2 Activity 3.3 Activity 3.4 Activity 3.5 Summative assessment: Module 3	12 hours

Week	Module	Topic	Activities	Hours
4–5	Module 4 Differential equations	4.1 First order linear differential equations 4.2 Second order differential equations	Activity 4.1 Activity 4.2 Activity 4.3 Summative assessment: Module 4	12 hours
5–7	Module 5 Areas and volumes	5.1 Areas 5.2 The volume of a solid of revolution	Activity 5.1 Activity 5.2 Activity 5.3 Activity 5.4 Activity 5.5 Activity 5.6 Summative assessment: Module 5	15 hours
7–8	Module 6 Centroids and centre of gravity	6.1 Centroids 6.2 Centre of gravity	Activity 6.1 Activity 6.2 Summative assessment: Module 6	10 hours
8–9	Module 7 Second moment of area and moment of inertia (second moment of mass)	7.1 Second moment of area 7.2 Moment of inertia 7.3 Depth of centre of fluid pressure	Activity 7.1 Activity 7.2 Activity 7.3 Summative assessment: Module 7	15 hours

Week	Module	Topic	Activities	Hours
9–10	Module 8 Combinations of differentiation and integration	8.1 The length of a curve 8.2 The area of a surface of revolution	Activity 8.1 Activity 8.2 Activity 8.3 Summative assessment: Module 8	12 hours
TOTAL				100 hours

MODULE

1 Differentiation

After they have completed this module, students should be able to:

- apply differentiation to first and second order partial derivatives by:
  - partially differentiating a function consisting of two or more variables with respect to one variable only;
  - using successive differentiation to obtain the second derivative(s) of a function consisting of two variables;
  - calculating specific values of the first and second order partial derivative(s) at specified coordinates;
- apply differentiation to practical (real-life) problems by analy sing, recreating and applying partial differentiation then interpreting results; and
- apply differentiation to first and second order parametric equations by:
  - differentiating two functions consisting of the same variable (parameter);
  - using successive differentiation to obtain the second derivative of two functions consisting of the same variable (parameter);
  - calculating specific values of the derivative(s) at specified coordinates.

Introduction

Students have learnt that differentiation is an important fundamental concept of Mathematics. It allows us to find the rate of change of one variable with respect to another.

Throughout this module they will learn more about the basics of differentiation which include partial derivatives as well as parametric equations. This should give them a solid understanding of how differentiation works. In further modules they will learn about more advanced differential techniques and how to apply them.

Students need the following pre-knowledge to successfully complete this module.

Pre-knowledge

Students should already know:

- Basic principles of differentiation;
- Standard forms of derivatives, as found in the formulae sheet;
- Rules of differentiation including:
  - the chain rule:
    $\frac{dy}{dx} = \frac{dy}{du} . \frac{du}{dx}$
  - the product rule:
    If $y = u(x).v(x)$ then $\frac{dy}{dx} = u'(x).v(x) + u(x).v'(x)$
  - the quotient rule:
    If $y = \frac{u(x)}{v(x)}$ then $\frac{dy}{dx} = \frac{u'(x).v(x) - u(x).v'(x)}{(v(x))^2}$
- How to differentiate successively to obtain a second order derivative:
  $\frac{d}{dx}\left(\frac{dy}{dx}\right)$

Activity 1.1 — SB page 8

1. $\frac{\partial z}{\partial x} = 6x - 4y$

   $\frac{\partial z}{\partial y} = -4x + 30y^5$

2. $\frac{\partial z}{\partial x} = yx^{(y-1)}$

   $\frac{\partial z}{\partial y} = x^y \ln x$

3. $\frac{\partial z}{\partial x} = -y \sin xy$ • Chain rule

   $\frac{\partial z}{\partial y} = -x \sin xy$ • Chain rule

4. $\frac{\partial z}{\partial x} = \frac{2x(x^4 + y^2) - (x^2 - y^4)4x^3}{(x^4 + y^2)^2}$ • Quotient rule

   $= \frac{2x^5 + 2xy^2 - 4x^5 + 4x^3y^4}{(x^4 + y^2)^2}$

   $= \frac{-2x^5 + 2xy^2 + 4x^3y^4}{(x^4 + y^2)^2}$

   $\frac{\partial z}{\partial y} = \frac{-4y^3(x^4 + y^2) - (x^2 - y^4)2y}{(x^4 + y^2)^2}$ • Quotient rule

   $= \frac{-4y^5 - 4x^4y^3 - 2x^2y + 2y^5}{(x^4 + y^2)^2}$

   $= \frac{-2y^5 - 4x^4y^3 - 2x^2y}{(x^4 + y^2)^2}$

5. $\frac{\partial z}{\partial x} = \frac{4\sec^2(4x+3y)(x^2y) - 2xy\tan(4x+3y)}{(x^2y)^2}$ • Quotient rule and chain rule

$= \frac{4x^2y\sec^2(4x+3y) - 2xy\tan(4x+3y)}{x^4y^2}$

$\frac{\partial z}{\partial y} = \frac{3\sec^2(4x+3y)(x^2y) - x^2\tan(4x+3y)}{(x^2y)^2}$ • Quotient rule and chain rule

$= \frac{3x^2y\sec^2(4x+3y) - x^2\tan(4x+3y)}{x^4y^2}$

6. $\frac{\partial z}{\partial x} = ye^x$

$\frac{\partial z}{\partial y} = e^x$

7. $\frac{\partial z}{\partial x} = 4(x^4 - 2y) + (4x + 5y^4)4x^3$ • Product rule

$= 4x^4 - 8y + 16x^4 + 20x^3y^4$

$= 20x^4 + 20x^3y^4 - 8y$

If $x = -2$ and $y = 1$

$\frac{\partial z}{\partial x} = 20(-2)^4 + 20(-2)^3 1^4 - 8(1)$

$= 320 - 160 - 8$

$= 152$

$\frac{\partial z}{\partial y} = 20y^3(x^4 - 2y) + (4x + 5y^4)(-2)$ • Product rule

$= 20x^4y^3 - 40y^4 - 8x - 10y^4$

$= -50y^4 + 20x^4y^3 - 8x$

If $x = -2$ and $y = 1$

$\frac{\partial z}{\partial y} = -50(1)^4 + 20(-2)^4 1^3 - 8(-2)$

$= -50 + 320 + 16$

$= 286$

8. $\frac{\partial z}{\partial x} = -15x^2y^2 + 6xy$

$\therefore$ At coordinate (2; 1):

$\frac{\partial z}{\partial x} = -15(2)^2(-1)^2 + 6(2)(-1)$

$= -60 - 12$

$= -72$

$\frac{\partial z}{\partial y} = -10x^3y - 4y^3 + 3x^2$

$\therefore$ At coordinate (2; 1):

$$\frac{\partial z}{\partial y} = -10(2)^3(-1) - 4(-1)^3 + 3(2)^2$$

$$= 80 + 4 + 12$$

$$= 96$$

Activity 1.2 SB page 14

1. First find the first partial derivatives: $\frac{\partial u}{\partial x} = 15x^4$ and $\frac{\partial u}{\partial y} = 8y$

   Then:

   $$\frac{\partial^2 u}{\partial x^2} = 60x^3$$

   $$\frac{\partial^2 u}{\partial y^2} = 8$$

   $$\frac{\partial^2 u}{\partial y \partial x} = 0 = \frac{\partial^2 u}{\partial x \partial y}$$

2. First find the first partial derivatives: $\frac{\partial v}{\partial x} = 9x^2y^2 + 4y + 4y^2$ and $\frac{\partial v}{\partial y} = 6x^3y + 4x + 8xy$

   Then:

   $$\frac{\partial^2 v}{\partial x^2} = 18xy^2$$

   $$\frac{\partial^2 v}{\partial y^2} = 6x^3 + 8x$$

   $$\frac{\partial^2 v}{\partial y \partial x} = 18x^2y + 4 + 8y = \frac{\partial^2 v}{\partial x \partial y}$$

3. First find the first partial derivatives: $\frac{\partial z}{\partial x} = 2x\cos(x^2 - y^2)$ and $\frac{\partial z}{\partial y} = -2y\cos(x^2 - y^2)$

   Then:

   $$\frac{\partial^2 z}{\partial x^2} = 2\cos(x^2 - y^2) - 4x^2\sin(x^2 - y^2)$$

   $$\frac{\partial^2 z}{\partial y^2} = -2\cos(x^2 - y^2) - 4y^2\sin(x^2 - y^2)$$

   $$\frac{\partial^2 z}{\partial y \partial x} = 4xy\sin(x^2 - y^2) = \frac{\partial^2 z}{\partial x \partial y}$$

4. First find the first partial derivatives: $\frac{\partial z}{\partial x} = y^2\cos(y^2x)$ and $\frac{\partial z}{\partial y} = 2xy\cos(y^2x)$

   Then:

   $$\frac{\partial^2 z}{\partial x^2} = -y^4\sin(y^2x)$$

   $$\frac{\partial^2 z}{\partial y^2} = 2x\cos(y^2x) - 4x^2y^2\sin(y^2x)$$

   $$\frac{\partial^2 z}{\partial y \partial x} = 2y\cos(y^2x) - 2xy^3\sin(y^2x) = \frac{\partial^2 z}{\partial x \partial y}$$

5. First find the first partial derivatives: $\frac{\partial h}{\partial x} = 2x\, e^{x^2+y^2}$ and $\frac{\partial h}{\partial y} = 2y\, e^{x^2+y^2}$

Then:

$$\frac{\partial^2 h}{\partial x^2} = 2e^{x^2+y^2} + 4x^2 e^{x^2+y^2}$$

$$\frac{\partial^2 h}{\partial y^2} = 2e^{x^2+y^2} + 4y^2 e^{x^2+y^2}$$

$$\frac{\partial^2 h}{\partial y \partial x} = 4xye^{x^2+y^2} = \frac{\partial^2 h}{\partial x \partial y}$$

6. First find the first partial derivatives: $\frac{\partial u}{\partial x} = \frac{(x-y)-(x+y)}{(x-y)^2} = \frac{-2y}{(x-y)^2}$

and $\frac{\partial u}{\partial y} = \frac{(x-y)+(x+y)}{(x-y)^2} = \frac{2x}{(x-y)^2}$

Then:

$$\frac{\partial^2 u}{\partial x^2} = -2y(-2)(x-y)^{-3}$$

$$= \frac{4y}{(x-y)^3}$$

If $x = 1$ and $y = -1$:

$$\frac{\partial^2 u}{\partial x^2} = \frac{4(-1)}{(1-(-1))^3}$$

$$= \frac{-4}{8} = \frac{-1}{2}$$

$$\frac{\partial^2 u}{\partial y^2} = 2x(-2)(x-y)^{-3}(-1)$$

$$= \frac{4x}{(x-y)^3}$$

If $x = 1$ and $y = -1$:

$$\frac{\partial^2 u}{\partial y^2} = \frac{4(1)}{(1-(-1))^3}$$

$$= \frac{4}{8} = \frac{1}{2}$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{-2(x-y)^2 - 4y(x-y)}{(x-y)^4}$$

$$= \frac{-2}{(x-y)^2} + \frac{-4y}{(x-y)^3}$$

$$= \frac{-2x+2y-4y}{(x-y)^3}$$

$$= \frac{-2x-2y}{(x-y)^3}$$

$$= \frac{\partial^2 u}{\partial x \partial y}$$

If $x = 1$ and $y = -1$:

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{-2(1)-2(-1)}{(1-(-1))^3}$$

$$= \frac{0}{8} = 0$$

7. First find the first partial derivatives: $\frac{\partial f}{\partial x} = 6x^2y^5 + 12x^3y$ and $\frac{\partial f}{\partial y} = 10x^3y^4 + 3x^4$

Then:

$$\frac{\partial^2 f}{\partial x^2} = 12xy^5 + 36x^2y$$

If $x = 2$ and $y = -1$

$$\frac{\partial^2 f}{\partial x^2} = 12(2)(-1)^5 + 36(2)^2(-1)$$

$$= -24 - 144$$

$$= -168$$

$$\frac{\partial^2 f}{\partial y^2} = 40x^3y^3$$

If $x = 2$ and $y = -1$

$$\frac{\partial^2 f}{\partial y^2} = 40(2)^3(-1)^3$$

$$= -320$$

$$\frac{\partial^2 f}{\partial x \partial y} = 30x^2y^4 + 12x^3 = \frac{\partial^2 f}{\partial y \partial x}$$

If $x = 2$ and $y = -1$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = 30(2)^2(-1)^4 + 12(2)^3$$

$$= 120 + 96$$

$$= 216$$

8. First find the first partial derivatives: $\frac{\partial w}{\partial u} = \frac{3u}{\sqrt{u^2 + v^2}}$ and $\frac{\partial w}{\partial v} = \frac{3v}{\sqrt{u^2 + v^2}}$

Then:

$$\frac{\partial^2 w}{\partial u^2} = \frac{3v^2}{(u^2 + v^2)^{\frac{3}{2}}}$$

If $u = 4$ and $v = -3$

$$\frac{\partial^2 w}{\partial u^2} = \frac{3(-3)^2}{((4)^2 + (-3)^2)^{\frac{3}{2}}}$$

$$= \frac{27}{125}$$

$$= 0{,}216$$

$$\frac{\partial^2 w}{\partial v^2} = \frac{3u^2}{(u^2 + v^2)^{\frac{3}{2}}}$$

If $u = 4$ and $v = -3$

$$\frac{\partial^2 w}{\partial v^2} = \frac{3(4)^2}{((-3)^2 + (4)^2)^{\frac{3}{2}}}$$

$$= \frac{48}{125}$$

$$= 0{,}384$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{-3uv}{(u^2 + v^2)^{\frac{3}{2}}} = \frac{\partial^2 f}{\partial y \partial x}$$

If $u = 4$ and $v = -3$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{-3(4)(-3)}{((-3)^2 + (4)^2)^{\frac{3}{2}}}$$

$$= \frac{36}{125}$$

$$= 0{,}288$$

Activity 1.3 SB page 20

1. The surface area, S = wd, therefore $\frac{\partial S}{\partial w} = d$ and $\frac{\partial S}{\partial d} = w$.

$$\Delta S \approx \frac{\partial S}{\partial w}\Delta w + \frac{\partial S}{\partial d}\Delta d$$
$$\approx (d)\Delta w + (w)\Delta d$$
$$\approx (21)(0{,}72) + (27)(0{,}5)$$
$$\approx 15{,}12 + 13{,}5$$
$$\approx 28{,}62 \text{ mm}^2$$

The surface area increased by approximately 28,62 mm².

2. $\Delta P \approx \frac{\partial P}{\partial I}\Delta I + \frac{\partial P}{\partial R}\Delta R$

P = I^2R, therefore $\frac{\partial P}{\partial I} = 2IR$ and $\frac{\partial P}{\partial R} = I^2$

$$\Delta P \approx (2IR)\Delta I + (I^2)\Delta R$$
$$\approx 2(2)(100)(0{,}2) + (2)^2(-5)$$
$$\approx 80 - 20$$
$$\approx 60 \text{ W}$$

The power increased by approximately 60 W.

3. $e = \frac{1}{2}mv^2$, therefore $\frac{\partial e}{\partial m} = \frac{1}{2}v^2$ and $\frac{\partial e}{\partial v} = mv$

$$\Delta e \approx \frac{\partial e}{\partial m}\Delta m + \frac{\partial e}{\partial v}\Delta v$$
$$\approx \left(\frac{1}{2}v^2\right)\Delta m + (mv)\Delta v$$
$$\approx \frac{1}{2}(12\ 500)^2(-1) + (1\ 930)(12\ 500)(100)$$
$$\approx 2{,}334 \times 10^9 J$$

(Note that the mass decreases, so $\Delta m = -1$)

The spacecraft's kinetic energy increased by approximately 2,334 GJ.

4. $I = \frac{V}{R}$

$\therefore I = VR^{-1}$

$\Delta I = \frac{\partial I}{\partial V}\Delta V + \frac{\partial I}{\partial R}\Delta R$

$= R^{-1}\Delta V - VR^{-2}\,\Delta R$

$= \frac{1}{R}\Delta V - \frac{V}{R^2}\,\Delta R$

$= \frac{1}{(10)}(-0{,}2) - \frac{20}{(10)^2}\,(-0{,}6)$

$= -\frac{0{,}2}{10} + \frac{1{,}2}{10}$

$= \frac{1}{10}$

$= 0{,}1$ A

The current increases by 0,1 A.

5. $V = \pi r^2 h$

$\Delta V = \frac{\partial V}{\partial r}\Delta r + \frac{\partial V}{\partial h}\Delta h$

$= 2\pi rh\,\Delta r + \pi r^2\,\Delta h$

$= 2\pi(5)(15)(0{,}1) + \pi(5)^2(0{,}5)$

$= 15\pi + 12{,}5\pi$

$= 27{,}5\pi$

$= 86{,}394\ \text{cm}^3$

The volume of the cylinder would change by approximately 86 cm^3. This is a significant change considering that 1 cm^3 is equivalent to 1 ml. The cylinder would be able to hold considerably more.

Activity 1.4 SB page 26

1. 1.1 Given $x = t$, $y = 2t$

Then $y = 2x$

$\frac{dy}{dx} = 2$

$\frac{d^2y}{dx^2} = 0$

1.2 Given $x = \sin t$, $y = -\cos 2t$

Then $\frac{dx}{dt} = \cos t$

$\frac{dy}{dt} = 2 \sin 2t$

$\frac{dy}{dx} = \frac{dy}{dt} . \frac{dt}{dx}$

$= \frac{2 \sin 2t}{\cos t}$

$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) . \frac{dt}{dx}$

$= \frac{d}{dt}\left(\frac{2 \sin 2t}{\cos t}\right) . \frac{1}{\cos t}$

$= \left[\frac{(4 \cos 2t)\cos t - (2 \sin 2t)(-\sin t)}{(\cos t)^2}\right] . \frac{1}{\cos t}$

$= \frac{[4 \cos 2t \cos t] + [(2 \sin 2t)(\sin t)]}{\cos^3 t}$

1.3 Given $x = 1 + \frac{1}{t}$, $y = \ln t$

Then $\frac{dx}{dt} = -t^{-2} \therefore \frac{dt}{dx} = -t^2$

$\frac{dy}{dt} = \frac{1}{t}$

$\frac{dy}{dx} = \frac{dy}{dt} . \frac{dt}{dx}$

$= \frac{1}{t} . -t^2$

$= -t$

$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) . \frac{dt}{dx}$

$= \frac{d}{dt} . (-t) . -t^2$

$= \frac{d}{dt} . (t^3)$

$= 3t^2$

2. Given $x = \sin \theta$, $y = 1 - \cos \theta$

Then $\frac{dx}{d\theta} = \cos \theta$

$\frac{dy}{d\theta} = \sin \theta$

$\frac{dy}{dx} = \frac{dy}{d\theta} . \frac{d\theta}{dx}$

$= \sin(\theta) \div \cos(\theta)$

$= \tan \theta$

When $\theta = \frac{\pi}{4}$

$$\frac{dy}{dx} = \tan\left(\frac{\pi}{4}\right) = 1$$

$$\frac{d^2y}{dx^2} = \frac{d}{d\theta}\left(\frac{dy}{dx}\right).\frac{d\theta}{dx}$$

$$= \sec^2\theta \div \cos\theta$$

$$= \frac{1}{\cos^3\theta}$$

When $\theta = \frac{\pi}{4}$

$$\frac{d^2y}{dx^2} = \frac{1}{\cos^3\left(\frac{\pi}{4}\right)}$$

$$= 2\sqrt{2}$$

3. $x = t^2$ and $y = 2t^5$

$\therefore \frac{dx}{dt} = 2t$ $\therefore \frac{dy}{dt} = 10t^4$

$$\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx}$$

$$= \frac{dy}{dt} \div \frac{dx}{dt}$$

$$= 10t^4 \div 2t$$

$$= 5t^3$$

When $t = 5$:

$$\frac{dy}{dx} = 5(5)^3$$

$$= 625$$

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right).\frac{dt}{dx}$$

$$= \frac{d}{dt}(5t^3).\frac{1}{2t}$$

$$= \frac{15t^2}{2t}$$

$$= 7{,}5\ t$$

When $t = 5$:

$$\frac{d^2y}{dx^2} = 7{,}5\ t$$

$$= 37{,}5$$

4. $x = 2\sqrt{s}$ and $y = \frac{2}{\sqrt{s}}$

$\therefore \frac{dx}{ds} = s^{-\frac{1}{2}}$ $\therefore \frac{dy}{ds} = -s^{-\frac{3}{2}}$

$\frac{dy}{dx} = \frac{dy}{ds} \div \frac{dx}{ds}$

$= -s^{-\frac{3}{2}} \div s^{-\frac{1}{2}}$

$= -s^{-1}$

$\frac{d^2y}{dx^2} = \frac{d}{ds}\left(\frac{dy}{dx}\right).\frac{ds}{dx}$

$= \frac{d}{ds}(-st^{-1}).\frac{1}{s^{-\frac{1}{2}}}$

$= s^{-2}.s^{\frac{1}{2}}$

$= s^{-\frac{3}{2}}$

5. $x = 4e^{2v}$ and $y = e^{3v} - 5$

$\therefore \frac{dx}{dv} = 8e^{2v}$ $\therefore \frac{dy}{dv} = 3e^{3v}$

$\frac{dy}{dx} = \frac{dy}{dv} \div \frac{dx}{dv}$

$= 3e^{3v} \div 8e^{2v}$

$= \frac{3}{8}e^{v}$

$\frac{d^2y}{dx^2} = \frac{d}{dv}\left(\frac{dy}{dx}\right).\frac{dv}{dx}$

$= \frac{d}{dv}\left(\frac{3}{8}e^{v}\right).\frac{1}{8e^{2v}}$

$= \frac{3}{8}e^{v}.\frac{1}{8e^{2v}}$

$= \frac{3}{64}e^{-v}$

Summative assessment: Module 1 SB page 27

1. Given: $y = t^3$ and $x = e^{2t}$

$\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx}$

$\frac{dy}{dt} = 3t^2; \frac{dx}{dt} = 2e^{2t}$

$\therefore \frac{dy}{dx} = (3t^2) \div (2e^{2t})$

$= \frac{3}{2}t^2e^{-2t}$ (3)

2. Given: $z = \sin x \ln y$

$$\frac{\partial z}{\partial y} = \sin x \frac{1}{y}$$

$$= \frac{\sin x}{y}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\cos x}{y}$$

When $(x; y) = \frac{\pi}{6}; \frac{1}{4}$:

$$\frac{\partial z}{\partial y} = \frac{\sin\left(\frac{\pi}{6}\right)}{\left(\frac{1}{4}\right)} = 2$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\cos\left(\frac{\pi}{6}\right)}{\left(\frac{1}{4}\right)} = 2\sqrt{3}$$ (4)

3. Given: $t = x^2 + xy^3 - 2y + 3x^2y^2$

3.1 $\frac{\partial t}{\partial x} = 2x + y^3 + 6xy^2$ (2)

3.2 $\frac{\partial t}{\partial y} = 3xy^2 - 2 + 6x^2y$ (2)

3.3 $\frac{\partial^2 t}{\partial x \partial y} = 3y^2 + 12xy$ (2)

4. Given: $y = 2\tan\theta + \pi$ and $x = \sec\theta$

4.1 $\frac{dy}{d\theta} = 2\sec^2\theta;$

$$\frac{dx}{d\theta} = \tan\theta \sec\theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx}$$

$$= 2\sec^2\theta \div \tan\theta \sec\theta$$

$$= \frac{2\sec\theta}{\tan\theta}$$

$$= 2\operatorname{cosec}\theta$$

When $\theta = \frac{\pi}{4}$:

$$\frac{dy}{dx} = 2\operatorname{cosec}\left(\frac{\pi}{4}\right)$$

$$= 2\sqrt{2}$$ (6)

4.2 $\frac{d^2y}{dx^2} = \frac{d}{d\theta}\left(\frac{dy}{dx}\right) \cdot \frac{d\theta}{dx}$

$$\frac{d}{d\theta}(2\operatorname{cosec}\theta) = -2\cot\theta\operatorname{cosec}\theta$$

$$\therefore \frac{d^2y}{dx^2} = (-2 \cot \theta \operatorname{cosec} \theta) \div (\tan \theta \sec \theta)$$

$$= -2\left(\frac{\cos \theta}{\sin \theta}\right)\left(\frac{1}{\sin \theta}\right)\left(\frac{\cos \theta}{\sin \theta}\right)\left(\frac{\cos \theta}{1}\right)$$

$$= -2\left(\frac{\cos \theta}{\sin \theta}\right)^3$$

$$= -2 \cot^3\theta$$

When $\theta = \frac{\pi}{4}$:

$$\frac{d^2y}{dx^2} = -2 \cot^3\frac{\pi}{4}$$

$$= -2$$

(4)

5. Given: $y = \theta \sin 3\theta$ and $x = \theta(\theta^2 + 3)$

$$\frac{dy}{dx} = \frac{dy}{d\theta}\cdot\frac{d\theta}{dx}$$

$$\frac{dy}{d\theta} = \sin 3\theta + 3\theta \cos 3\theta$$

$$\frac{dx}{d\theta} = 3\theta^2 + 3$$

$$\therefore \frac{dy}{dx} = (\sin 3\theta + 3\theta \cos 3\theta) \div (3\theta^2 + 3)$$

$$= \frac{\frac{1}{3}\sin 3\theta + \theta \cos \theta}{\theta^2 + 1}$$

When $\theta = \pi$ rad

$$\frac{dy}{dx} = \frac{0 + \pi}{\pi^2 + 1}$$

$$= 0{,}289 \text{ rad}$$

(4)

6. $V = \frac{1}{3}\pi r^2 h$

Therefore $\frac{dV}{dh} = \frac{1}{3}\pi r^2$ and $\frac{dV}{dr} = \frac{2}{3}\pi rh$

$$\Delta V \approx \frac{\partial V}{\partial r}\Delta r + \frac{\partial V}{\partial h}\Delta h$$

$$= \frac{2}{3}\pi rh\Delta r + \frac{1}{3}\pi r^2 \Delta h$$

$$= \frac{1}{3}\pi r(2h\Delta r + r\Delta h)$$

Since $\frac{r}{h} = \frac{4}{6}$,

Then $\frac{\Delta r}{\Delta h} = \frac{4}{6}$

so $\Delta r = \frac{4}{6}\Delta h$

$$= \frac{4}{6}(0{,}01)$$

$$= 0{,}0067 \text{ m}$$

So, the radius changes by 0,0067 m for every 1 cm increase in depth.

6.1 When $h = 5$, then $r = \frac{4}{6}h = 3{,}333$ m

Therefore

$$\Delta V \approx \frac{1}{3}\pi r(2h\Delta r + r\Delta h)$$
$$= \frac{1}{3}\pi(3{,}333)[2(5)(0{,}0067) + (3{,}333)(0{,}01)]$$
$$= 0{,}105 \text{ m}^3$$

The volume increases by 0,105 m^3 for every 1 cm increase in depth at a depth of 5 m. (8)

6.2 When $h = 1$, then $r = \frac{4}{6}h = 0{,}667$ m

Therefore

$$\Delta V \approx \frac{1}{3}\pi r(2h\Delta r + r\Delta h)$$
$$= \frac{1}{3}\pi(0{,}667)[2(1)(0{,}0067) + (0{,}667)(0{,}01)]$$
$$= 0{,}021 \text{ m}^3$$

The volume increases by 0,021 m^3 for every 1 cm increase in depth at a depth of 1 m. (2)

6.3 The change in volume is more accurate when $h = 5$, (question 6.1) because the approximation works best for small relative changes. A change of 1 cm is less significant when the tank is fuller $\left(\frac{0{,}01}{5} \times 100\%\right)$ than when it is emptier $\left(\frac{0{,}01}{1} \times 100\%\right)$. (2)

7. $r = \sqrt{x^2 + y^2}$

Therefore,

$$\frac{\partial r}{\partial x} = \frac{2x}{2\sqrt{x^2 + y^2}}$$
$$= \frac{x}{\sqrt{x^2 + y^2}}$$

And similarly,

$$\frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}}$$

So

$$\Delta r \approx \frac{\partial r}{\partial x}\Delta x + \frac{\partial r}{\partial y}\Delta y$$
$$= \frac{x}{\sqrt{x^2 + y^2}}\Delta x + \frac{y}{\sqrt{x^2 + y^2}}\Delta y$$
$$= \frac{x\,\Delta x + y\Delta y}{\sqrt{x^2 + y^2}}$$

Substitute the given values:

$$= \frac{(16)(1) + (12)(1)}{\sqrt{(16)^2 + (12)^2}}$$

= 1,4 cm

At those dimensions, the diagonal increases by approximately 1,4 cm for each 1 cm increase in width and in length. (6)

8. Given $r = 5$ cm, $h = 15$ cm and $\Delta r = 0{,}5$ cm, $\Delta h = 2{,}5$ cm

$V = \pi r^2 h$

$$\Delta V \approx \frac{\partial V}{\partial r}\Delta r + \frac{\partial V}{\partial h}\Delta h$$

$$= 2\pi rh\Delta r + \pi r^2(\Delta h)$$

$$= 2\pi(5)(15)(0{,}5) + \pi 5^2(2{,}5)$$

$$= 431{,}969 \text{ cm}^3$$ (5)

TOTAL: [50]

MODULE

2 Integration techniques

After they have completed this module, students should be able to:

- use integration by parts to integrate the product of a function and the derivative of another function, where neither is a derivative of the other;
- apply specific integration techniques to powers of trigonometric functions:
  - $\sin^m ax$ and $\cos^n ax$;
  - $\tan^m ax$ and $\cot^n ax$;
  - $\sin^m ax.\cos^n ax$; and
- integrate by means of completing the square, applied to the functions:
  - $\frac{1}{\sqrt{ax^2 + bx + c}}$;
  - $\frac{1}{ax^2 + bx + c}$;
  - $\frac{1}{c + bx - ax^2}$;
  - $\frac{1}{\sqrt{c + bx - ax^2}}$.

Introduction

Previously, students have studied the basic concepts of integration. This module expands on that knowledge to explore further integration techniques. Students will learn more about how to integrate by using integration by parts; integrate trigonometric functions; and integrate by means of completing the square.

Students need the following pre-knowledge to successfully complete this module.

Pre-knowledge

Students should already know:

- Standard forms of integrals:

Table 2.1: Standard integrals

$\int cf(x)\,dx = c\int f(x)\,dx$	$\int f(x)\,dx + \int g(x)\,dx = \int [f(x) + g(x)]\,dx$
$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, (n \neq 1)$	$\int a^x\,dx = \frac{a^x}{\ln(a)} + C$
$\int \frac{1}{x}\,dx = \ln\lvert x\rvert + C$	$\int e^x\,dx = e^x + C$
$\int \sin(x)\,dx = -\cos(x) + C$	$\int \cos(x) = \sin(x) + C$
$\int \sec^2(x)\,dx = \tan(x) + C$	$\int \text{cosec}^2(x)\,dx = -\cot(x) + C$
$\int \sec(x)\tan(x)\,dx = \sec(x) + C$	$\int \text{cosec}(x)\cot(x)\,dx = -\text{cosec}(x) + C$
$\int \tan(x)\,dx = \ln\lvert\sec(x)\rvert + C$	$\int \cot(x)\,dx = \ln\lvert\sin(x)\rvert + C$

- Integration by using antiderivatives and known integral forms:
  $\int \left[\frac{d}{dx}f(x)\right]dx = \frac{d}{dx}\left[\int f(x)\,dx\right] = f(x)$
- Integration by inspection, identifying the following forms:
  $\int [f(x)]^n.f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1} + C; (n \neq -1)$
  $\int \frac{f'(x)}{f(x)}\,dx = \ln f(x) + C$
- Integrating composite functions by algebraic substitution:
  If $\int f(g(x))g'(x)\,dx$, set $g(x) = u$ then $g'(x) = du$ and $\int f(u)\,du$
- Integration by parts to integrate the product of two functions, where one function is not the derivative of the other:
  $\int f(x).g'(x)\,dx = f(x).g(x) - \int f'(x).g(x)\,dx$
- Trigonometric identities and how to manipulate them:

Table 2.2: Trigonometric identities

$\sin^2(x) + \cos^2(x) = 1$	$\tan^2(x) + 1 = \sec^2(x)$
$1 + \cot^2(x) = \text{cosec}^2(x)$	$\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$
$\cos^2(x) = \frac{1}{2}(1 + \cos(2x)$	

- How to use algebraic substitution to integrate trigonometric functions:
  $\int \sin[f(x)].f'(x)\,dx = -\cos[f(x)] + C$
  $\int \cos[f(x)].f'(x)\,dx = \sin[f(x)] + C$
  $\int \sec^2[f(x)].f'(x)\,dx = \tan[f(x)] + C$
  $\int \text{cosec}^2[f(x)].f'(x)\,dx = \cot[f(x)] + C$
  $\int \sec[f(x)].\tan[f(x)].f'(x)\,dx = \sec[f(x)] + C$
  $\int \text{cosec}[f(x)].\cot[f(x)].f'(x)\,dx = \text{cosec}[f(x)] + C$
- How to use trigonometric substitutions to simplify certain integrals:

Table 2.3: Trigonometric substitutions

Substitution	Integral
$bx = a\tan\theta$	$\int \frac{1}{a^2 + b^2x^2}\,dx = \frac{1}{ab}\tan^{-1}\left(\frac{b}{a}x\right) + C$
$bx = a\sin\theta$	$\int \frac{1}{\sqrt{a^2 - b^2x^2}}\,dx = \frac{1}{b}\sin^{-1}\left(\frac{b}{a}x\right) + C$
$bx = a\sin\theta$	$\int \sqrt{a^2 - b^2x^2}\,dx = \frac{a^2}{2b}\sin^{-1}\left(\frac{b}{a}x\right) + \frac{1}{2}x\sqrt{a^2 - b^2x^2} + C$

- How to complete the square by writing $ax^2 + bx + c$ in the form $a(x - h)^2 + k$.

Activity 2.1 SB page 41

1. $\int(x + 4)\sin(x)\,dx = \int x\sin(x)\,dx + \int 4\sin(x)\,dx$
   Let $f(x) = x$, then $f'(x) = 1$
   Let $g'(x) = \sin(x)$, then $g(x) = -\cos(x)$
   $\int x\sin(x)\,dx = -x\cos(x) + \sin(x) + C$
   and $\int 4\sin(x)\,dx = -4\cos(x) + C$
   $\int(x + 4)\sin(x)\,dx = -x\cos(x) + \sin(x) - 4\cos(x) + C$
   $= \sin(x) - (x + 4)\cos(x) + C$
   $\therefore \int(x + 4)\sin(x)\,dx = \sin(x) - (x + 4)\cos(x) + C$

2. $\int(2x + 3)\cos(x)\,dx = \int 2x\cos(x)\,dx + \int 3\cos(x)\,dx$
   Let $f(x) = x$, then $f'(x) = 1$
   Let $g'(x) = \cos(x)$, then $g(x) = \sin(x)$
   $\int 2x\cos(x)\,dx = 2[x\sin(x) + \cos(x) + C]$
   and $\int 3\cos(x)\,dx = 3\sin(x) + C$
   $\therefore \int(2x + 3)\cos(x)\,dx = 2\cos(x) + (2x + 3)\sin(x) + C$

3. Let $f(x) = x$, then $f'(x) = 1$
   Let $g'(x) = \cos(x)$, then $g(x) = \sin(x)$
   $\therefore \int x \cos(x)\, dx = x \sin(x) + \cos(x) + C$

4. Let $f(x) = x^2$, then $f'(x) = 2x$
   Let $g'(x) = \cos(x)$, then $g(x) = \sin(x)$
   $\int x^2 \cos(x)\, dx = x^2 \sin(x) - \int 2x \sin(x)\, dx \quad \ldots(1)$

   Need to do integration by parts twice, therefore let:
   $f(x) = 2x$, then $f'(x) = 2$
   $g'(x) = \sin(x)$, then $g(x) = -\cos(x)$
   $\int 2x \sin(x)\, dx = -2x \cos(x) - \int -2 \cos(x)\, dx$
   $\int 2x \sin(x)\, dx = -2x \cos(x) + 2 \sin(x) + C$

   Substituting back into equation (1):
   $\int x^2 \cos(x)\, dx = x^2 \sin(x) + 2x \cos(x) - 2 \sin(x) + C$
   $\therefore \int x^2 \cos(x)\, dx = (x^2 - 2) \sin(x) + 2x \cos(x) + C$

5. Let $f(x) = x^3$, then $f'(x) = 3x^2$
   Let $g'(x) = \cos(x)$, then $g(x) = \sin(x)$
   $\int x^3 \cos(x)\, dx = x^3 \sin(x) - 3\int x^2 \sin(x)\, dx \quad \ldots(1)$

   Need to perform integration by parts again therefore let:
   $f(x) = x^2$, then $f'(x) = 2x$
   $g'(x) = \sin(x)$, then $g(x) = -\cos(x)$
   $\int x^2 \sin(x)\, dx = -x^2 \cos(x) - \int -2x \cos(x)\, dx \quad \ldots(2)$

   Need to perform integration by parts again therefore let:
   $f(x) = x$, then $f'(x) = 1$
   $g'(x) = \cos(x)$, then $g(x) = \sin(x)$
   $\int -2x \cos(x)\, dx = -2[x \sin(x) - \int \sin(x)\, dx]$
   $\int -2x \cos(x)\, dx = -2[x \sin(x) - (-\cos(x))] + C$

   Substituting back into equation (2):
   $\int x^2 \sin(x)\, dx = -x^2 \cos(x) + 2[x \sin(x) + \cos(x)] + C$

   Substituting back into equation (1):
   $\int x^3 \cos(x)\, dx = x^3 \sin(x) - 3[-x^2 \cos(x) + 2[x \sin(x) + \cos(x)]] + C$
   $\int x^3 \cos(x)\, dx = x^3 \sin(x) + 3x^2 \cos(x) - 6x \sin(x) - 6 \cos(x) + C$
   $\therefore \int x^3 \cos(x)\, dx = (x^2 - 6)x \sin(x) + 3(x^2 - 2) \cos(x) + C$

6. Let $f(x) = e^x$, then $f'(x) = e^x$
   Let $g'(x) = \cos(x)$, then $g(x) = \sin(x)$
   $\int e^x \cos(x)\, dx = e^x \sin(x) - \int e^x \sin(x)\, dx \quad \ldots(1)$

Need to perform integration by parts again therefore let:
$f(x) = e^x$, then $f'(x) = e^x$
$g'(x) = \sin(x)$, then $g(x) = -\cos(x)$
$\int e^x \sin(x)\, dx = -e^x \cos(x) - [\int -e^x \cos(x)\, dx]$
$\int e^x \sin(x)\, dx = -e^x \cos(x) + [\int e^x \cos(x)\, dx]$

Substituting back into equation (1):
$\int e^x \cos(x)\, dx = e^x \sin(x) - [-e^x \cos(x) + \int e^x \cos(x)\, dx]$
$\int e^x \cos(x)\, dx = e^x \sin(x) + e^x \cos(x) - \int e^x \cos(x)\, dx$

Adding $\int e^x \cos(x)\, dx$ to both sides:
$2\int e^x \cos(x)\, dx = e^x \sin(x) + e^x \cos(x) + C$
$\therefore \int e^x \cos(x)\, dx = \dfrac{e^x(\sin(x) + \cos(x))}{2} + C$

7. Let $f(x) = x$, then $f'(x) = 1$
Let $g'(x) = \text{cosec}^2(x)$, then $g(x) = -\cot(x)$
$\int x\, \text{cosec}^2(x)\, dx = -x \cot(x) - \int (1)(-\cot(x))\, dx$
$\int x\, \text{cosec}^2(x)\, dx = -x \cot(x) + \int \cot(x) dx$
$\therefore \int x\, \text{cosec}^2(x)\, dx = -x \cot(x) + \ln|\sin(x)| + C$

8. Let $f(x) = x^3$, then $f'(x) = 3x^2$
Let $g'(x) = \sin(x)$, then $g(x) = -\cos(x)$
$\int x^3 \sin(x)\, dx = -x^3 \cos(x) - \int -3x^2 \cos(x)\, dx \qquad \ldots(1)$
$\int x^3 \sin(x)\, dx = -x^3 \cos(x) + 3\int x^2 \cos(x)\, dx$

Need to perform integration by parts again therefore let:
$f(x) = x^2$, then $f'(x) = 2x$
$g'(x) = \cos(x)$, then $g(x) = \sin(x)$
$\int x^2 \cos(x)\, dx = x^2 \sin(x) - 2\int x \sin(x)\, dx \qquad \ldots(2)$

Need to perform integration by parts again therefore let:
$f(x) = x$, then $f'(x) = 1$
$g'(x) = \sin(x)$, then $g(x) = -\cos(x)$
$\int x \sin(x)\, dx = -x \cos(x) - \int -\cos(x)\, dx$
$\int x \sin(x)\, dx = -x \cos(x) + \int \cos(x)\, dx$
$\int x \sin(x)\, dx = -x \cos(x) + \sin(x) + C$

Substituting back into equation (2):
$\int x^2 \cos(x)\, dx = x^2 \sin(x) - 2[-x \cos(x) + \sin(x)] + C$

Substituting back into equation (1):
$\int x^3 \sin(x)\, dx = -x^3 \cos(x) + 3[x^2 \sin(x) - 2[-x \cos(x) + \sin(x)]] + C$
$\int x^3 \sin(x)\, dx = -x^3 \cos(x) + 3x^2 \sin(x) + 6x \cos(x) - 6 \sin(x) + C$
$\therefore \int x^3 \sin(x)\, dx = (6 - x^2)x \cos(x) + 3(x^2 - 2) \sin(x) + C$

9. Let $f(x) = x$, then $f'(x) = 1$

   Let $g'(x) = \sec^2(x)$, then $g(x) = \tan(x)$

   $\int x \sec^2(x)\, dx = x\tan(x) - \int \tan(x)\, dx$

   $\therefore \int x \sec^2(x)\, dx = x\tan(x) - \ln|\sec(x)| + C$

10. Let $f(x) = \ln(x)$, then $f'(x) = \frac{1}{x}$

    Let $g'(x) = 1$, then $g(x) = x$

    $\int \ln(x)\, dx = x\ln(x) - \int \frac{1}{x}.x\, dx$

    $\int \ln(x)\, dx = x\ln(x) - \int 1\, dx$

    $\therefore \int \ln(x)\, dx = x\ln(x) - x + C$

Activity 2.2 SB page 55

1. $\int \tan^2(3x)\, dx$

   $u = 3x, \frac{du}{dx} = 3, dx = \frac{du}{3}$

   $\frac{1}{3}\int \tan^2(u)\, du$

   $= \frac{1}{3}\int \left(\sec^2(u) - 1\right) du$ [standard integral]

   $= \frac{1}{3}\int (\tan(u) - u)\, du$

   $= \frac{\tan(3x)}{3} - \frac{3x}{3}$

   $= \frac{\tan(3x)}{3} - x + C$

2. Let: $u = 2x$, then $du = 2\, dx$

   $\int \sin^3(2x)\, dx = \int \sin^3(u)\frac{1}{2}\, du$

   and

   $\sin^2(u) = \left(1 - \cos^2(u)\right)$

   $\int \sin^3(u)\frac{1}{2}\, du = \frac{1}{2}\int \left(1 - \cos^2(u)\right)(\sin(u))\, du$

   Let: $v = \cos(u)$, then $dv = -\sin(u)\, du$

   $\frac{1}{2}\int \left(1 - \cos^2(u)\right)(\sin(u))\, du = \frac{1}{2}\int -(1 - v^2)\, dv$

   $\frac{1}{2}\int -(1 - v^2)\, dv = \frac{1}{2}\left(\frac{v^3}{3} - v\right) + C$

   Substituting the values back in gives:

   $\int \sin^3(2x)\, dx = \frac{\cos^3(2x)}{6} - \frac{\cos(2x)}{2} + C$

3. $\int \sin^5(2x)\, dx$

   Let $u = 2x, \frac{du}{dx} = 2, dx = \frac{du}{2}$

   $\frac{1}{2}\int \sin^5(u)\, du$

   $= \frac{1}{2}\int (1 - \cos^2(u))^2 \sin(u)\, du$

Now let $v = \cos(u)$, $\frac{dv}{du} = -\sin(u)$, $du = -\frac{dv}{\sin(u)}$

$\frac{1}{2}\int -(1 - v^2)^2\, dv$

$= -\frac{1}{2}\int (v^2 - 1)^2\, dv$

$= -\frac{1}{2}\int (v^4 - 2v^2 + 1)\, dv$

$= -\frac{1}{2}\left[\frac{v^5}{5} - \frac{2v^3}{3} + v\right]$

$= -\frac{v^5}{10} + \frac{v^3}{3} - \frac{v}{2}$

$= -\frac{\cos(2x)}{2} + \frac{\cos^3(2x)}{3} - \frac{\cos^5(2x)}{10} + C$

4. $\int \cot^3(x)\, dx = \int \cot^2(x)\cot(x)\, dx$

$\int \left(\operatorname{cosec}^2(x) - 1\right)\cot(x)\, dx = \int \operatorname{cosec}^2(x)\cot(x)\, dx - \int \cot(x)\, dx$

Let: $u = \cot(x)$, then $du = -\operatorname{cosec}^2\, dx$

$\int \operatorname{cosec}^2(x)\cot(x)\, dx = \int u(-du)$

$\int u(-du) = -\frac{u^2}{2} + C$

Substituting:

$-\frac{u^2}{2} + C = -\frac{\cot^2(x)}{2} + C$

and

$\int \cot(x)\, dx = \ln|\sin(x)| + C$

$\therefore \int \cot^3(x)\, dx = -\frac{\cot^2(x)}{2} - \ln|\sin(x)| + C$

5. $\int \cos^5(2x + 5)\sin^2(2x + 5)\, dx$

Let $u = 2x + 5$ $\quad \frac{du}{dx} = 2 \quad dx = \frac{du}{2}$

Then we have $\frac{1}{2}\int \cos^5(u)\sin^2(u)\, du$

$= \frac{1}{2}\int \cos(u)(1 - \sin^2(u))^2\sin^2(u)\, du$

Now let $v = \sin(u)$, $\frac{dv}{du} = \cos(u)$, $du = \frac{dv}{\cos(u)}$

We have $\frac{1}{2}\int (1 - v^2)^2 v^2\, dv$

$= \frac{1}{2}\int (1 - 2v^2 + v^4)v^2\, dv$

$= \frac{1}{2}\left[\int v^2\, dv - \int 2v^4\, dv + \int v^6\, dv\right]$

$= \frac{1}{2}\left(\frac{v^3}{3} - \frac{2v^5}{5} + \frac{v^7}{7}\right)$

Substituting back: $\frac{1}{2}\left(\frac{\sin^3(2x + 5)}{3} - \frac{2\sin^5(2x + 5)}{5} + \frac{\sin^7(2x + 5)}{7}\right)$

$= \frac{\sin^3(2x + 5)}{6} - \frac{\sin^5(2x + 5)}{5} + \frac{\sin^7(2x + 5)}{14} + C$

6. $\int \cot^4(x)\, dx$

$= \int (\text{cosec}^2(x) - 1)^2\, dx$

$= \int (\text{cosec}^4(x) - 2\,\text{cosec}^2(x) + 1)\, dx$

$= \int \text{cosec}^4(x)\, dx - 2\int \text{cosec}^2(x)\, dx + \int 1\, dx$

$= \int \text{cosec}^4(x)\, dx + 2\cot(x) + x$

$\int \text{cosec}^4(x)\, dx = \int (\cot^2(x) + 1)\,\text{cosec}^2(x)\, dx$

Let $u = \cot(x), \frac{du}{dx} = -\text{cosec}^2(x), dx = -\frac{du}{\text{cosec}^2(x)}$

Then: $\int -(u^2 + 1)\, du$

$= \frac{u^3}{3} - u$

$= \frac{-\cot^3(x)}{3} - \cot(x)$

$\therefore \int \cot^4(x)\, dx = -\frac{\cot^3(x)}{3} - \cot(x) + 2\cot(x) + x$

$= -\frac{\cot^3(x)}{3} + \cot(x) + x + C$

7. $\int \tan^4(x)\, dx = \int \tan^2(x)\left[\sec^2(x) - 1\right] dx$

$\int \tan^2(x)\left[\sec^2(x) - 1\right] dx = \int \tan^2(x)\sec^2(x)\, dx - \int \tan^2(x)\, dx \qquad \ldots(1)$

Let:

$u = \tan(x)$, then $du = sec^2(x)\, dx$

$\int \tan^2(x)\sec^2(x)\, dx = u^2\, du$

$\int u^2\, du = \frac{u^3}{3} + C$

Substituting:

$\frac{u^3}{3} + C = \frac{\tan^3(x)}{3} + C$

and

$\int \tan^2(x)\, dx = \int \left[\sec^2(x) - 1\right] dx$

$\int \left[\sec^2(x) - 1\right] dx = -\tan(x) - x + C$

Substituting back into equation (1):

$\int \tan^2(x)\sec^2(x)\, dx - \int \tan^2(x)\, dx = \frac{\tan^3(x)}{3} - (\tan(x) - x) + C$

$\therefore \int \tan^4(x)\, dx = \frac{\tan^3(x)}{3} - x - \tan(x) + C$

8. $\int_{\frac{\pi}{2}}^{0} \cos^4(x) \sin^3(x)\, dx = \int_{\frac{\pi}{2}}^{0} \cos^4(x)(1 - \cos^2(x)) \sin(x)\, dx$

Let:

$u = \cos(x)$, then $du = -\sin(x)\, dx$

$\int_{\frac{\pi}{2}}^{0} \cos^4(x)(1 - \cos^2(x)) \sin(x)\, dx = \int_{\frac{\pi}{2}}^{0} -u^4(1 - u^2)\, du$

$$\int_{\frac{\pi}{2}}^{0} -u^4(1 - u^2)\, du = -\int_{\frac{\pi}{2}}^{0} (u^4 - u^6)\, du$$
$$= \frac{u^7}{7} - \frac{u^5}{5}$$
$$= \frac{\cos^7(x)}{7} - \frac{\cos^5(x)}{5}$$
$$= \left[\frac{\cos^7(0)}{7} - \frac{\cos^5(0)}{5}\right] - \left[\frac{\cos^7\left(\frac{\pi}{2}\right)}{7} - \frac{\cos^5\left(\frac{\pi}{2}\right)}{5}\right]$$
$$= \left[\frac{1}{7} - \frac{1}{5}\right]$$

$\int_{\frac{\pi}{2}}^{0} \cos^4(x) \sin^3(x)\, dx = -\frac{2}{35}$

Activity 2.3 — SB page 59

1. $x^2 + 2x - 1 = 0$

$(x + 1)^2 = x^2 + 2x + 1$

So $x^2 + 2x - 1 = (x + 1)^2 - 2$

$\therefore (x + 1)^2 - 2 = 0$

$x + 1 = \pm \sqrt{2}$

$x = \pm \sqrt{2} - 1$

2. $x^2 + 6x + 5 = 0$

$(x + 3)^2 = x^2 + 6x + 9$

So $x^2 + 6x + 5 = (x + 3)^2 - 4$

$\therefore (x + 3)^2 - 4 = 0$

$x + 3 = \pm \sqrt{4}$

$x = \pm 2 - 3$

$\therefore x = -5$ or $x = -1$

3. $4x^2 + 28x + 40 = 0$

$(2x + 7)^2 = 4x^2 + 28x + 49$

So $4x^2 + 28x + 40 = (2x + 7)^2 - 9$

$\therefore (2x + 7)^2 - 9 = 0$

$2x + 7 = \pm \sqrt{9}$

$2x = \pm \sqrt{9} - 7$

$2x = \pm 3 - 7$

$\therefore x = -5$ or $x = -2$

4. $9x^2 - 24x + 7 = 0$

$(3x - 4)^2 = 9x^2 - 24x + 16$

So $9x^2 - 24x + 7 = (3x - 4)^2 - 9$

$\therefore (3x - 4)^2 - 9 = 0$

$3x - 4 = \pm 3$

$3x = \pm 3 + 4$

$\therefore x = \frac{7}{3}$ or $x = \frac{1}{3}$

5. $4x^2 - 24x + 20$

$(2x - 6)^2 = 4x^2 - 24x + 36$

So $4x^2 - 24x + 20 = (2x - 6)^2 - 16$

$\therefore (2x - 6)^2 - 16 = 0$

$2x - 6 = \pm 4$

$2x = \pm 4 + 6$

$\therefore x = 5$ or $x = 1$

Activity 2.4 SB page 67

1. This is of the form $\int \frac{1}{mx^2 + nx + p}\, dx$.

   Complete the square: $\int \frac{1}{4\left(x + \frac{3}{2}\right) + 4}\, dx$

   Now use the formula $\int \frac{1}{a^2 + b^2x^2}\, dx = \frac{1}{ab} \tan^{-1}\left(\frac{b}{a}x\right) + C,$

   where $a^2 = 4$; $b^2 = 4$ and $\left(x + \frac{3}{2}\right)$ is substituted into x.

   $$\int \frac{1}{4x^2 + 12x + 13}\, dx = \frac{1}{(2)(2)} \tan^{-1}\left[\frac{(2)}{(2)}\left(x + \frac{3}{2}\right)\right] + C$$

   $$= \frac{1}{4} \tan^{-1}\left(x + \frac{3}{2}\right) + C$$

2. This is of the form $\int \frac{1}{\sqrt{-mx^2 + nx + p}}\, dx$.

   Complete the square: $\int \frac{1}{\sqrt{-(x + 4)^2 + 25}}\, dx$

   Now use the formula $\int \frac{1}{\sqrt{a^2 - b^2x^2}}\, dx = \frac{1}{a} \sin^{-1}\left(\frac{b}{a}x\right) + C,$

   where $a^2 = 25$; $b^2 = 1$ and $(x + 4)$ is substituted into x.

   $$\int \frac{1}{\sqrt{9 - 8x - x^2}}\, dx = \frac{1}{(5)} \sin^{-1}\left[\frac{1}{(5)}(x + 4)\right] + C = \frac{1}{5} \sin^{-1}\left(\frac{x + 4}{5}\right) + C$$

3. This is of the form $\int \frac{1}{-mx^2 + nx + p}\, dx$.

   Complete the square: $\frac{1}{2}\int \frac{1}{-4(x - 1)^2 + 9}\, dx$

   Now use the formula $\int \frac{1}{a^2 - b^2x^2}\, dx = \frac{1}{2ab} \ln \left|\frac{a + bx}{a - bx}\right| + C,$

   where $a^2 = 9$; $b^2 = 4$ and $(x - 1)$ is substituted into x.

   $$\int \frac{2}{5 - 4x^2 + 8x}\, dx = \frac{1}{2(3)(2)} \ln \left|\frac{3 + 2(x - 1)}{3 - 2(x - 1)}\right| + C$$

   $$= \frac{1}{12} \ln \left|\frac{1 + 2x}{5 - 2x}\right| + C$$

4. This is of the form $\int \frac{1}{\sqrt{mx^2 + nx + p}}\, dx$.

   Complete the square: $\int \frac{6}{\sqrt{(x + 3)^2 - 12}}\, dx$

   Now use the formula $\int \frac{1}{\sqrt{b^2x^2 \pm a^2}}\, dx = \frac{1}{b} \ln \left|bx + \sqrt{b^2x^2 \pm a^2}\right| + C,$

   where $a^2 = 12$; $b^2 = 1$ and $(x + 3)$ is substituted into x.

   $$\int \frac{6}{\sqrt{x^2 + 6x - 3}}\, dx = \frac{1}{(1)} \ln \left|(1)(x + 3) + \sqrt{(1)(x + 3)^2 - (12)}\right| + C$$

   $$= \ln \left|x + 3 + \sqrt{x^2 + 6x - 3}\right| + C$$

5. This is of the form $\int \frac{1}{-mx^2 + nx + p}\,dx$.

   Complete the square: $\int \frac{1}{16 - (x+2)^2}\,dx$

   Now use the formula $\int \frac{1}{a^2 - b^2x^2}\,dx = \frac{1}{2ab}\ln\left|\frac{a+bx}{a-bx}\right| + C$,

   where $a^2 = 16$; $b^2 = 1$ and $(x + 2)$ is substituted into x.

   $$\int \frac{1}{12 - 4x - x^2}\,dx = \frac{1}{2(4)}\ln\left|\frac{4+(x+2)}{4-(x+2)}\right| + C$$

   $$= \frac{1}{8}\ln\left|\frac{6+x}{2+x}\right| + C$$

6. First simplify the denominator:

   $(x+5)^2 - 12x$
   $= x^2 + 10x + 25 - 12x$
   $= x^2 - 2x + 25$

   Now complete the square:
   $= x^2 - 2x + 1 + 25 - 1$
   $= (x-1)^2 + 24$

   $$\therefore \int \frac{1}{(x+5)^2 - 12x}\,dx = \int \frac{1}{(x-1)^2 + 24}\,dx$$

   This is of the form $\int \frac{1}{mx^2 + nx + p}\,dx$.

   Complete the square: $\int \frac{1}{24 + (x-1)^2}\,dx$

   Now use the formula $\int \frac{1}{a^2 + b^2x^2}\,dx = \frac{1}{ab}\tan^{-1}\left(\frac{b}{a}x\right) + C$,

   where $a^2 = 24$; $b^2 = 1$ and $(x - 1)$ is substituted into x.

   $$\therefore \int \frac{1}{24 + (x-1)^2}\,dx = \frac{1}{\sqrt{24}.1}\tan^{-1}\left(\frac{x-1}{\sqrt{24}}\right) + C$$

   $$= \frac{1}{\sqrt{24}}\tan^{-1}\left(\frac{x-1}{\sqrt{24}}\right) + C$$

Summative assessment: Module 2 — SB page 68

1. $\int \cos^3(u)\sin^4(u)\,du$
   $= \int \cos(u)(1 - \sin^2(u))\sin^4(u)\,du$

   Let $v = \sin(u)$, $\frac{dv}{du} = \cos(u)$, $du = \frac{dv}{\cos(u)}$

   $\int \cos(u)(1 - v^2)v^4 \frac{dv}{\cos(u)}$

   $= \int v^4 - v^6\,dv$

   $= \frac{v^5}{5} - \frac{v^7}{7}$

   $= \frac{\sin^5(u)}{5} - \frac{\sin^7(u)}{7} + C$ (5)

2. $\int \cos^3(3x + 4)\, dx$

Let $u = 3x + 4, \frac{du}{dx} = 3, dx = \frac{du}{3}$

$\int \cos^3(u) \frac{du}{3}$

$= \frac{1}{3} \int \cos^3(u)\, du$

$= \frac{1}{3} \int \cos(u)(1 - \sin^2(u))\, du$

Let $v = \sin(u), \frac{dv}{du} = \cos(u)$

$= \frac{1}{3} \int \cos(u)(1 - v^2) \frac{dv}{\cos(u)}$

$= \frac{1}{3} \int (1 - v^2)\, dv$

$= \frac{1}{3}\left(v - \frac{v^3}{3}\right)$

$= \frac{1}{3}\left(\sin(u) - \frac{\sin^3(u)}{3}\right)$

$= \frac{\sin(u)}{3} - \frac{\sin^3(u)}{9}$

$= \frac{\sin(3x + 4)}{3} - \frac{\sin^3(3x + 4)}{9} + C$ (5)

3. $\int e^x \sin(x)\, dx$

$= e^x \sin(x) - \int e^x \cos(x)4\, dx$ [Integrate by parts]

$= e^x \sin(x) - \int e^x \cos(x) - \sin(x)\, e^x\, dx$ [Integrate by parts again]

$= e^x \sin(x) - e^x \cos(x) + \int -e^x \sin(x)\, e^x\, dx$

$= -\int e^x \sin(x)\, dx + e^x \sin(x) - e^x \cos(x)$

$= \frac{e^x \sin(x) - e^x \cos(x)}{2} + C$ (3)

4. $\int xe^x\, dx$

$= xe^x - \int 1.e^x\, dx$ [Integrate by parts]

$= xe^x - e^x + C$ (2)

5. $\int \frac{1}{(x + 3)^2 - 8x}\, dx$

Rewrite the term under the line in the form $k + a(x \pm h)^2$:

$(x + 3)^2 - 8x = x^2 - 2x + 9 = 8 + (x - 1)^2$

Since:

$\int \frac{1}{k + a(x \pm h)^2}\, dx = \frac{1}{\sqrt{ka}} \tan^{-1}\left(\sqrt{\frac{a}{k}}(x \pm h)\right) + C$

You can now substitute $k = 8, a = 1, h = 1$:

$\therefore \int \frac{1}{8 + 1(x - 1)^2}\, dx$

$= \frac{1}{\sqrt{8.1}} \tan^{-1}\left(\sqrt{\frac{1}{8}}(x - 1)\right) + C$

$= \frac{1}{2\sqrt{2}} \tan^{-1}\left(\frac{x - 1}{2\sqrt{2}}\right) + C$ (4)

6. $\int \frac{1}{(s^2 + 9)}\, ds$

Rewrite the term under the line in the form $k + a(s \pm h)^2$:

$s^2 + 9 = 9 + 1(s + 0)^2$

Since:

$\int \frac{1}{k + a(s \pm h)^2}\, ds = \frac{1}{\sqrt{ka}} \tan^{-1}\left(\sqrt{\frac{a}{k}}(s \pm h)\right) + C$

You can now substitute $k = 9$, $a = 1$, $h = 0$:

$\therefore \int \frac{1}{9 + 1(s + 0)^2}\, ds$

$= \frac{1}{\sqrt{9.1}} \tan^{-1}\left(\sqrt{\frac{1}{9}}(s + 0)\right) + C$

$= \frac{1}{3} \tan^{-1}\left(\frac{s}{3}\right) + C$ (4)

7. $\int \frac{1}{\sqrt{4 - x^2}}\, dx$

Let $u = \frac{x}{2}, \frac{du}{dx} = \frac{1}{2}, dx = 2\, du$

$\int \frac{1}{\sqrt{4 - (2u^2)}}\, 2\, du$

$= \int \frac{1}{\sqrt{4 - 4u^2}}\, 2\, du$

$= \int \frac{1}{\sqrt{1 - u^2}}\, du$

$= \sin^{-1}(u)$ [Standard integration]

$= \sin^{-1}\frac{x}{2} + C$ (4)

8. $\int \sin^3(x + 4) \cos(x + 4)\, dx$

Let $u = \sin(x + 4); \frac{du}{dx} = \cos(x + 4);$

$dx = \frac{du}{\cos(x + 4)}$

Then we have $\int u^3\, du$

$= \frac{u^4}{4}$

$= \frac{\sin^4(u + 4)}{4} + C$ (4)

9. $\int x^2 \ln(x)\, dx$

$= \frac{x^3 \ln(x)}{3} - \int \frac{x^3}{3} . \frac{1}{x}\, dx$

$= \frac{x^3 \ln(x)}{3} - \int \frac{x^3}{3}\, dx$

$= \frac{x^3 \ln(x)}{3} - \frac{x^3}{9} + C$ (3)

10. $\int \ln(x)\, dx$
$= x \ln(x) - \int \frac{1}{x} x\, dx$
$= x \ln(x) - \int 1\, dx$
$= x \ln(x) - x + C$ (2)

11. $\int x \tan^2(x)\, dx$
$= \int x(\sec^2(x) - 1)\, dx$
$= \int x \sec^2(x)\, dx - \int x\, dx$
$= x \tan(x) - \int \tan(x)\, dx - \int x\, dx$
$= x \tan(x) - \ln \sec(x) = \frac{x^2}{2} + C$ (2)

12. $\int v^3 e^v\, dv$
$= v^3 e^v - 3 \int e^v . v^2\, dv$
$= v^3 e^v - 3(v^2 . e^v - 2v^2 . e^v)$
$= v^3 e^v - 3v^2 . e^v + 6 \int e^v . v\, dv$
$= v^3 e^v - 3v^2 . e^v + 6(v . e^v - \int e^v\, dv)$
$= v^3 e^v - 3v^2 . e^v + 6v . e^v - 6e^v + C$ (4)

13. $\int \cos^2(v) . \sin^5(v)\, dv$
$= \int \cos^2(v) . (1 - \cos^2(v))^2 . \sin(v)\, dv$
$= -\int u^2 . (1 - u^2)\, du$
$= -\int u^2 . (1 - 2u^2 + u^4)\, du$
$= -\left(\int u^2\, du - \int 2u^4\, du + \int u^6\, du\right)$
$= -\frac{u^3}{3} + \frac{2u^5}{5} - \frac{u^7}{7} + C$
$= -\frac{\cos^3(v)}{3} + \frac{2 \cos^5(v)}{5} - \frac{\cos^7(v)}{7} + C$ (4)

14. $\int \cos^4(x) . \sin^3(x)\, dx$
$= \int \cos^4(x) . (1 - \cos^2(x)) . \sin(x)\, dx$
$= -\int u^4 (1 - u^2)\, du$
$= -\int u^4\, du + \int u^6\, du$
$= -\frac{u^5}{5} + \frac{u^7}{7} + C$
$= -\frac{\cos^5(x)}{5} + \frac{\cos^7(x)}{7} + C$ (4)

TOTAL: [50]

MODULE

3 Partial fractions

After they have completed this module, students should be able to:

- apply the process of partial fraction decomposition to proper fractions where the denominator has:
  - a single recursive factor;
  - two recursive factors;
  - a trinomial factor and recursive factors; and
- apply the process of partial fraction decomposition to improper rational fractions, after using polynomial long division, where the denominator has:
  - two recursive factors;
  - a trinomial factor and recursive factors.

Introduction

We know that we can add or subtract algebraic fractions by finding the lowest common denominator. But how do we reverse this process? That's what partial fraction decomposition is about.

Partial fraction decomposition is a method used to break apart fractions containing polynomials. It separates a fraction with multiple factors in the denominator, into its initial polynomial fractions with "uncommon denominators".

This method makes it possible to integrate many rational fractions, by changing a denominator of a higher degree to ones of lower degrees to which the integrals are known. These simpler fractions are usually much easier to integrate.

Students need the following pre-knowledge to successfully complete this module.

Pre-knowledge

Students should already know how to:

- Solve for unknown coefficients, either with simultaneous equations or by equating coefficients.
- Apply polynomial long division.
- Factorise polynomials such as cubic or quadratic functions.
- Decompose rational fractions into partial fractions.
- Integrate simple fractions such as:
  - $\int \frac{f'(x)}{f(x)}\, dx = \ln f(x) + C$
  - $\int [f(x)]^n . f'(x)\, dx = \frac{[f(x)]^{n+1}}{n+1} + C \qquad (n < -1)$
  - $\int \frac{f'(x)}{1 + [f(x)]^2}\, dx = \tan^{-1} f(x) + C$

Activity 3.1 SB page 75

1. 1.1 Given $\frac{x+4}{x^2 - x - 6}$

$x^2 - x - 6 = (x-3)(x+2)$ • Factorise

$\frac{x+4}{x^2 - x - 6} = \frac{A}{x+2} + \frac{B}{x-3}$ • Write in partial fraction form

$x + 4 = A(x-3) + B(x+2)$ • Simplify

$A = -\frac{2}{5};\ B = \frac{7}{5}$ • Find values of A and B

Therefore:

$\frac{x+4}{x^2 - x - 6} = -\frac{2}{5(x+2)} + \frac{7}{5(x-3)}$

1.2 Given $\frac{x+14}{-x^2 - 2x + 8}$

$-x^2 - 2x + 8 = -(x-2)(x+4)$ • Factorise

$\frac{x+14}{-(x-2)(x+4)} = -\frac{A}{x-2} + \frac{B}{x+4}$ • Write in partial fraction form

$x + 14 = -A(x+4) + B(x-2)$ • Simplify

$A = -\frac{8}{3};\ B = \frac{5}{3}$ • Find values of A and B

Therefore:

$\frac{x+14}{-x^2 - 2x + 8} = -\frac{8}{3(x-2)} + \frac{5}{3(x+4)}$

1.3 Given $\frac{5x - 19}{x^2 - 7x + 10}$

$x^2 - 7x + 10 = (x - 2)(x - 5)$ • Factorise

$\frac{5x - 19}{x^2 - 7x + 10} = \frac{A}{x - 2} + \frac{B}{x - 5}$ • Write in partial fraction form

$5x - 19 = A(x - 5) + B(x - 2)$ • Simplify

$A = 3, B = 2$ • Find values of A and B

Therefore:

$$\frac{5x - 19}{x^2 - 7x + 10} = \frac{3}{x - 2} + \frac{2}{x - 5}$$

1.4 Given $\frac{31 - 2x}{x^2 - x - 6}$

$x^2 - x - 6 = (x - 3)(x + 2)$ • Factorise

$\frac{31 - 2x}{x^2 - x - 6} = \frac{A}{x - 3} + \frac{B}{x + 2}$ • Write in partial fraction form

$31 - 2x = A(x + 2) + B(x - 3)$ • Simplify

$A = 5, B = -7$ • Find values of A and B

Therefore:

$$\frac{31 - 2x}{x^2 - x - 6} = \frac{5}{x - 3} - \frac{7}{x + 2}$$

2. 2.1 Decompose denominator into partial fractions:

$3x^2 - x - 2 = (x - 1)(3x + 2)$

$\frac{13x + 2}{3x^2 - x - 2} = \frac{A}{x - 1} + \frac{B}{3x + 2}$ • Write in partial fraction form

$13x + 2 = A(3x + 2) + B(x - 1)$ • Simplify

Solve for the unknown coefficients:

$A = 3; B = 4$ • Solve for the unknown coefficients

Therefore:

$$\frac{13x + 2}{3x^2 - x - 2} = \frac{3}{(x - 1)} + \frac{4}{(3x + 2)}$$

The integral of the result:

$$\int \frac{13x + 2}{3x^2 - x - 2}\,dx = \int \frac{3}{x - 1}dx + \int \frac{4}{3x + 2}\,dx$$

$$= 3 \ln |x - 1| + \frac{4}{3} \ln |3x + 2| + C$$

2.2 Decompose denominator into partial fractions:

$x^3 + x^2 - 2x = x(x - 1)(x + 2)$

$\frac{12x^2 + 3x - 9}{x^3 + x^2 - 2x} = \frac{A}{x} + \frac{B}{x - 1} + \frac{C}{x + 2}$ • Write in partial fraction form

$12x^2 + 3x - 9 = A(x - 1)(x + 2) + B(x)(x + 2) + C(x)(x - 1)$

$A = \frac{9}{2}$; $B = 2$; $C = \frac{11}{2}$ • Solve for the unknown coefficients

Therefore:

$$\frac{12x^2 + 3x - 9}{x^3 + x^2 - 2x} = \frac{9}{2x} + \frac{2}{x - 1} + \frac{11}{2(x + 2)}$$

The integral of the result:

$$\int \frac{12x^2 + 3x - 9}{x^3 + x^2 - 2x}\, dx = \int \left(\frac{9}{2x} + \frac{2}{x - 1} + \frac{11}{2(x + 2)} \right) dx$$

$$= \int \frac{9}{2x}\, dx + \int \frac{2}{x - 1}\, dx + \int \frac{11}{2(x + 2)}\, dx$$

$$= \frac{9}{2} \ln |x| + 2 \ln |x - 1| + \frac{11}{2} \ln |x + 2| + C$$

2.3 Decompose denominator into partial fractions:

$$4x^2 - 9 = (2x + 3)(2x - 3)$$

$\frac{1}{4x^2 - 9} = \frac{A}{2x + 3} + \frac{B}{2x - 3}$ • Write in partial fraction form

$1 = A(2x - 3) + B(2x + 3)$ • Simplify

$A = -\frac{1}{6}$; $B = \frac{1}{6}$ • Solve for the unknown coefficients

Therefore:

$$\frac{1}{4x^2 - 9} = -\frac{1}{6(2x + 3)} + \frac{1}{6(2x - 3)}$$

The integral of the result:

$$\int \frac{1}{4x^2 - 9}\, dx = \int \left(-\frac{1}{6(2x + 3)} + \frac{1}{6(2x - 3)} \right) dx$$

$$= \int -\frac{1}{6(2x + 3)}\, dx + \int \frac{1}{6(2x - 3)}\, dx$$

$$= -\frac{1}{12} \ln |2x + 3| + \frac{1}{12} \ln |2x - 3| + C$$

2.4 Decompose denominator into partial fractions:

$$x^2 - 7x + 10 = (x - 2)(x - 5)$$

$\frac{31 - 2x}{x^2 - 7x + 10} = \frac{A}{x - 2} + \frac{B}{x - 5}$ • Write in partial fraction form

$31 - 2x = A(x - 5) + B(x - 2)$ • Simplify

Solve for unknown coefficients:
$A = -9$, $B = 7$

Therefore:

$$\frac{31 - 2x}{x^2 - 7x + 10} = -\frac{9}{x - 2} + \frac{7}{x - 5}$$

The integral of the result:

$$\int \frac{31 - 2x}{x^2 - 7x + 10}\, dx = -\int \frac{9}{x - 2}\, dx + \int \frac{7}{x - 5}\, dx$$

$$= -9 \ln |x - 2| + 7 \ln |x - 2| + C$$

Activity 3.2 SB page 78

1. Write as partial fractions: $\frac{3x+5}{x^2-2x+1}$

$x^2 - 2x + 1 = (x-1)^2$

Then:

$$\frac{3x+5}{x^2-2x+1} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2}$$

Simplified:

$3x + 5 = A(x-1) + B$

Values of A and B: A = 3; B = 8

Therefore:

$$\frac{3x+5}{x^2-2x+1} = \frac{3}{(x-1)} + \frac{8}{(x-1)^2}$$

The integral of the result:

$$\int \frac{3x+5}{x^2-2x+1}\,dx = \int \frac{3}{(x-1)} + \frac{8}{(x-1)^2}\,dx$$

$$= \int \frac{3}{(x-1)}\,dx + \int \frac{8}{(x-1)^2}\,dx$$

$$= 3\ln|x-1| - \frac{8}{x-1} + C$$

2. Write as partial fractions: $\frac{5+x}{x^3+3x^2+3x+1}$

$$\frac{5+x}{x^3+3x^2+3x+1} = \frac{5+x}{(x+1)^3}$$

Then:

$$\frac{5+x}{(x+1)^3} = \frac{A}{(x+1)} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}$$

Simplified:

$5 + x = A(x+1)^2 + B(x+1) + C$

Values of A, B, and C: A = 0; B = 1; C = 4

Therefore:

$$\frac{5+x}{(x+1)^3} = \frac{1}{(x+1)^2} + \frac{4}{(x+1)^3}$$

The integral of the result:

$$\int \frac{5+x}{x^3+3x^2+3x+1}\,dx = \int \frac{1}{(x+1)^2}\,dx + \int \frac{4}{(x+1)^3}\,dx$$

$$= -\frac{1}{(x+1)} - \frac{2}{(x+1)^2} + C$$

$$= -\frac{x+3}{(x+1)^2} + C$$

3. Write as partial fractions: $\frac{2x-3}{(x+3)^3}$

Then:

$$\frac{2x-3}{(x+3)^3} = \frac{A}{(x+3)} + \frac{B}{(x+3)^2} + \frac{C}{(x+3)^3}$$

Simplified:

$2x - 3 = A(x + 3)^2 + B(x + 3) + C$

Values of A, B, and C: A = 0; B = 2; C = −9

Therefore:

$$\frac{2x-3}{(x+3)^3} = \frac{2}{(x+3)^2} - \frac{9}{(x+3)^3}$$

The integral of the result:

$$\int \frac{2x-3}{(x+3)^3}\,dx = \int \left(\frac{2}{(x+3)^2} - \frac{9}{(x+3)^3}\right) dx$$

$$= \int \frac{2}{(x+3)^2}\,dx + \int -\frac{9}{(x+3)^3}\,dx$$

$$= -\frac{2}{x+3} + \frac{9}{2(x+3)^2} + C$$

4. Write as partial fractions: $\frac{7x-11}{x^2-4x+4}$

$x^2 - 4x + 4 = (x - 2)^2$

Then:

$$\frac{7x-11}{x^2-4x+4} = \frac{A}{x-2} + \frac{B}{(x-2)^2}$$

Simplified:

$7x - 11 = A(x - 2) + B$

Values of A and B: A = 7, B = 3

Therefore:

$$\frac{7x-11}{x^2-4x+4} = \frac{7}{x-2} + \frac{3}{(x-2)^2}$$

The integral of the result:

$$\int \frac{7x-11}{x^2-4x+4}\,dx = \int \frac{7}{x-2} + \frac{3}{(x-2)^2}\,dx$$

$$= \int \frac{7}{x-2}\,dx + \int \frac{3}{(x-2)^2}\,dx$$

$$= 7 \ln |x-2| - \frac{3}{(x-2)} + C$$

5. Write as partial fractions: $\frac{3x^2-24x+54}{x^3-6x^2+9x}$

$x^3 - 6x^2 + 9x = x(x - 3)^2$

Then:

$$\frac{3x^2-24x+54}{x^3-6x^2+9x} = \frac{A}{x} + \frac{B}{x-3} + \frac{C}{(x-3)^2}$$

Simplified:

$3x^2 - 24x + 54 = A(x - 3)^2 + Bx(x - 3) + Cx$

Values of A, B and C: A = 6, B = −3, C = 3

Therefore:

$$\frac{3x^2 - 24x + 54}{x^3 - 6x^2 + 9x} = \frac{6}{x} - \frac{3}{x - 3} + \frac{3}{(x - 3)^2}$$

The integral of the result:

$$\int \frac{3x^2 - 24x + 54}{x^3 - 6x^2 + 9x}\,dx = \int \frac{6}{x} - \frac{3}{x - 3} + \frac{3}{(x - 3)^2}\,dx$$

$$= \int \frac{6}{x}\,dx - \int \frac{3}{x - 3}\,dx + \int \frac{3}{(x - 3)^2}\,dx$$

$$= 6 \ln |x| - 3 \ln |x - 3| - \frac{3}{(x - 3)} + C$$

Activity 3.3 SB page 79

1. Write as partial fractions: $\frac{12x}{(x^2 - 9)^2}$

$(x^2 - 9)^2 = (x + 3)^2(x - 3)^2$

Then:

$$\frac{12x}{(x^2 - 9)^2} = \frac{A}{(x + 3)} + \frac{B}{(x + 3)^2} + \frac{C}{(x - 3)} + \frac{D}{(x - 3)^2}$$

Simplified:

$12x = A(x + 3)(x - 3)^2 + B(x - 3)^2 + C(x - 3)(x + 3)^2 + D(x + 3)^2$

Values of A, B, C and D: A = 0; B = −1; C = 0; D = 1

Therefore:

$$\frac{12x}{(x^2 - 9)^2} = -\frac{1}{(x + 3)^2} + \frac{1}{(x - 3)^2}$$

The integral of the result:

$$\int \frac{12x}{(x^2 - 9)^2}\,dx = \frac{1}{(x + 3)} - \frac{1}{(x - 3)} + C$$

$$= \frac{-6}{(x^2 - 9)} + C$$

2. Write as partial fractions:

$$\frac{2x^2 + 8x - 1}{(2x + 1)^2(x - 1)^2}$$

Then:

$$\frac{2x^2 + 8x - 1}{(2x + 1)^2(x - 1)^2} = \frac{A}{(2x + 1)} + \frac{B}{(2x + 1)^2} + \frac{C}{(x - 1)} + \frac{D}{(x - 1)^2}$$

Simplified:

$2x^2 + 8x - 1 = A(2x + 1)(x - 1)^2 + B(x - 1)^2 + C(x - 1)(2x + 1)^2 + D(2x + 1)^2$

Values of A, B, C and D: A = 0; B = −2; C = 0; D = 1

Therefore:

$$\frac{2x^2 + 8x - 1}{(2x + 1)^2 (x - 1)^2} = -\frac{2}{(2x + 1)^2} + \frac{1}{(x - 1)^2}$$

The integral of the result:

$$\int \frac{2x^2 + 8x - 1}{(2x + 1)^2 (x - 1)^2}\, dx = -\int \frac{2}{(2x + 1)^2}\, dx + \int \frac{1}{(x - 1)^2}\, dx$$

$$= \frac{1}{(2x + 1)} - \frac{1}{(x - 1)} + C$$

$$= -\frac{x + 2}{(2x + 1)(x - 1)} + C$$

3. Write as partial fractions:

$$\frac{x + 1}{x^2 (x + 2)^2}$$

Then:

$$\frac{x + 1}{x^2 (x + 2)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{(x + 2)} + \frac{D}{(x + 2)^2}$$

Simplified:

$$x + 1 = A(x)(x + 2)^2 + B(x + 2)^2 + C(x + 2)(x)^2 + D(x)^2$$

Values of A, B, C and D: A = 0; B = $\frac{1}{4}$; C = 0; D = $-\frac{1}{4}$

Therefore:

$$\frac{x + 1}{x^2 (x + 2)^2} = \frac{1}{4(x)^2} - \frac{1}{4(x + 2)^2}$$

The integral of the result:

$$\int \frac{x + 1}{x^2 (x + 2)^2}\, dx = -\frac{1}{4x} + \frac{1}{4(x + 2)} + C$$

4. Write as partial fractions: $\frac{-x^2 + 3x + 4}{x^2 (1 - 2x)^2}$

$$x^2 (1 - 2x)^2 = x.x.(1 - 2x).(1 - 2x)$$

Then:

$$\frac{-x^2 + 3x + 4}{x^2 (1 - 2x)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{1 - 2x} + \frac{D}{(1 - 2x)^2}$$

Simplified:

$$-x^2 + 3x + 4 = Ax(1 - 2x)^2 + B(1 - 2x)^2 + Cx^2(1 - 2x) + Dx^2$$

Values of A, B, C and D: A = 3, B = 4, C = 6, D = 21

Therefore:

$$\frac{-x^2 + 3x + 4}{x^2 (1 - 2x)^2} = \frac{3}{x} + \frac{4}{x^2} + \frac{6}{1 - 2x} + \frac{21}{(1 - 2x)^2}$$

The integral of the result:

$$\int \frac{-x^2 + 3x + 4}{x^2(1-2x)^2}\,dx = \int \frac{3}{x} + \frac{4}{x^2} + \frac{6}{1-2x} + \frac{21}{(1-2x)^2}\,dx$$

$$= \int \frac{3}{x}\,dx + \int \frac{4}{x^2}\,dx + \int \frac{6}{1-2x}\,dx + \int \frac{21}{(1-2x)^2}\,dx$$

$$= 3\ln|x| - \frac{4}{x} - 3\ln|1-2x| + \frac{21}{(2-4x)} + C$$

5. Write as partial fractions: $\frac{4}{(x+1)^2(x^2-2x+1)}$

$(x+1)^2(x^2-2x+1) = (x+1)^2(x-1)^2$

Then:

$$\frac{4}{(x+1)^2(x^2-2x+1)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-1} + \frac{D}{(x-1)^2}$$

Simplified:

$4 = A(x+1)(x-1)^2 + B(x-1)^2 + C(x-1)(x+1)^2 + D(x+1)^2$

Values of A, B, C and D: A = 1, B = 1, C = –1, D = 1

Therefore:

$$\frac{4}{(x+1)^2(x^2-2x+1)} = \frac{1}{x+1} + \frac{1}{(x+1)^2} - \frac{1}{x-1} + \frac{1}{(x-1)^2}$$

The integral of the result:

$$\int \frac{4}{(x+1)^2(x^2-2x+1)}\,dx = \int \frac{1}{x+1} + \frac{1}{(x+1)^2} - \frac{1}{x-1} + \frac{1}{(x-1)^2}\,dx$$

$$= \int \frac{1}{x+1}\,dx + \int \frac{1}{(x+1)^2}\,dx - \int \frac{1}{x-1}\,dx + \int \frac{1}{(x-1)^2}\,dx$$

$$= \ln|x+1| - \frac{1}{x+1} - \ln|x-1| - \frac{1}{x-1} + C$$

Activity 3.4 — SB page 82

1. No long division required. Factorise the denominator.

$x^4 - 2x^3 + 2x^2 - 2x + 1 = (x-1)^2(x^2+1)$

Write as a partial fraction.

$$\frac{x^3 - 4x^2 - x - 2}{(x-1)^2(x^2+1)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{Cx + D}{(x^2+1)}$$

Solving for the unknown coefficients: A = 0; B = –3; C = 1; D = 1

$$\frac{x^3 - 4x^2 - x - 2}{(x-1)^2(x^2+1)} = -\frac{3}{(x-1)^2} + \frac{x}{(x^2+1)} + \frac{1}{(x^2+1)}$$

Integrate.

$$\int \frac{x^3 - 4x^2 - x - 2}{x^4 - 2x^3 + 2x^2 - 2x + 1}\,dx = \frac{3}{x-1} + \frac{1}{2}\ln(x^2+1) + \tan^{-1}(x) + C$$

2. No long division required, and the denominator is already factorised.

   Write as a partial fraction.

   $$\frac{4x^2+4x+7}{(x^2+1)(x+2)^2} = \frac{Ax+B}{(x^2+1)} + \frac{C}{(x+2)} + \frac{D}{(x+2)^2}$$

   Solving for the unknown coefficients: A = 0; B = 1; C = 0; D = 3

   $$\frac{4x^2+4x+7}{(x^2+1)(x+2)^2} = \frac{1}{(x^2+1)} + \frac{3}{(x+2)^2}$$

   Integrate.

   $$\int \frac{4x^2+4x+7}{(x^2+1)(x+2)^2}\,dx = \tan^{-1}(x) - \frac{3}{x+2} + C$$

3. No long division required, and the denominator is already factorised.

   Write as a partial fraction.

   $$\frac{x^3+4x^2+5x+2}{(x^2+2x+9)(x-1)^2} = \frac{Ax+B}{(x^2+2x+9)} + \frac{C}{(x-1)} + \frac{D}{(x-1)^2}$$

   Solving for the unknown coefficients: A = 0; B = 2; C = 1; D = 1

   $$\frac{x^3+4x^2+5x+2}{(x^2+2x+9)(x-1)^2} = \frac{2}{(x^2+2x+9)} + \frac{1}{(x-1)} + \frac{1}{(x-1)^2}$$

   To integrate $\frac{2}{(x^2+2x+9)}$, first complete the square:

   $$x^2 + 2x + 9 = (x+1)^2 + 8$$

   Integrate.

   $$\int \frac{x^3+4x^2+5x+2}{(x^2+2x+9)(x-1)^2}\,dx = \frac{2}{\sqrt{8}}\tan^{-1}\left(\frac{x+1}{\sqrt{8}}\right) + \ln(x-1) - \frac{1}{(x-1)} + C$$

4. Factorise the denominator.

   $$(x^2-1)(x^2+4) = (x+1)(x-1)(x^2+4)$$

   Write as a partial fraction:

   $$\frac{x^2+9}{(x^2-1)(x^2+4)} = \frac{A}{(x+1)} + \frac{B}{(x-1)} + \frac{Cx+D}{(x^2+4)}$$

   Solving for the unknown coefficients:

   A = −1; B = 1; C = 0; D = −1

   $$\frac{x^2+9}{(x^2-1)(x^2+4)} = \frac{-1}{(x+1)} + \frac{1}{(x-1)} + \frac{-1}{(x^2+4)}$$

   Integrate:

   $$\int \frac{x^2+9}{(x^2-1)(x^2+4)}\,dx = -\ln|x+1| + \ln|x-1| - \frac{1}{2}\tan^{-1}\frac{x}{2} + C$$

5. Factorise the denominator.

   $$(x^2+1)(x-1)^2 = (x-1)(x-1)(x^2+1)$$

   Write as a partial fraction:

   $$\frac{-2x+4}{(x^2+1)(x-1)^2} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{Cx+D}{(x^2+1)}$$

Solving for the unknown coefficients:

A = –2; B = 1; C = 2; D = 1

$$\frac{-2x+4}{(x^2+1)(x-1)^2} = \frac{-2}{(x-1)} + \frac{1}{(x-1)^2} + \frac{2x+1}{(x^2+1)}$$

Integrate:

$$\int \frac{-2x+4}{(x^2+1)(x-1)^2}\,dx = \int \frac{-2}{(x-1)} + \frac{1}{(x-1)^2} + \frac{2x+1}{(x^2+1)}\,dx$$

$$= \int \frac{-2}{(x-1)}\,dx + \int \frac{1}{(x-1)^2}\,dx + \int \frac{2x}{(x^2+1)}\,dx + \int \frac{1}{(x^2+1)}\,dx$$

$$= -2\ln|x-1| - \frac{1}{(x-1)} + \ln\left[|x|^2+1\right] + \tan^{-1}x + C$$

Activity 3.5 SB page 87

1. Using long division:

$$\begin{array}{r} x+1 \\ x^2+x-2\,\overline{)\,x^3+2x^2+6x+3} \\ \underline{x^3+x^2-2x} \\ x^2+8x+3 \\ \underline{x^2+x-2} \\ 7x+5 \end{array}$$

Therefore: $\frac{x^3+2x^2+6x+3}{x^2+x-2} = x+1+\frac{7x+5}{x^2+x-2}$

Factorise the denominator:

$x^2+x-2 = (x-1)(x+2)$

Partial fraction decomposition: $\frac{7x+5}{x^2+x-2} = \frac{7x+5}{(x-1)(x+2)}$

$$= \frac{A}{(x-1)} + \frac{B}{(x+2)}$$

To find A and B: $7x+5 = A(x+2) + B(x-1)$

Let $x = 1$: A = 4

Let $x = -2$: B = 3

$\therefore \frac{7x+5}{x^2+x-2} = \frac{4}{(x-1)} + \frac{3}{(x+2)}$

So, $\frac{x^3+2x^2+6x+3}{x^2+x-2} = 1+x+\frac{3}{x+2}+\frac{4}{x-1}$

2. Using long division, $\frac{2x^3+x^2-9x-3}{x^3-x^2-4x+4} = 2+\frac{3x^2-x-11}{x^3-x^2-4x+4}$

$$\frac{3x^2-x-11}{(x-1)(x+2)(x-2)} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{x-2}$$

$$= 3x^2-x-11$$

$$= A(x+2)(x-2) + B(x-1)(x-2) + C(x-1)(x+2)$$

$x = 2$:
$-1 = 0 + 0 + 4C$
$\therefore C = -\frac{1}{4}$

$x = 1$:
$-9 = -3A + 0 + 0$
$\therefore A = 3$

$x = -2$:
$3 = 0 + B(12) + 0$
$\therefore B = \frac{1}{4}$

$$\int \frac{2x^3 + x^2 - 9x - 3}{x^3 - x^2 - 4x + 4}\,dx = \int 2 + \frac{3}{x-1} + \frac{1}{4(x+2)} - \frac{1}{4(x-2)}\,dx$$
$$= 2x + 3\ln|x-1| + \frac{1}{4}\ln|x+2| - \frac{1}{4}\ln|x-2| + C$$

3. Using long division:

$$\begin{array}{r|l} & x + 1 \\ \hline 2x - 3 & 2x^2 - x - 5 \\ & \underline{2x^2 - 3x} \\ & \quad 2x - 5 \\ & \quad \underline{2x - 3} \\ & \qquad -2 \end{array}$$

Therefore: $\frac{2x^2 - x - 5}{2x - 3} = x + 1 + \frac{-2}{2x - 3}$

The fraction is already in an irreducible form.

$$\int \frac{2x^2 - x - 5}{2x - 3}\,dx = \int x + 1 + \frac{-2}{2x - 3}\,dx$$

$$= \int x\,dx + \int 1\,dx + \int \frac{-2}{2x - 3}\,dx$$

$$= \frac{x^2}{2} + x - \ln|2x - 3| + C$$

4. $\int \frac{3x^5 - 5x^4 + x^3 + 2x^2 + 2x - 1}{x^4 - 2x^3 + x^2}\,dx$

Long division:

$$\begin{array}{r|l} & 3x + 1 \\ \hline x^4 - 2x^3 + x^2 & 3x^5 - 5x^4 + x^3 + 2x^2 + 2x - 1 \\ & \underline{3x^5 - 6x^4 + 3x^3} \\ & \qquad x^4 - 2x^3 + 2x^2 + 2x - 1 \\ & \qquad \underline{x^4 - 2x^3 + x^2} \\ & \qquad\qquad x^2 + 2x - 1 \end{array}$$

$$\therefore \frac{3x^5 - 5x^4 + x^3 + 2x^2 + 2x - 1}{x^4 - 2x^3 + x^2} = 3x + 1 + \frac{x^2 + 2x - 1}{x^4 - 2x^3 + x^2}$$

Factorise the denominator:

$x^4 - 2x^3 + x^2 = x^2(x - 1)^2$

Partial fraction decomposition:

$$\frac{x^2+2x-1}{x^2(x-1)^2}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{(x-1)}+\frac{D}{(x-1)^2}$$

Multiplying by the denominator and equating coefficients:

A = 0; B = −1; C = 0; D = 2

$$\therefore \frac{x^2+2x-1}{x^4-2x^3+x^2}=-\frac{1}{x^2}+\frac{2}{(x-1)^2}$$

Integrating:

$$\int\frac{3x^5-5x^4+x^3+2x^2+2x-1}{x^4-2x^3+x^2}\,dx=\int 3x\,dx+\int 1\,dx-\int\frac{1}{x^2}\,dx+\int\frac{2}{(x-1)^2}\,dx$$

$$=\frac{3}{2}x^2+x+\frac{1}{x}-\frac{2}{x-1}+C$$

5. $$\int\frac{-x^6+x^4+3x^3+7x^2-3}{x^5-3x^3}\,dx$$

Long division:

$$\begin{array}{r|l} & -x \\ x^5-3x^3 & -x^6+x^4+3x^3+7x^2-3 \\ & -x^6+3x^4 \\ \hline & -2x^4+3x^3+7x^2-3 \end{array}$$

$$\therefore \frac{-x^6+x^4+3x^3+7x^2-3}{x^5-3x^3}=-x+\frac{-2x^4+3x^3+7x^2-3}{x^5-3x^3}$$

Factorise the denominator:

$$x^5-3x^3=x^3(x^2-3)$$

Partial fraction decomposition:

$$\frac{-2x^4+3x^3+7x^2-3}{x^3(x^2-3)}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x^3}+\frac{Dx+E}{x^2-3}$$

Multiplying by the denominator and equating coefficients:

A = −2; B = 0; C = 1; D = 0; E = 3

$$\therefore \frac{-2x^4+3x^3+7x^2-3}{x^5-3x^3}=-\frac{2}{x}+\frac{1}{x^3}+\frac{3}{x^2-3}$$

Integrating:

$$\int\frac{-x^6+x^4+3x^3+7x^2-3}{x^5-3x^3}\,dx=-\int x\,dx-\int\frac{2}{x}\,dx+\int\frac{1}{x^3}\,dx+\int\frac{3}{x^2-3}\,dx$$

$$=-\frac{1}{2}x^2-2\ln x-\frac{1}{2x^2}-\frac{3}{2\sqrt{3}}\ln\left(\frac{x+\sqrt{3}}{x-\sqrt{3}}\right)+C$$

Summative assessment: Module 3 **SB page 88**

1. 1.1 Start with long division:

$$\frac{x^4 - 2x^2 - x}{x^3 - x^2 + x - 1} = x + 1 + \frac{-2x^2 - x + 1}{x^3 - x^2 + x - 1}$$

Write as a partial fraction: $\frac{-2x^2 - x + 1}{x^3 - x^2 + x - 1}$

Factorise the denominator:

$x^3 - x^2 + x - 1 = (x - 1)(x^2 + 1)$

Then:

$$\frac{-2x^2 - x + 1}{x^3 - x^2 + x - 1} = \frac{A}{(x - 1)} + \frac{Bx + C}{(x^2 + 1)}$$

Simplified:

$-2x^2 - x + 1 = A(x^2 + 1) + (Bx + C)(x - 1)$

Values of A, B, and C:

A = −1, B = −1, C = −2

Therefore:

$$\frac{-2x^2 - x + 1}{x^3 - x^2 + x - 1} = -\frac{1}{(x - 1)} - \frac{(x + 2)}{(x^2 + 1)}$$

and

$$\frac{x^4 - 2x^2 - x}{x^3 - x^2 + x - 1} = x + 1 - \frac{1}{(x - 1)} - \frac{(x + 2)}{(x^2 + 1)}$$ (4)

1.2 Factorise the denominator:

$x^2 - 1 = (x + 1)(x - 1)$

Then:

$$\frac{4x}{x^2 - 1} = \frac{A}{x + 1} + \frac{B}{x - 1}$$

Simplified:

$4x = A(x - 1) + B(x + 1)$

Values of A and B:

A = 2; B = 2

Therefore:

$$\frac{4x}{x^2 - 1} = \frac{2}{x + 1} + \frac{2}{x - 1}$$ (4)

1.3 Factorise the denominator:

$-x^3 + 2x + 4 = (2 - x)(x^2 + 2x + 2)$

Then

$$\frac{x^2 + 7x + 12}{(2 - x)(x^2 + 2x + 2)} = \frac{A}{(2 - x)} + \frac{Bx + C}{(x^2 + 2x + 2)}$$

Simplified:

$x^2 + 7x + 12 = A(x^2 + 2x + 2) + (Bx + C)(2 - x)$

Values of A and B, and C:

A = 3, B = 2, C = 3

Therefore:

$$\frac{x^2 + 7x + 12}{(2 - x)(x^2 + 2x + 2)} = \frac{3}{(2 - x)} + \frac{2x + 3}{(x^2 + 2x + 2)} \quad (4)$$

1.4 Factorise the denominator:

$x^3 - 3x^2 + 3x - 1 = (x - 1)^3$

Then:

$$\frac{2x + 1}{(x - 1)^3} = \frac{A}{(x - 1)} + \frac{B}{(x - 1)^2} + \frac{C}{(x - 1)^3}$$

Simplified:

$2x + 1 = A(x - 1)^2 + B(x - 1) + C$

Values of A and B, and C:

A = 0, B = 1, C = 3

Therefore:

$$\frac{2x + 1}{(x - 1)^3} = \frac{2}{(x - 1)^2} + \frac{3}{(x - 1)^3} \quad (4)$$

1.5 Factorise the denominator:

$x^3 + x^2 + 2x = x(x^2 + x + 2)$

Then:

$$\frac{4x^2 + 3x + 6}{x(x^2 + x + 2)} = \frac{A}{x} + \frac{Bx + C}{x^2 + x + 2}$$

Simplified:

$4x^2 + 3x + 6 = A(x^2 + x + 2) + (Bx + C)x$

Values of A and B, and C:

A = 3, B = 1, C = 0

Therefore:

$$\frac{4x^2 + 3x + 6}{x(x^2 + x + 2)} = \frac{3}{x} + \frac{x}{x^2 + x + 2} \quad (4)$$

2. 2.1 After long division and factorising the denominator:

$$\frac{u^3 - u - 2}{u^3 - u^2 + u - 1} = 1 + \frac{u^2 - 2u - 1}{(u - 1)(u^2 + 1)}$$

Partial fraction decomposition:

$$\frac{u^2 - 2u - 1}{(u - 1)(u^2 + 1)} = \frac{A}{(u - 1)} + \frac{Bu + C}{(u^2 + 1)}$$

Solving for the unknown coefficients: A = –1; B = 0; C = –2

$$\frac{u^3 - u - 2}{u^3 - u^2 + u - 1} = 1 - \frac{1}{(u-1)} - \frac{2}{(u^2+1)}$$

Integrating:

$$\int \frac{u^3 - u - 2}{u^3 - u^2 + u - 1}\, du = u - \ln(u-1) - 2\tan^{-1}(u) + C \qquad (6)$$

2.2 Partial fraction decomposition:

$$\frac{3v^3 - 2v^2 + 16v - 12}{(v^2+4)(1-v)v} = \frac{Av + B}{(v^2+4)} + \frac{C}{(v-1)} + \frac{D}{v}$$

Solving for the unknown coefficients: A = 1; B = 0; C = –1; D = –3

$$\frac{3v^3 - 2v^2 + 16v - 12}{(v^2+4)(v-v^2)} = \frac{v}{(v^2+4)} - \frac{1}{(v-1)} - \frac{3}{v}$$

Integrating:

$$\int \frac{3v^3 - 2v^2 + 16v - 12}{(v^2+4)(v-v^2)}\, dv = \frac{1}{2}\ln\left(v^2+4\right) - \ln(v-1) - 3\ln(v) + C \qquad (6)$$

2.3 Partial fraction decomposition:

$$\frac{9x^2 - 6x + 4}{(3x-1)^3} = \frac{A}{(3x-1)} + \frac{B}{(3x-1)^2} + \frac{C}{(3x-1)^3}$$

Solving for the unknown coefficients: A = 1; B = 0; C = 3

$$\frac{9x^2 - 6x + 4}{(3x-1)^3} = \frac{1}{(3x-1)} + \frac{3}{(3x-1)^3}$$

Integrating:

$$\int \frac{9x^2 - 6x + 4}{(3x-1)^3}\, dx = \frac{1}{3}\ln(3x-1) - \frac{1}{2(3x-1)^2} + C \qquad (6)$$

2.4 Partial fraction decomposition:

$$\frac{4x^4 + 3x^3 - 9x^2 + 25x + 9}{(2x+2)^2(x-1)^3} = \frac{A}{(2x+2)} + \frac{B}{(2x+2)^2} + \frac{C}{(x-1)} + \frac{D}{(x-1)^2} + \frac{E}{(x-1)^3}$$

Solving for the unknown coefficients: A = 0; B = 3; C = 1; D = 0; E = 2

$$\frac{4x^4 + 3x^3 - 9x^2 + 25x + 9}{(2x+2)^2(x-1)^3} = \frac{3}{(2x+2)^2} + \frac{1}{(x-1)} + \frac{2}{(x-1)^3}$$

Integrating:

$$\int \frac{4x^4 + 3x^3 - 9x^2 + 25x + 9}{(2x+2)^2(x-1)^3}\, dx = -\frac{3}{(4x+4)} + \ln(x-1) - \frac{1}{(x-1)^2} + C \qquad (6)$$

2.5 Simplify by taking out 4 on the denominator and numerator

$$\frac{4\left(x^5 - 4x^4 + 5x^3 + x^2 - 4x + 4\right)}{4\left(x^5 - 4x^4 + 4x^3\right)}$$

After long division and factorising the denominator:

$$\frac{x^5 - 4x^4 + 5x^3 + x^2 - 4x + 4}{x^5 - 4x^4 + 4x^3} = 1 + \frac{x^3 + x^2 - 4x + 4}{x^3(x-2)^2}$$

Partial fraction decomposition:

$$\frac{x^3 + x^2 - 4x + 4}{x^3(x-2)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{(x-2)} + \frac{E}{(x-2)^2}$$

Solving for the unknown coefficients: A = 0; B = 0; C = 1; D = 0; E = 1

$$\frac{x^5 - 4x^4 + 5x^3 + x^2 - 4x + 4}{x^5 - 4x^4 + 4x^3} = 1 + \frac{1}{x^3} + \frac{1}{(x-2)^2}$$

Integrating:

$$\int \frac{4x^5 - 16x^4 + 20x^3 + 4x^2 - 16x + 16}{4x^5 - 16x^4 + 16x^3}\,dx = x - \frac{1}{2x^2} - \frac{1}{(x-2)} + C$$ (6)

TOTAL: [50]

MODULE

4 *Differential equations*

After they have completed this module, students should be able to:

- write first order differential equations in standard form:
  $\frac{dy}{dx} + \mathrm{P}(x)y = \mathrm{Q}(x)$, where P and Q are continuous functions;
- calculate the integrating factor of a first order differential equation:
  $\mathrm{I} = e^{\int \mathrm{P}\,dx}$;
- determine the general solution of a first order differential equation:
  $ye^{\int \mathrm{P}\,dx} = \int \mathrm{Q}e^{\int \mathrm{P}\,dx}\,dx$;
- write second order differential equations in standard form:
  $\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = \mathrm{R}(x)$, where a and b are real numbers;
- use the auxiliary function of a second order differential equation to determine the complementary function: $m^2 + am + b = 0$; and
- determine the particular function, R(x), of a second order differential equation, whether it is constant, linear, quadratic or exponential.

Introduction

A differential equation is an equation that involves the derivatives of a function. It describes the relationship between a function and its derivatives. Students have done some introductory work on differential equations previously. This module will further their knowledge of linear differential equations of both the first and second order.

Students need the following pre-knowledge to successfully complete this module.

Pre-knowledge

Students should already know how to:

- Apply the product rule: If $y = u(x).v(x)$, then $\frac{dy}{dx} = v.\frac{du}{dx} + u.\frac{dv}{dx}$
- Calculate partial derivatives (see Module 1)
- Differentiate parametric equations (see Module 1)
- Find the antiderivative by applying the fundamental theorem of calculus. Integration and differentiation are inverse functions.
  $\frac{d}{dx}e^x = e^x \quad \Leftrightarrow \quad \int e^x\,dx = e^x + C$
- Find the inverses of logarithms and exponential functions: $\ln e^x = x = e^{\ln x}$
- Use complex numbers to manipulate the square root of a negative number: $-1 = i^2$

Activity 4.1 — SB page 94

1. $\frac{dy}{dx} + 3x^2 y = 9x^2$

   $P = 3x^2$ and $Q = 9x^2$

   $I = e^{\int P.dx}$

   $= e^{\int 3x^2.dx}$

   $= e^{x^3}$

   $yI = \int QI\,dx$

   $e^{x^3}y = \int 9x^2 e^{x^3}.dx$

   $e^{x^3}y = 3e^{x^3} + C$

   $y = 3 + Ce^{-x^3}$

2. $\frac{dy}{dx} = x - y$

   $P = 1$ and $Q = x$

   $I = e^{\int P.dx}$

   $= e^{\int 1.dx}$

   $= e^x$

   $yI = \int QI\,dx$

   $e^x y = \int x e^x.dx$

   $e^x y = xe^x - e^x + C$

   $y = x - 1 + Ce^{-x}$

3. $x\frac{dy}{dx} = y + x^2 \sin(x)$

$P = -\frac{1}{x}$ and $Q = x \sin x$

$I = e^{\int P.dx}$

$= e^{\int -\frac{1}{x}.dx}$

$= e^{-\ln(x)}$

$= \frac{1}{x}$

$yI = \int QI\ dx$

$y\left(\frac{1}{x}\right) = \int (x \sin(x))\left(\frac{1}{x}\right)\ dx$

$= \int \sin(x)\ dx$

$= -\cos(x) + C$

$y = -x \cos(x) + Cx$

4. $\frac{dy}{dx} = \sin x - 2y \tan x$

$\frac{dy}{dx} + 2y \tan x = \sin x$

$\Rightarrow P = 2 \tan x;\ Q = \sin x$

$I = e^{\int P.dx}$

$= e^{\int 2 \tan x\ dx}$

$= e^{2 \ln \sec x} = \sec^2 x$

$yI = \int QI\ dx$

$y \sec^2 x = \int \sin x \sec^2 x\ dx$

$= \int \tan x \sec x\ dx$

$= \sec x + C$

$y = \cos x + C \cos^2 x$

5. $xy' + 6y = 3x^3$

$y' + \frac{6}{x}y = 3x^2$

$P = \frac{6}{x}$ and $Q = 3x^2$

$I = e^{\int P.dx}$

$\int P.dx = \int \frac{6}{x}.dx$

$= 6\int \frac{1}{x}.dx$

$= 6 \ln x$

$\therefore I = e^{\int P.dx} = e^{6 \ln x} = e^{\ln x^6}$

$= x^6$

$yI = \int QI\,dx$

$x^6 y = \int 3x^2.x^6\,dx$

$= 3\int x^8\,dx$

$= 3\frac{x^9}{9} + C$

$\therefore y = \frac{3x^9}{9x^6} + \frac{C}{x^6}$

$= \frac{1}{3}\frac{x^3}{x^6} + \frac{C}{x^6}$

$= \frac{1}{3x^3} + \frac{C}{x^6}$

Activity 4.2 SB page 103

1. $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 12y = 0$

   The equation has an auxiliary equation of the form:

   $r^2 + 4r - 12 = 0$

   $(r - 2)(r + 6) = 0$

   $r_1 = 2$ and $r_2 = -6$

   $y = Ae^{2x} + Be^{-6x}$

2. $\frac{1}{3}.\frac{d^2y}{dx^2} + \frac{2}{3}.\frac{dy}{dx} - (1)y = 0$

   The equation has an auxiliary equation of the form:

   $\frac{1}{3}r^2 + \frac{2}{3}r - 1 = 0$

   Multiplying by 3:

   $r^2 + 2r - 3 = 0$

   $(r - 1)(r + 3) = 0$

   $r_1 = 1$ and $r_2 = -3$

   $y = Ae^{x} + Be^{-3x}$

3. $4\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + (1)y = 0$

   The equation has an auxiliary equation of the form:

   $4r^2 - 4r + 1 = 0$

   Multiplying the equation by $\frac{1}{4}$:

   $r^2 - r + \frac{1}{4} = 0$

   $\left(r - \frac{1}{2}\right)\left(r - \frac{1}{2}\right) = 0$

   $r_1 = r_2 = \frac{1}{2}$

   $y = (A + Bx)e^{\frac{1}{2}x}$

4. $y'' - 4y' + 13y = 0$

   Rewrite as: $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 13y = 0$

   The equation has an auxiliary equation of the form:

   $m^2 - 4m + 13 = 0$

   $(m - 2)^2 + 9 = 0$

   $\therefore m = 2 \pm 3i$

   For complex roots $m = s \pm ti$ so, $s = 2$ and $t = 3$

   $\therefore y = e^{2x}[\text{A} \cos(3x) + \text{B} \sin(3x)]$

5. $\frac{d^2y}{dx^2} = 4\frac{dy}{dx} - 4y$

   Rewrite as: $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$

   The equation has an auxiliary equation of the form:

   $m^2 - 4m + 4 = 0$

   $(m - 2)^2 = 0$

   $m_1 = m_2 = 2$

   $\therefore y = (\text{A} + \text{B}x)e^{2x}$

Activity 4.3 — SB page 112

1. Solve the differential equation: $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 8y = 2x + 1$

   The equation has an auxiliary equation of the form:

   $r^2 - 2r - 8 = 0$

   $(r - 4)(r + 2) = 0$

   $r_1 = 4$ and $r_2 = -2$

   $y = \text{A}e^{4x} + \text{B}e^{-2x}$

   The partial solution takes the form:

   $y = \text{C}x + \text{D}$

   $y' = \text{C}$

   $y'' = 0$

   Substituting in the original equation:

   $0 - 2\text{C} - 8(\text{C}x + \text{D}) = 2x + 1$

   Grouping like terms together:

   $-2\text{C} - 8\text{D} = 1$

   $-8\text{C}x = 2x$

   The equation then works as follows:

   $y = -\frac{1}{4}x + \frac{1}{16}$

The general solution is therefore:

$y = Ae^{4x} + Be^{-2x} - \frac{1}{4x} + \frac{1}{16}$

2. Determine the particular solution of the following equation:

$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 2e^{-x}$ given $y(0) = 1$ and $y'(0) = 0$

The equation has an auxiliary equation of the form:

$r^2 - r - 6 = 0$

$(r - 3)(r + 2) = 0$

$r_1 = 3$ and $r_2 = -2$

$y = Ae^{3x} + Be^{-2x}$

The particular solution takes the form:

$y = Ce^{-x}$

$y' = -Ce^{-x}$

$y'' = Ce^{-x}$

Substituting in the original equation:

$Ce^{-x} - (-Ce^{-x}) - 6(Ce^{-x}) = 2e^{-x}$

$-4Ce^{-x} = 2e^{-x}$

$C = -\frac{1}{2}$

The general solution is therefore:

$y = Ae^{3x} + Be^{-2x} - \frac{1}{2}e^{-x}$

Find the unique solution of $y = Ae^{3x} + Be^{-2x} - \frac{1}{2}e^{-x}$ for $y(0) = 1$ and $y'(0) = 0$.

$y' = 3Ae^{3x} - 2Be^{-2x} + \frac{1}{2}e^{-x}$

When $y(0) = 1$:

$1 = A + B - \frac{1}{2}$

$A + B = \frac{3}{2}$

When $y'(0) = 0$:

$0 = 3A - 2B + \frac{1}{2}$

$3A - 2B = -\frac{1}{2}$

Solving simultaneously, $A = \frac{8}{12}$ and $B = \frac{10}{12}$

Thus the unique solution is:

$y = \frac{2}{3}e^{3x} + \frac{5}{6}e^{-2x} - \frac{1}{2}e^{-x}$

3. Solve the differential equation: $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = \sin x$

The equation in question has an auxiliary equation of the form:

$r^2 + 2r + 2 = 0$

There are complex roots:

$r_1 = -1 + 2i$ and $r_2 = -1 - 2i$

$y = e^{-x}(\text{A} \cos 2x + \text{B} \sin 2x)$

The particular solution takes the form:

$y = \text{C} \cos x + \text{D} \sin x$

$y' = -\text{C} \sin x + \text{D} \cos x$

$y'' = -\text{C} \cos x - \text{D} \sin x$

Substituting in the original equation:

$-\text{C} \cos x - \text{D} \sin x + 2(- \text{C} \sin x + \text{D} \cos x) + 2(\text{C} \cos x + \text{D} \sin x) = \sin x$

Grouping like terms together:

$\text{C} \cos x + 2\text{D} \cos x = 0$

$\text{D} \sin x - 2\text{C} \sin x = \sin x$

Doing transposing and divisions:

$\text{C} = -2\text{D}$

$\text{D} - 2\text{C} = 1$

Solving both equations simultaneously:

$\text{D} = \frac{1}{5} \qquad \text{C} = \frac{-2}{5}$

Therefore, the general solution is as follows:

$y = e^{-x}\left(\text{A} \cos 2x + \text{B} \sin 2x\right) - \frac{2}{5}\cos x + \frac{1}{5}\sin x$

4. $2\frac{d^2y}{dx^2} + 18y = 12\frac{dy}{dx} + 36e^{-3x}$

Rewrite as: $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 18e^{-3x}$

The equation has an auxiliary equation of the form:

$m^2 - 6m + 3 = 0$

$(m - 3)^2 = 0$

$m_1 = m_2 = 3$

$\therefore y = (\text{A} + \text{B}x)e^{3x}$

The particular solution takes the form:

$y = \text{C}e^{-3x}$

$y' = -3\text{C}e^{-3x}$

$y'' = 9\text{C}e^{-3x}$

Substituting in the original equation:

$9\text{C}e^{-3x} - 6(-3\text{C}e^{-3x}) + 9(\text{C}e^{-3x}) = 18e^{-3x}$

$36Ce^{-3x} = 18e^{-3x}$

$\therefore C = \frac{1}{2}$

The general solution is therefore:

$y = (A + Bx)e^{3x} + \frac{1}{2}e^{-3x}$

5. $\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 6y = 6x + 5$

The equation has an auxiliary equation of the form:

$m^2 - 7m + 6 = 0$

$(m - 6)(m - 1) = 0$

$m_1 = 6; m_2 = 1$

$\therefore y = Ae^{6x} + Be^{x}$

The particular solution takes the form:

$y = Cx + D$

$y' = C$

$y'' = 0$

Substituting in the original equation:

$0 - 7C + 6(Cx + D) = 6x + 5$

$6C = 6$

$\therefore C = 1$

$-7C + 6D = 5$

$\therefore D = 2$

The general solution is therefore:

$y = Ae^{6x} + Be^{x} + x + 2$

Find the unique solution when $y(0) = 1$ and $y'(0) = 2$

$y' = 6Ae^{6x} + Be^{x} + 1$

When $y(0) = 1$:

$1 = A + B + 0 + 2$

$A + B = -1$

When $y'(0) = 2$:

$2 = 6A + B + 1$

$6A + B = 1$

Solving simultaneously, $A = \frac{2}{5}$ and $B = -\frac{7}{5}$

Thus the unique solution is:

$y = \frac{2}{5}e^{6x} - \frac{7}{5}e^{x} + x + 2$

Summative assessment: Module 4 **SB page 113**

1. 1.1 $x\frac{du}{dx} = x^2 + 3u$ $\quad P = -\frac{3}{x}; Q = x$

$\frac{du}{dx} - \frac{3}{x}u = x$

$\therefore e^{-3\int\frac{1}{x}dx} = e^{-3\ln(x)} = I$

$e^{-3\ln(x)} = e^{-\ln(x)^3} = e^{\ln\frac{1}{x^3}} = \frac{1}{x^3}$

$u.e^{-3\ln(x)} = \int x.\frac{1}{x^3}\,dx$

$u.\frac{1}{x^3} = \int\frac{1}{x^2}\,dx$

$u.\frac{1}{x^3} = -\frac{1}{x} + C$

$u = -x^2 + x^3.C$

1.2 $x^2\frac{dy}{dx} + 2xy - \ln(x) = 0$

$\frac{dy}{dx} + \frac{2}{x}.y - \frac{\ln(x)}{x^2} = 0$

$\frac{dy}{dx} + \frac{2}{x}.y = \frac{\ln(x)}{x^2}$

$P = \frac{2}{x}; Q = \frac{\ln(x)}{x^2}$

$\therefore e^{\int\frac{2}{x}.dx} = e^{2\int\frac{1}{x}.dx} = e^{2\ln x} = x^2 = I$

$yI = \int QI\,dx$

$x^2y = \int\frac{\ln x}{x^2}x^2\,dx$

$= \int\ln x\,dx$

$= x(\ln x - 1) + C$

$\therefore y = \frac{(\ln x - 1)}{x} + \frac{C}{x^2}$

$= \frac{\ln x}{x} - \frac{1}{x} + \frac{C}{x^2}$

1.3 $\frac{(1+t)\,dy}{dt} + y = 1 + t$

$\frac{dy}{dt} + \frac{y}{1+t} = 1$

$P = \frac{1}{1+t}; Q = 1$

$\therefore I = e^{\int\frac{1}{1+t}dt}$

Let $u = 1 + t$; $du = dt$

$e^{\int \frac{1}{u} du}$

$= e^{\ln(u)} = u = 1 + t$

$\frac{dy}{dt}(1 + t) + y = 1 + t$

$\frac{d}{dt}(y(1 + t)) = \int (1 + t)\, dt$

$y(1 + t) = \int dt + \int t\, dt$

$y(1 + t) = t + \frac{t^2}{2} + C$

$y = \frac{\frac{1}{2}t^2 + t + C}{1 + t}$

1.4 $\frac{dy}{dx} = x + 4y$

$\frac{dy}{dx} - 4y = x$

$P = -4$; $Q = x$

$\therefore I = e^{\int -4\, dx} = e^{-4\int dx} = e^{-4x}$

$\frac{dy}{dx}.e^{-4x}y = xe^{-4x}$

$\frac{d}{dx}(y.e^{-4x}) = xe^{-4x}$

$\int \frac{d}{dx}(ye^{-4x})\, dx = \int xe^{-4x}\, dx$

Let $u = -4x$

$\therefore x = \frac{u}{-4}$

Then $du = -4dx$; $dx = \frac{du}{-4}$

$y.e^{u} = \int \left(\frac{u}{-4}\right).e^{u}.\left(\frac{du}{-4}\right)$

$= \frac{1}{16}\int u.e^{u}\, du$

Let $f(x) = u$; $g'(x) = e^{u}$

Then $f'(x) = du$; $g(x) = e^{u}$

$y.e^{u} = \frac{1}{16}(u.e^{u} - \int e^{u}\, du)$

$y.e^{u} = \frac{1}{16}u.e^{u} - \frac{e^{u}}{16} + C$

$y = \frac{u}{16} - \frac{1}{16} + \frac{C}{e^{u}}$

$y = \frac{-x}{4} - \frac{1}{16} + \frac{C}{e^{-4x}}$

1.5 $\frac{dy}{dx} = xe^{-\sin(x)} - y\cos(x)$

$\frac{dy}{dx} + \cos(x)y = x.e^{-\sin(x)}$

$P = \cos(x)$; $Q = x.e^{-\sin(x)}$

$\therefore I = e^{\int \cos(x)\, dx} = e^{\sin(x)}$

$\frac{dy}{dx}\, e^{\sin(x)}.y = x.e^{-\sin(x)}.e^{\sin(x)}$

$\frac{d}{dx}\,(ye^{\sin(x)}) = x$

$\int \frac{d}{dx}\,(y.e^{\sin(x)})\; dx = \int x\; dx$

$ye^{\sin(x)} = \frac{1}{2}x^2 + \text{C}$

$y = \left(\frac{1}{2}x^2 + \text{C}\right) e^{-\sin(x)}$ (4)

2. 2.1 $9\frac{d^2y}{dx^2} - 12\frac{dy}{dx} + 4y = 0$

Auxiliary equation is: $9r^2 - 12r + 4 = 0$

Divide by 9 to get the monic polynomial form:

$r^2 - \frac{12r}{9} + \frac{4}{9} = 0$

By quadratic formula:

$\frac{\frac{12}{9} \pm \sqrt{\left(-\frac{12}{9}\right)^2 - 4\left(\frac{4}{9}\right)}}{2} = r$

$r = \frac{2}{3}$

$\therefore \left(r - \frac{2}{3}\right)^2 = 0$

$r = \frac{2}{3}$

General solution is:

$y = e^{\frac{2}{3}x}\,(\text{A} + \text{B}x)$

2.2 $\frac{d^2y}{dx^2} = \frac{dy}{dx}$

$\frac{d^2y}{dx^2} - \frac{dy}{dx} = 0$

Auxiliary equation is:

$r^2 - r = 0$

$r(r - 1) = 0$

$r = 0$ or $r = 1$

General solution is:

$y = \text{A}e^{(0)x} + \text{B}^{(1)x}$

$y = \text{A} + \text{B}e^x$

2.3 $3\frac{d^2y}{dx^2} = y - 2\,\frac{dy}{dx}$

$3\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - y = 0$

Auxiliary equation is: $3r^2 + 2r - 1 = 0$

Divide by 3 to get the monic polynomial form

$r^2 + \frac{2}{3}r - \frac{1}{3} = 0$

By quadratic formula:

$$\frac{-\frac{2}{3} \pm \sqrt{\left(\frac{2}{3}\right)^2 - 4\left(\frac{1}{-3}\right)}}{2} = -\frac{-\frac{2}{3} \pm \frac{4}{3}}{2}$$

$\left(r + \frac{1}{3}\right)(r - 1) = 0$

$r = -\frac{1}{3}$ or $r = 1$

General solution:

$y = \mathrm{A}e^{-\frac{1}{3}x} + \mathrm{B}e^{x}$ (5)

3. 3.1 $\frac{d^2y}{dx^2} - 2y = x^2 - \frac{dy}{dx}$

$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = x^2$

Auxiliary equation is:

$r^2 + r - 2 = 0$

$(r - 1)(r + 2) = 0$

$\therefore r = 1$ or $r = -2$

Characteristic equation is: $y = \mathrm{A}e^{x} + \mathrm{B}e^{-2x}$

Particular integral has form:

$y = \mathrm{C}x^2 + \mathrm{D}x + \mathrm{E}$

$\frac{dy}{dx} = 2\mathrm{C}x + \mathrm{D}$

$\frac{d^2y}{dx^2} = 2\mathrm{C}$

Substitute in D, E:

$2\mathrm{C} + (2\mathrm{C}x + \mathrm{D}) - 2(\mathrm{C}x^2 + \mathrm{D}x + \mathrm{E}) = x^2$

Gathering coefficients of like terms:

$-2\mathrm{C} = 1$

$2\mathrm{C} - 2\mathrm{D} = 0$

$2\mathrm{C} + \mathrm{D} - 2\mathrm{E} = 0$

$\therefore \mathrm{C} = -\frac{1}{2}; \mathrm{D} = -\frac{1}{2}; \mathrm{E} = -\frac{3}{4}$

Particular integral is: $y = -\frac{1}{2}x^2 - \frac{1}{2}x - \frac{3}{4}$

$y = \mathrm{A}e^{x} + \mathrm{B}e^{-2x} - \frac{1}{2}x^2 - \frac{1}{2}x - \frac{3}{4}$

3.2 $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = \cos(2x)$

Auxiliary equation is:

$r^2 - 2r - 3 = 0$

$(r - 3)(r + 1) = 0$

$\therefore r = 3$ or $r = -1$

Characteristic equation is:

$y = Ae^{3x} + Be^{-x}$

Particular integral has form:

$y = C\cos(2x) + D\sin(2x)$

$\frac{dy}{dx} = -2C\sin(2x) + 2D\cos(2x)$

$\frac{d^2y}{dx^2} = -4C\cos(2x) - 4D\sin(2x)$

Substitute in D, E:

$-4C\cos(2x) - 4D\sin(2x) - 2(-2C\sin(2x) + 2D\cos(2x)) - 3(C\cos(2x) + D\sin(2x))$
$= \cos(2x)$

Gathering coefficients in like terms:

$-7C - 4D = 1$ … ①

$-7D + 4C = 0$ … ②

$\therefore C = \frac{7}{4}D$ (from ②)

$-7\left(\frac{7}{4D}\right) - 4D = 1$ sub ② in ①

$\therefore D = -\frac{4}{65}$

$\therefore C = \frac{7}{4}\left(-\frac{4}{65}\right) = -\frac{7}{65}$

Particular integral is

$y = -\frac{7}{65}\cos(2x) - \frac{4}{65}\sin(2x)$

General solution is: $y = Ae^{3x} + Be^{-x} - \frac{7}{65}\cos(2x) - \frac{4}{65}\sin(2x)$

3.3 $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^{2x}$

Auxiliary equation is:

$r^2 - 2r + r = 0$

$(r - 1)^2 = 0$

$\therefore r = 1$

Characteristic equation is:

$y = e^x(A + Bx)$

Particular integral has form:

$y = Ce^{2x}$

$\frac{dy}{dx} = 2Ce^{2x}$

$\frac{d^2y}{dx^2} = 4Ce^{2x}$

Substitute in D, E:

$4Ce^{2x} - 2(2Ce^{2x}) + Ce^{2x} = e^{2x}$

$Ce^{2x} = e^{2x}$

$\therefore C = 1$

Particular integral is: $y = e^{2x}$

$\therefore$ General solution is:

$y = e^{x}(A + Bx) + e^{2x}$ (5)

TOTAL: [50]

MODULE

5 *Areas and volumes*

After they have completed this module, students should be able to:

- sketch a function on a given interval;
- calculate the areas and volumes of a given function using a definite integral;
- calculate the points of intersection of areas and volumes under two functions;
- sketch the points of intersection of areas and volumes under two functions; and
- calculate the areas and volumes of two given functions using a definite integral.

Introduction

Students have already encountered and used different integration strategies to calculate the areas and volumes of functions in N5 Mathematics. In this module we will expand on the topic of determining areas and volumes of spaces that are bordered by algebraic and trigonometric functions.

Students need the following pre-knowledge to successfully complete this module.

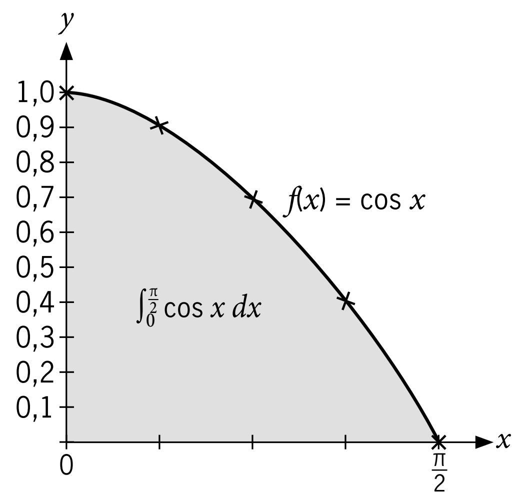
Pre-knowledge

Students should already know how to:

- Use different strategies to solve integrals.
- Solve definite integrals.
- Find the roots and intersection points of functions.

Activity 5.1 SB page 123

1. $\int_0^{\frac{\pi}{2}} \cos x\, dx = \sin x\Big|_0^{\frac{\pi}{2}} = 1 - 0 = 1$ units^2



2. The roots are: $y^2 - y - 12 = 0$

$(y - 4)(y + 3) = 0$

$\therefore y = 4$ and $y = -3$

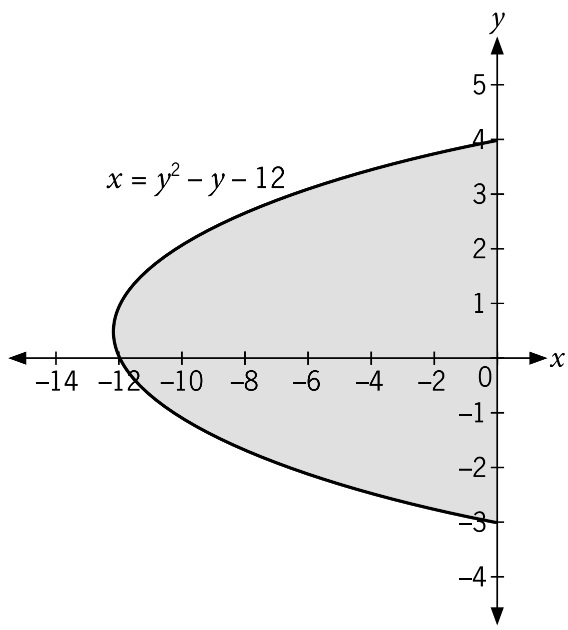
$A = \int_{-3}^{4} y^2 - y - 12\, dy$

$= \left(\frac{1}{3}y^3 - \frac{1}{2}y^2 - 12y\right)\Big|_{-3}^{4}$

$= \frac{64}{3} - \frac{16}{2} - 12.4 + \frac{27}{3} + \frac{9}{2} - 12.3$

$= -57{,}167$ units^2

$= 57{,}167$ units^2



3. The roots are:

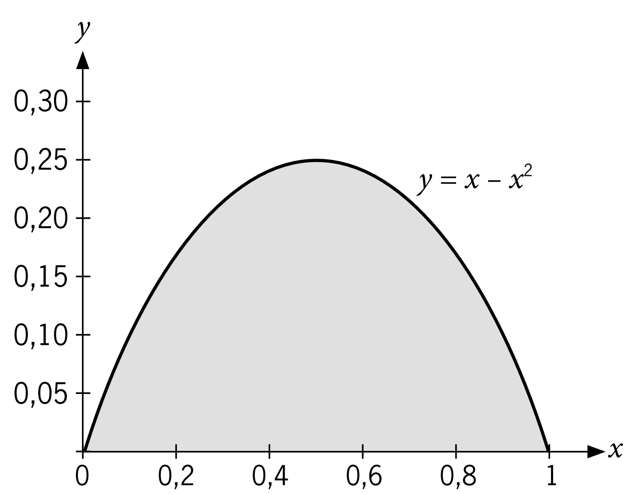
$x - x^2 = 0$

$x(1 - x) = 0$

$\therefore x = 0$ and $x = 1$

$A = \int_0^1 (x - x^2)\, dx = \frac{x^2}{2} - \frac{x^3}{3}\Big|_0^1$

$= \frac{1}{6}$ units^2



4. $y = 5x - x^2$

$= x(5 - x)$

The roots are $x = 0$ and $x = 5$.

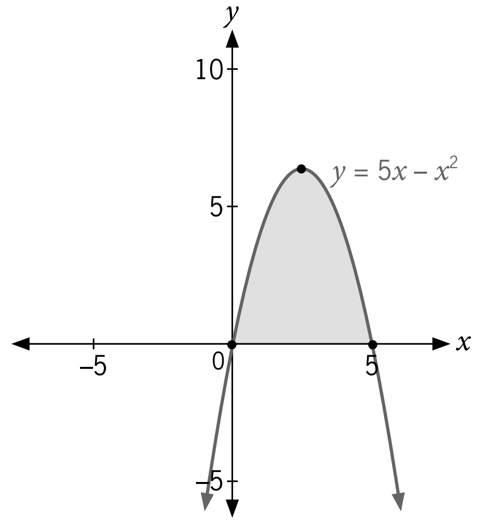
$A = \int_0^5 5x - x^2 \, dx$

$= \left[\frac{5x^2}{2} - \frac{x^3}{3}\right]_0^5$

$= \left[\frac{5(5)^2}{2} - \frac{(5)^3}{3}\right] - \left[\frac{5(0)^2}{2} - \frac{(0)^3}{3}\right]$

$= \frac{125}{2} - \frac{125}{3} - 0$

$= \frac{125}{6}$ units^2



5. $y = x^3$

We want the area in relation to the y-axis, so write the function in terms of y:

$x = \sqrt[3]{y} = y^{\frac{1}{3}}$

The limits are given as $y = 0$ and $y = 8$.

$A = \int_0^8 y^{\frac{1}{3}} \, dy$

$= \left[\frac{y^{\frac{4}{3}}}{\frac{4}{3}}\right]_0^8$

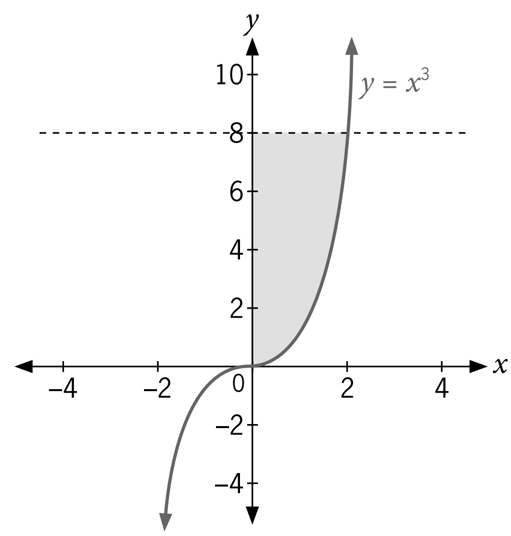
$= \frac{3}{4}\left[y^{\frac{4}{3}}\right]_0^8$

$= \frac{3}{4}\left[y^{\frac{4}{3}}\right]_0^8$

$= \frac{3}{4}\left[\sqrt[3]{(8)}^4 - \sqrt[3]{(0)}^4\right]$

$= \frac{3}{4}\left[\sqrt[3]{(8)}^4\right]$

$= 12$ units^2



Activity 5.2 SB page 125

1. $x^2 + 2x + 3 = x + 9$

   $x^2 + x - 6 = 0$

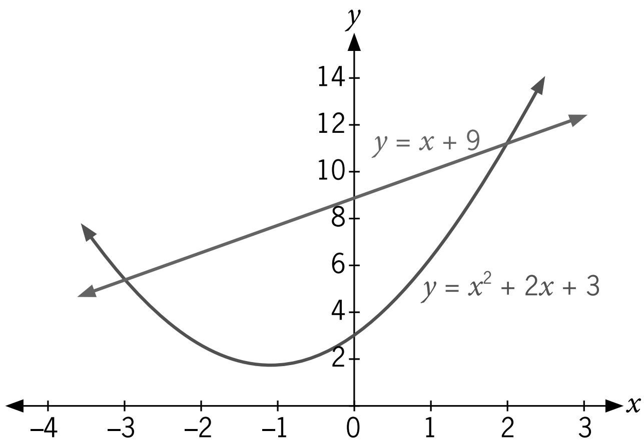
   $(x + 3)(x - 2) = 0$

   $\therefore x = -3, x = 2$

   If $x = -3, y = x + 9 = -3 + 9 = 6$

   If $x = 2, y = x + 9 = 2 + 9 = 11$

   $\therefore (x; y) = (-3; 6), (2; 11)$



2. $-4x^2 + 3x + 6 = 3x + 2$

   $-4x^2 + 4 = 0$

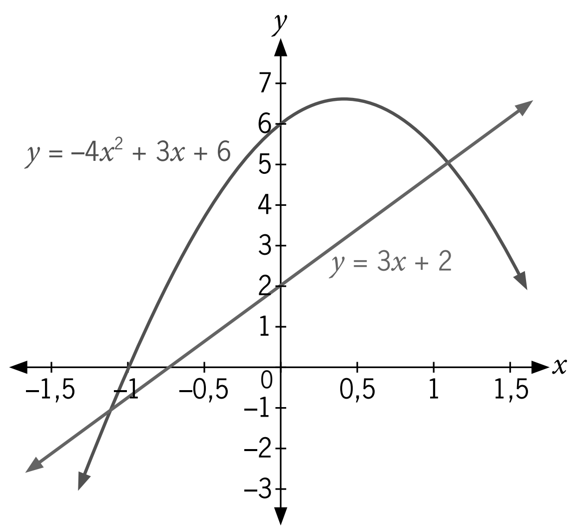
   $x^2 = 1$

   $\therefore x = \pm 1$

   If $x = -1, y = 3x + 2 = 3(-1) + 2 = -1$

   If $x = 1, y = 3x + 2 = 3(1) + 2 = 5$

   $\therefore (x; y) = (-1; -1), (1; 5)$



3. $3x^2 - 3x + 2 = -2x + 4$

   $3x^2 - x - 2 = 0$

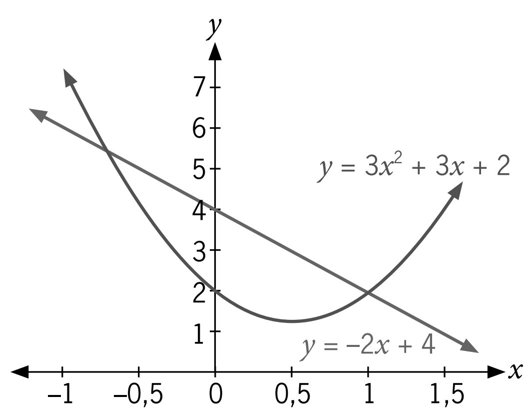
   $(3x + 2)(x - 1) = 0$

   $\therefore x = -\frac{2}{3}, x = 1$

   If $x = -\frac{2}{3}, y = -2x + 4 = -2\left(-\frac{2}{3}\right) + 4 = 5\frac{1}{3}$

   If $x = 1, y = -2x + 4 = -2(1) + 4 = 2$

   $\therefore (x; y) = \left(-\frac{2}{3}; 5\frac{1}{3}\right), (1; 2)$



4. $-3x^2 + 6x + 8 = x^2 - 4x + 2$

$-4x^2 + 10x + 6 = 0$

$(4x + 2)(-x + 3) = 0$

$\therefore x = -\frac{1}{2}, x = 3$

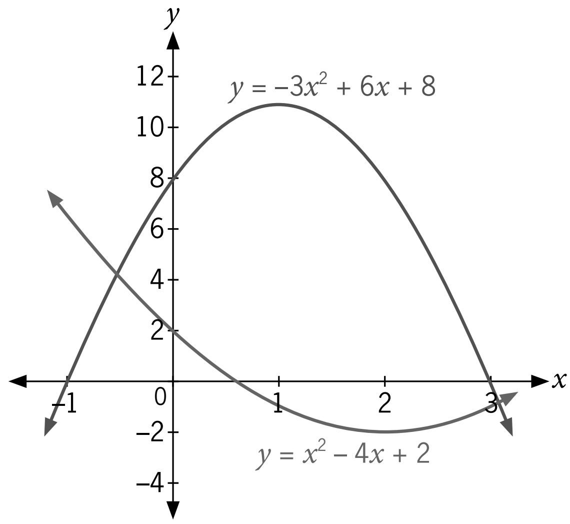
If $x = 3$, $y = x^2 - 4x + 2$

$= 3^2 - 4(3) + 2 = -1$

If $x = -\frac{1}{2}$, $y = \left(-\frac{1}{2}\right)^2 - 4\left(-\frac{1}{2}\right) + 2$

$= \frac{1}{4} + 2 + 2 = 4\frac{1}{4}$

$\therefore (x; y) = \left(-\frac{1}{2}; 4\frac{1}{4}\right), (3; -1)$



5. $5x - x^2 = x + 4$

$-x^2 + 4x - 4 = 0$

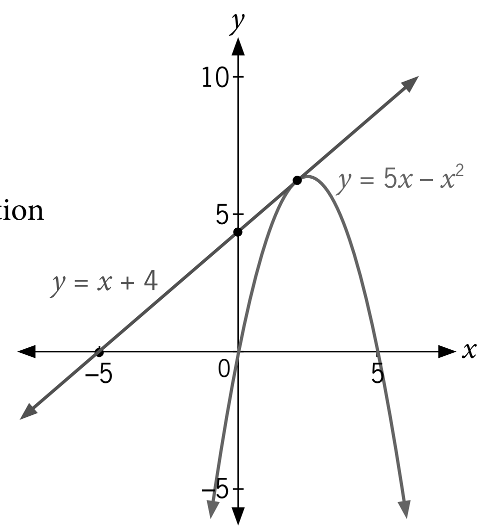
$x^2 - 4x + 4 = 0$

$(x - 2)(x - 2) = 0$

$\therefore x = 2$ and there is only one point of intersection

If $x = 2$, $y = x + 4 = 6$

$\therefore (x; y) = (2; 6)$



6. $5x + x^2 = 9x - 4$

$5x + x^2 - 9x + 4 = 0$

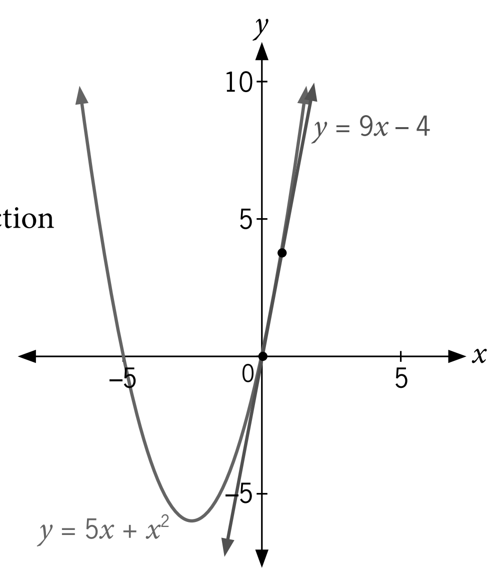
$x^2 - 4x + 4 = 0$

$(x - 2)(x - 2) = 0$

$\therefore x = 2$ and there is only one point of intersection

If $x = 2$, $y = 9x + 4 = 22$

$\therefore (x; y) = (2; 22)$



Activity 5.3 SB page 129

1. The intersection points are where:

$x^2 - x - 1 = -x^2 + 2x + 1$

$2x^2 - 3x - 2 = 0$

$(2x + 1)(x - 2) = 0$

$x = -\frac{1}{2}, x = 2$

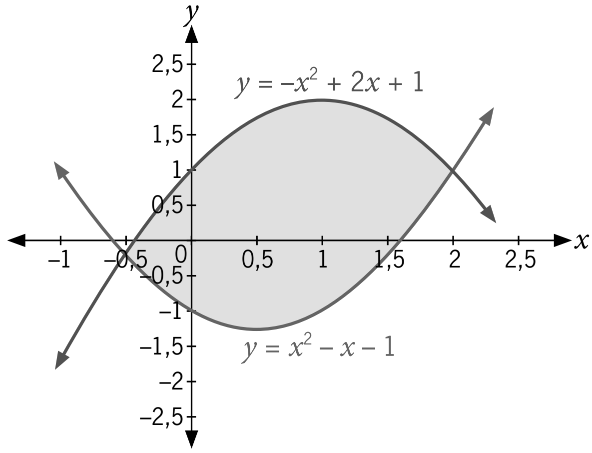
$A = \int_{-\frac{1}{2}}^{2} \left(-x^2 + 2x + 1 - x^2 + x + 1\right) dx$

$= \int_{-\frac{1}{2}}^{2} \left(-2x^2 + 3x + 2\right) dx$

$= \left(-\frac{2}{3}x^3 + \frac{3}{2}x^2 + 2x\right)\Big|_{-\frac{1}{2}}^{2}$

$= -\frac{2}{3}.8 + \frac{3}{2}.4 + 4 - \frac{2}{3}.\frac{1}{8} - \frac{3}{2}.\frac{1}{4} + 1$

$= \frac{125}{24} \approx 5{,}208 \text{ units}^2$



2. The intersection points are where:

$\frac{5}{x^2 + 1} = 1$

$\therefore x^2 = 4x = \pm 2$

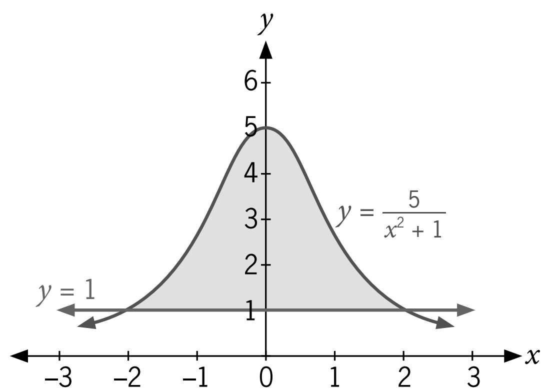
$A = \int_{-2}^{2} \left(\frac{5}{x^2 + 1}\right) dx - \int_{-2}^{2} 1\, dx$

$= \left(5 \tan^{-1} x - x\right)\Big|_{-2}^{2}$

$= 5\left[\tan^{-1}(2) - \tan^{-1}(-2)\right] - 4$

$= 10 \tan^{-1} 2 - 4$

$= 7{,}07 \text{ units}^2$



3. The intersection points are where:

$6 - x = \frac{5}{x}$

$\therefore x^2 - 6x + 5 = 0$

$(x - 1)(x - 5) = 0$

$x = 1, x = 5$

$A = \int_{1}^{5} 6 - x - \frac{5}{x} dx = 6x - \frac{x^2}{2} - 5 \ln x \Big|_{1}^{5}$

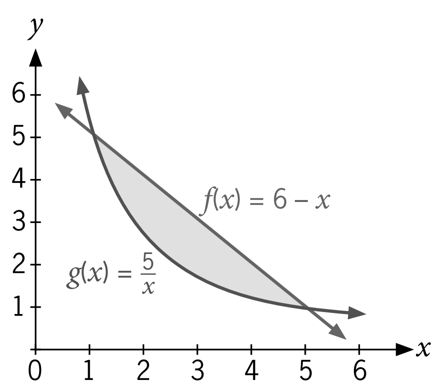
$= \left(6(5) - \frac{5^2}{2} - 5 \ln 5\right) - \left(6(1) - \frac{1^2}{2} - 5 \ln 1\right)$

$= 30 - \frac{25}{2} - 5 \ln 5 - 6 + \frac{1}{2} + 5(0)$

$= 12 - 5 \ln 5$

$\approx 3{,}953 \text{ units}^2$

x	1	2	3	4	5
***f*(*x*)**	5	4	3	2	1
***g*(*x*)**	5	2,5	$1\frac{2}{3}$	$1\frac{1}{4}$	1



4. The intersection points are where:

$3x = x^2 + 2x - 6$

$3x - x^2 - 2x + 6 = 0$

$x^2 - x - 6 = 0$

$(x - 3)(x + 2) = 0$

$\therefore x = 3$ or $x = -2$

$A = \int_{-2}^{3} 3x - (x^2 + 2x - 6)\, dx$

$= \int_{-2}^{3} -x^2 + x + 6\, dx$

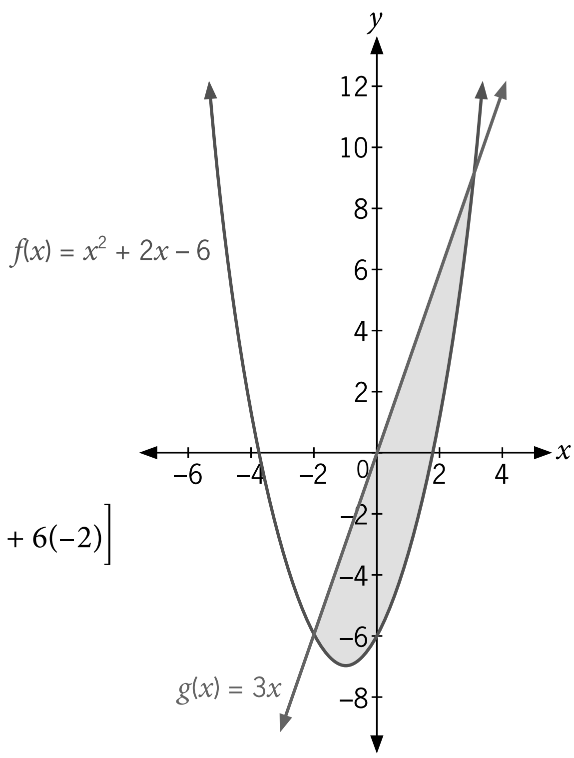
$= \left[-\frac{x^3}{3} + \frac{x^2}{2} + 6x\right]_{-2}^{3}$

$= \left[-\frac{(3)^3}{3} + \frac{(3)^2}{2} + 6(3)\right] - \left[-\frac{(-2)^3}{3} + \frac{(-2)^2}{2} + 6(-2)\right]$

$= \left[-\frac{27}{3} + \frac{9}{2} + 18\right] - \left[\frac{8}{3} + \frac{4}{2} - 12\right]$

$= \left[-\frac{35}{3} - \frac{7}{2} + 30\right]$

$= \frac{125}{6}$ units^2



5. The intersection points are where:

$5x + 1 = x^2 + 3x - 2$

$5x + 1 - x^2 - 3x + 2 = 0$

$x^2 - 2x - 3 = 0$

$(x - 3)(x + 1) = 0$

$\therefore x = 3$ or $x = -1$

$A = \int_{-1}^{3} 5x + 1 - (x^2 + 3x - 2)\, dx$

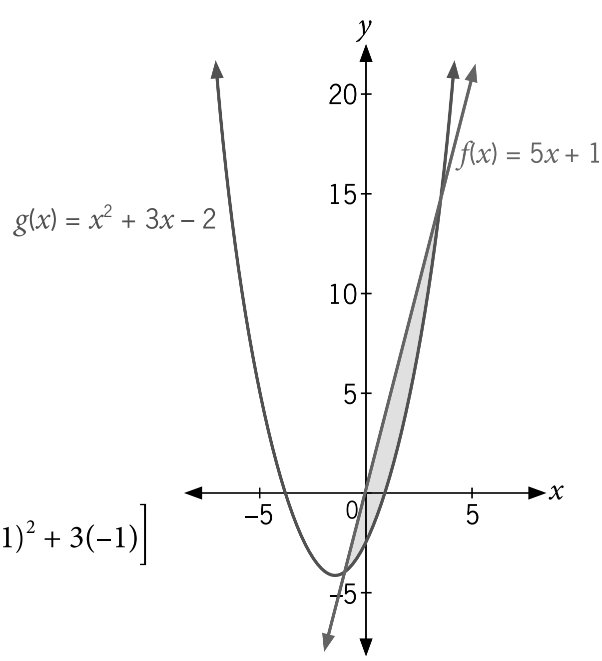
$= \int_{-1}^{3} -x^2 + 2x + 3\, dx$

$= \left[-\frac{x^3}{3} + x^2 + 3x\right]_{-1}^{3}$

$= \left[-\frac{(3)^3}{3} + (3)^2 + 3(3)\right] - \left[-\frac{(-1)^3}{3} + (-1)^2 + 3(-1)\right]$

$= [-9 + 9 + 9] - \left[\frac{1}{3} + 1 - 3\right]$

$= \frac{32}{3}$ units^2



Activity 5.4 SB page 136

1. 1.1 $V_x = \pi\int (f(x))^2\,dx$

$f(x) = 1 - x^2$ with limits $x = -1$ to $x = 1$:

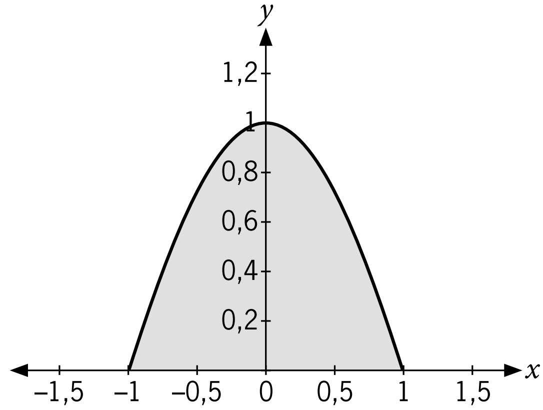
$\therefore V = \pi\int_{-1}^{1} (1 - x^2)^2\,dx$

$= \pi\int_{-1}^{1} 1 - 2x^2 + x^4\,dx$

$= \pi\left(x - \frac{2}{3}x^3 + \frac{x^5}{5}\right)\Big|_{-1}^{1}$

$= 2\pi\left(1 - \frac{2}{3} + \frac{1}{5}\right)$ (symmetrical about y-axis)

$= \frac{2.(15 - 10 + 3)}{15}\pi = \frac{16}{15}\pi$ units^3



1.2 $V_x = \pi\int (f(x))^2\,dx$

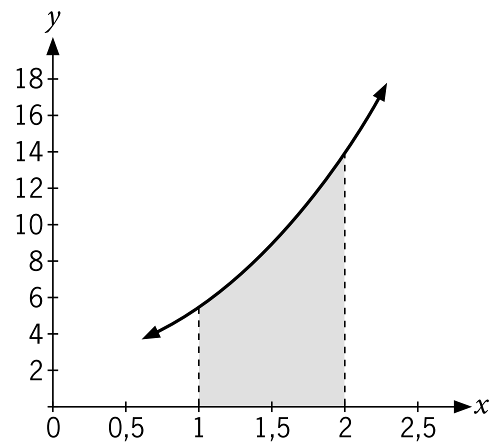
Substitute $f(x) = 2e^x$ with limits $x = 1$ to $x = 2$:

$\therefore V = \pi\int_{1}^{2} (2e^x)^2\,dx$

$= 4\pi\int_{1}^{2} e^{2x}\,dx$

$= 4\pi\left(\frac{1}{2}e^{2x}\right)\Big|_{1}^{2}$

$= 2\pi(e^4 - e^2) = 296{,}6$ units^3



1.3 $V_x = \pi\int (f(x))^2\,dx$

Substitute $f(x) = x - \frac{x^2}{3}$ with limits $x = 0$ to $x = 2$:

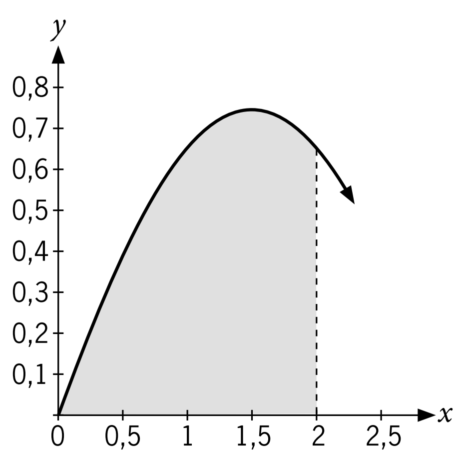
$\therefore V = \pi\int_{0}^{2} \left(x - \frac{x^2}{3}\right)^2\,dx$

$= \pi\int_{0}^{2} \left(x^2 - \frac{2}{3}x^3 + \frac{x^4}{9}\right)\,dx$

$= \pi\left(\frac{x^3}{3} - \frac{2x^4}{3.4} + \frac{x^5}{5.9}\right)\Big|_{0}^{2}$

$= \pi\left(\frac{8}{3} - \frac{16}{6} + \frac{32}{45}\right)$

$= \pi\left(\frac{32}{45}\right) = 2{,}23$ units^3



2. 2.1 Rewrite the function in terms of y:

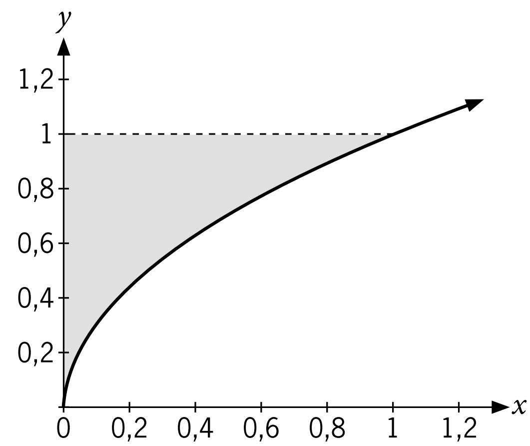
$x = y^2$

$V_y = \pi\int (f(y))^2\, dy$

$= \pi\int_0^1 (y^2)^2\, dy$

$= \pi\frac{y^5}{5}\Big|_0^1$

$= \frac{\pi}{5} = 0{,}63 \text{ units}^3$



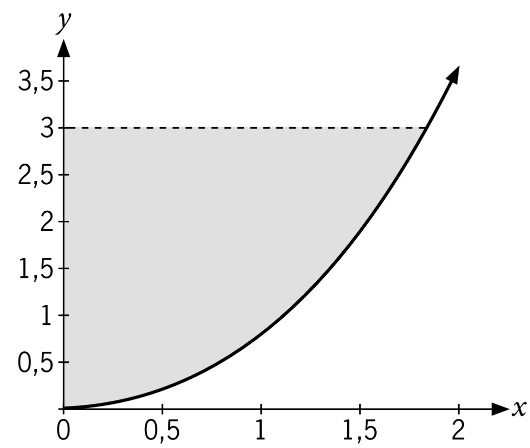
2.2 Rewrite the function in terms of y:

$x = \sqrt{y}$

$V_y = \pi\int (f(y))^2\, dy$

$= \pi\int_0^3 (\sqrt{y})^2\, dy$

$= \pi\frac{y^2}{2}\Big|_0^3 = \frac{9}{2}\pi = 14{,}1 \text{ units}^3$

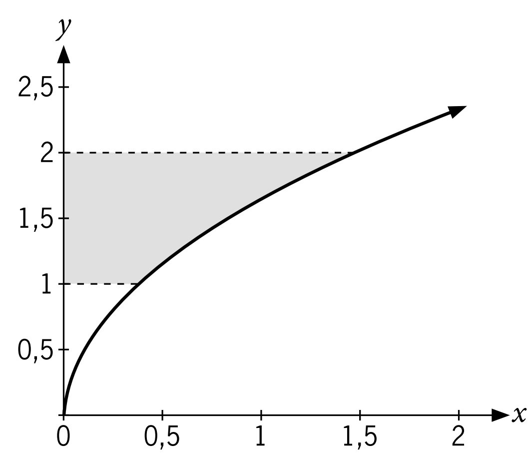


2.3 $V_y = \pi\int (f(y))^2\, dy$

$= \pi\int_1^2 \left(\frac{y^2}{3}\right)^2 dy$

$= \pi\frac{y^5}{9.5}\Big|_1^2$

$= \pi\left(\frac{32}{45} - \frac{1}{45}\right) = 2{,}16 \text{ units}^3$



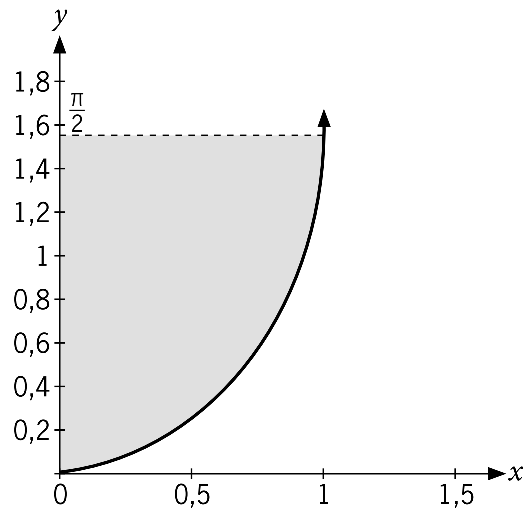
2.4 $V_y = \pi\int (f(y))^2\, dy$

$= \pi\int_0^{\frac{\pi}{2}} \left(\sqrt{\sin y}\right)^2 dy$

$= \pi\int_0^{\frac{\pi}{2}} \sin y\, dy$

$= -\pi.\cos y\Big|_0^{\frac{\pi}{2}}$

$= -\pi(0 - 1) = \pi$ units^3



Activity 5.5 SB page 144

1. $V_y = 2\pi\int_a^b xy\, dx$

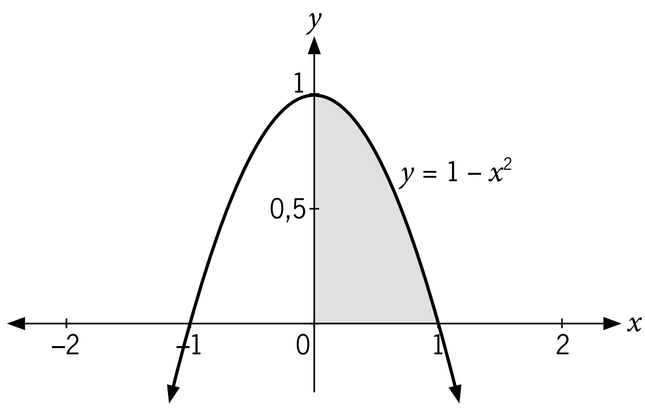
$V_y = 2\pi\int_0^1 x(1 - x^2)\, dx$

$= 2\pi\int_0^1 x - x^3\, dx$

$= 2\pi\left(\frac{1}{2}x^2 - \frac{1}{4}x^4\right)\Big|_0^1$

$= 2\pi\left[\left(\frac{1}{2} - \frac{1}{4}\right) - (0)\right]$

$= \frac{\pi}{2} = 6{,}283$ units^3



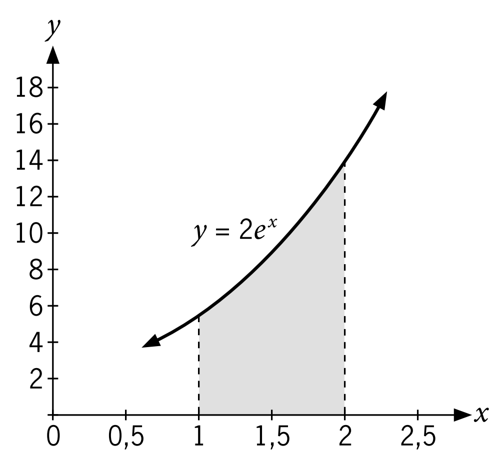
2. $V_y = 2\pi\int_a^b xy\, dx$

$= 2\pi\int_1^2 x.2e^x\, dx$

$= 4\pi(xe^x - e^x)\Big|_1^2$

$= 4\pi\left[(2e^2 - e^2) - (1e^1 - e^1)\right]$

$= 4\pi e^2 \approx 92{,}85$ units^3



3. $V_x = 2\pi\int_a^b xy\ dy$

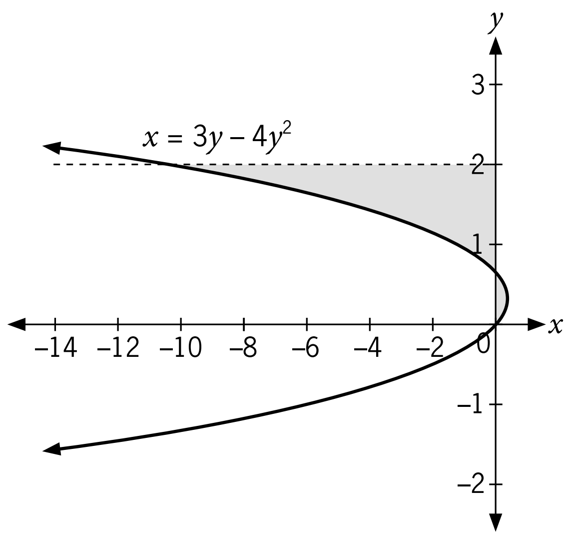
$= 2\pi\int_0^2 (3y - 4y^2).y\ dy$

$= 2\pi\int_0^2 3y^2 - 4y^3\ dy$

$= 2\pi(y^3 - y^4)\Big|_0^2$

$= 2\pi\left[(2^3 - 2^4) - 0\right]$

$= -16\pi \approx 50{,}266$ units^3



4. $V_y = 2\pi\int_a^b x.f(x)\ dx$

$= 2\pi\int_0^3 x.(3x^2 - x^3)\ dx$

$= 2\pi\int_0^3 (3x^3 - x^4)\ dx$

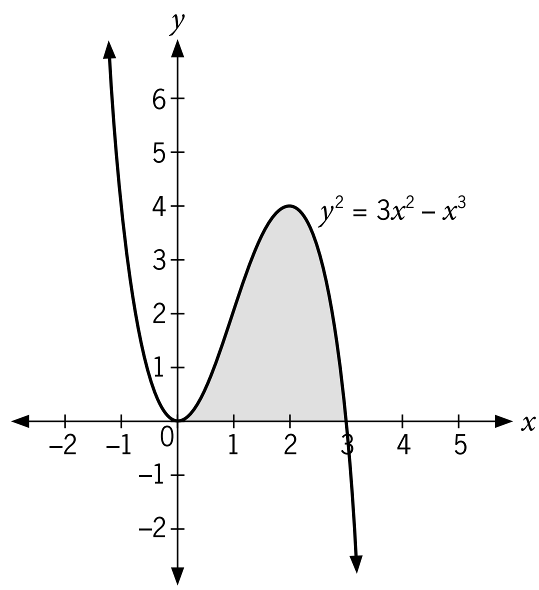
$= 2\pi\left[3\frac{x^4}{4} - \frac{x^5}{5}\right]_0^3$

$= 2\pi\left(3\frac{(3)^4}{4} - \frac{(3)^5}{5} - 0\right)$

$= 2\pi\left(3\frac{(3)^4}{4} - \frac{(3)^5}{5} - 0\right)$

$= 2\pi\left(\frac{243}{20}\right)$

$= 24{,}3\pi = 76{,}341$ units^3



5. $V_x = 2\pi\int_a^b y.f(y)\ dy$

$= 2\pi\int_1^2 y.(1 + y^2)\ dy$

$= 2\pi\int_1^2 (y + y^3)\ dy$

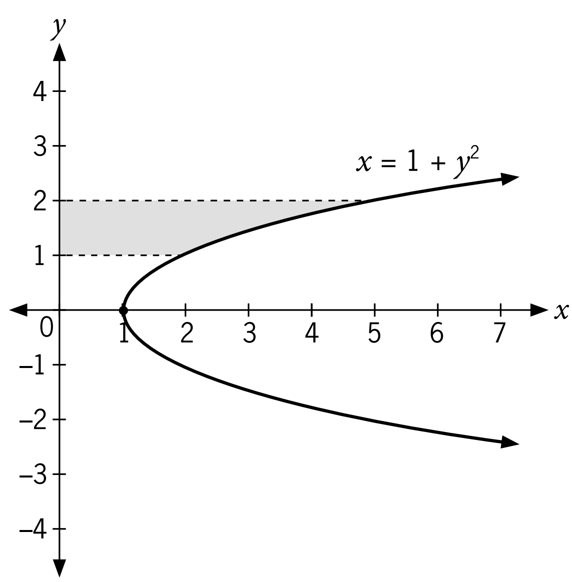
$= 2\pi\left[\frac{y^2}{2} + \frac{y^4}{4}\right]_1^2$

$= 2\pi\left[\left(\frac{(2)^2}{2} + \frac{(2)^4}{4}\right) - \left(\frac{(1)^2}{2} + \frac{(1)^4}{4}\right)\right]$

$= 2\pi\left(2 + 4 - \frac{3}{4}\right)$

$= 2\pi\left(\frac{21}{4}\right)$

$= \frac{21}{2}\pi = 32{,}987$ units^3



Activity 5.6 SB page 152

1. Determine the intersection points:

$x^2 = 2x$

$\therefore x^2 - 2x = 0$

$x(x - 2) = 0$

$x = 0$ and $x = 2$

$$V_x = \pi\int_c^d [f(x)^2 - g(x)^2\, dx$$

$$= \pi\int_0^2 (2x)^2 - \left(x^2\right)^2 dx$$

$$= \pi\int_0^2 4x^2 - x^4\, dx$$

$$= \pi\left(\frac{4}{3}x^3 - \frac{1}{5}x^5\right)\Big|_0^2$$

$$= \pi\left(\frac{32}{3} - \frac{32}{5}\right) = 13{,}404 \text{ units}^3$$

2. Determine the intersection points:

$2y^2 = \sqrt{\frac{y}{2}}$

$4y^4 = \frac{y}{2}$

$8y^4 - y = 0$

$y\left(8y^3 - 1\right) = 0$

$\therefore y = 0$ and $y^3 = \frac{1}{8} \Rightarrow y = \frac{1}{2}$

$$V_y = \pi\int_0^{\frac{1}{2}} \left(\sqrt{\frac{y}{2}}\right)^2 - \left(2y^2\right)^2 dy$$

$$= \pi\int_0^{\frac{1}{2}} \frac{y}{2} - 4y^4\, dy$$

$$= \pi\left(\frac{y^2}{4} - \frac{4y^5}{5}\right)\Big|_0^{\frac{1}{2}}$$

$$= \pi\left[\frac{1}{16} - \frac{4}{5}\left(\frac{1}{32}\right)\right]$$

$$= \pi\frac{10 - 4}{160} = \pi\frac{3}{80} = 0{,}118 \text{ units}^3$$

3. Determine the intersection points:

$x^2 - 2x + 2 = -x^2 - 3x + 3$

$\therefore 2x^2 + x - 1 = 0$

$(2x - 1)(x + 1) = 0$

$\therefore x = \frac{1}{2}$ and $x = -1$

$$V_x = \pi\int_c^d [f(x)^2 - g(x)^2]\ dx$$

$$= \pi\int_{-1}^{\frac{1}{2}} (-x^2 - 3x + 3)^2 - (x^2 - 2x + 2)^2\ dx$$

$$= \pi\int_{-1}^{\frac{1}{2}}(x^4 + 3x^3 - 3x^2 + 3x^3 + 9x^2 - 9x - 3x^2 - 9x + 9)$$

$$- (x^4 - 2x^3 + 2x^2 - 2x^3 + 4x^2 - 4x + 2x^2 - 4x + 4)\ dx$$

$$= \pi\int_{-1}^{\frac{1}{2}} 10x^3 - 5x^2 - 10x + 5\ dx$$

$$= \pi\left(10\frac{x^4}{4} - 5\frac{x^3}{3} - 10\frac{x^2}{2} + 5x\right)\Bigg|_{-1}^{\frac{1}{2}}$$

$$= \pi\left(\frac{10}{64} - \frac{5}{24} - \frac{10}{8} + \frac{5}{2}\right) - \pi\left(\frac{10}{4} + \frac{5}{3} - \frac{10}{2} - 5\right)$$

$$= \pi\left(\frac{15 - 20 - 120 + 240 - 240 - 160 + 480 + 480}{96}\right)$$

$$= \pi\frac{225}{32} = 22{,}089 \text{ units}^3$$

4. 4.1 Rotation about the x-axis

Find the points of intersection:

$$\frac{4}{x} = 9 - 2x$$

$$2x^2 - 9x + 4 = 0$$

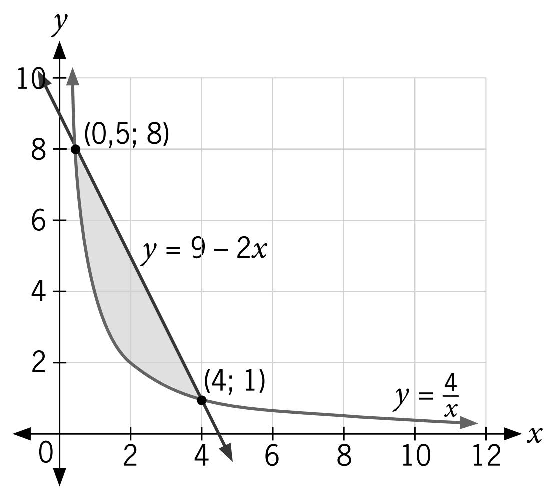
$$(2x - 1)(x - 4) = 0$$

$$\therefore x = 0{,}5 \text{ or } x = 4;\ y = 8 \text{ or } y = 1$$

Points of intersection: (0,5; 8); (4; 1)

$$y = \frac{4}{x} \therefore x = \frac{4}{y}$$

$$y = 9 - 2x \therefore x = \frac{9 - y}{2}$$



$$V_x = 2\pi\int_1^8 y\left[\frac{9 - y}{2} - \frac{4}{y}\right] dy$$

$$= \pi\int_1^8 y\left[\left(9 - y\right) - \left(\frac{8}{y}\right)\right] dy$$

$$= -\pi\int_1^8 y^2 - 9y + 8\ dy$$

$$= -\pi\left[\frac{y^3}{3} - 9\frac{y^2}{2} + 8y\right]_1^8$$

$$= -\pi\left[-\frac{160}{3} - \frac{23}{6}\right]$$

$$= \frac{343}{6}\pi = 57{,}167\pi = 179{,}594 \text{ units}^3$$

4.2 Rotation about the y-axis

We use the same graph and intersection points as for 4.1.

$$V_y = 2\pi\int_{0,5}^{4} x\left[(9 - 2x) - \frac{4}{x}\right] dx$$

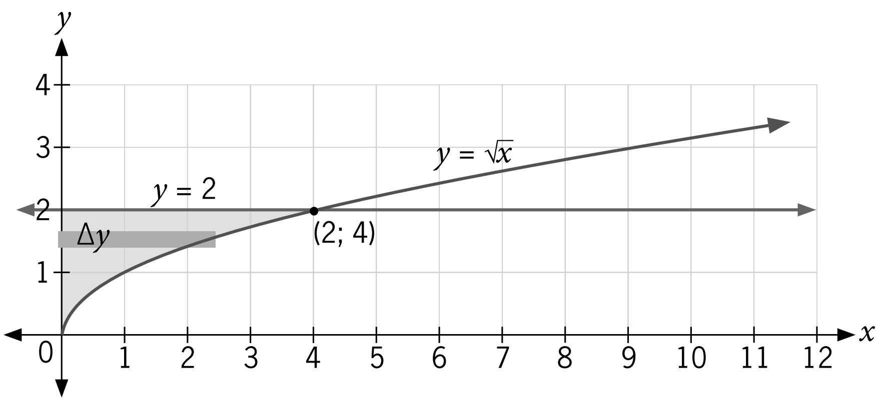
$$= 2\pi\int_{0,5}^{4} -2x^2 + 9x - 4\ dx$$

$$= 2\pi\left[-2\frac{x^3}{3} + 9\frac{x^2}{2} - 4x\right]_{0,5}^{4}$$

$$= 2\pi\left[\frac{40}{3} - (-0,958)\right]$$

$$= 28,583\pi = 89,796 \text{ units}^3$$

5. Rotation about the x-axis



The area is bounded by $y = 0$ and $y = 2$, these are the limits of the integral.

$y = \sqrt{x} \therefore x = y^2$

$$V_x = 2\pi\int_{0}^{2} y(x)\ dy$$

$$= 2\pi\int_{0}^{2} y\, y^2\ dy$$

$$= 2\pi\int_{0}^{2} y^3\ dy$$

$$= 2\pi\left[\frac{y^4}{4}\right]_{0}^{2}$$

$$= 2\pi[4]$$

$$= 8\pi = 25,133 \text{ units}^3$$

Summative assessment: Module 5 SB page 154

1. 1.1 $f(x) = x^2 - 3x - 2$ between $-2 \leq x \leq 5$

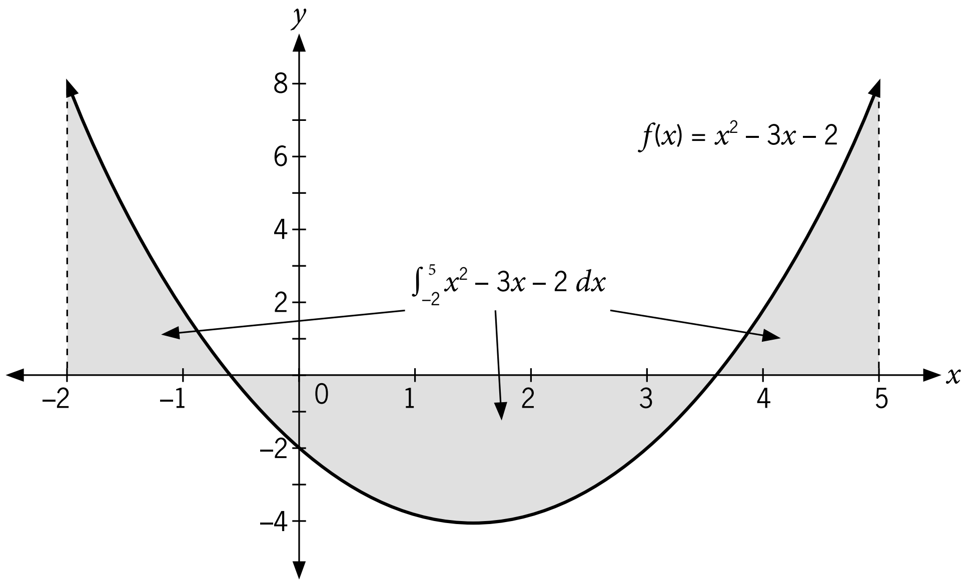
Limits:

$A_x = \int_{-2}^{5} x^2 - 3x - 2\, dx$

$= \frac{1}{3}x^3 - \frac{3}{2}x^2 - 2x \Big|_{-2}^{5}$

$= \frac{125}{3} - \frac{75}{2} - 10 + \frac{8}{3} + \frac{12}{2} - 4$

$= -\frac{7}{6} \approx 1{,}167 \text{ units}^2$



(5)

1.2 Limits:

$x^2 = x$

$\therefore x(x - 1) = 0$

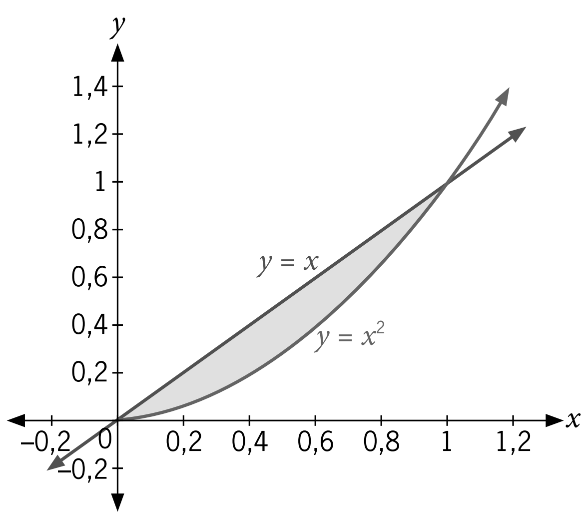
$\Rightarrow x = 0$ and $x = 1$

$A = \int_{0}^{1} x - x^2\, dx$

$= \frac{x^2}{2} - \frac{x^3}{3} \Big|_{0}^{1}$

$= \frac{1}{2} - \frac{1}{3} - 0$

$= \frac{1}{6} \approx 0{,}167 \text{ units}^2$



(5)

1.3 Limits:

$\frac{x^2}{4} = 1$

$x^2 = 4$

$\therefore x = \pm 2$

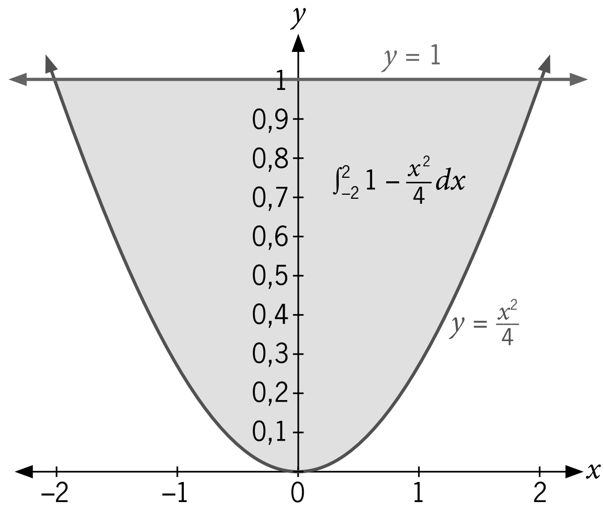
Area:

$A = \int_{-2}^{2} 1 - \frac{x^2}{4} dx$

$= \left(x - \frac{x^3}{12}\right)\Big|_{-2}^{2}$

$= 2 - \frac{8}{12} + 2 - \frac{8}{12}$

$= 2\frac{2}{3} \approx 2{,}667 \text{ units}^2$



(5)

1.4 Limits:

$4x^2 = x^4$

$4 = x^2$

$\therefore x = \pm 2$

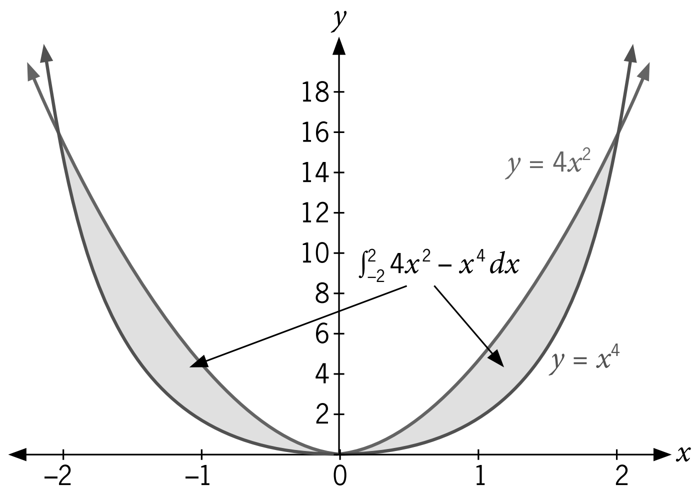
Area:

$A = \int_{-2}^{2} 4x^2 - x^4 \, dx$

$= \frac{4x^3}{3} - \frac{x^5}{5}\Big|_{-2}^{2}$

$= \frac{32}{3} - \frac{32}{5} + \frac{32}{3} - \frac{32}{5}$

$= \frac{128}{15} \approx 8{,}533 \text{ units}^2$



(5)

2. 2.1 Roots:

$-y^2 + 2y + 3 = 0$

$(y + 1)(-y + 3) = 0$

$\therefore y = -1 \text{ and } y = 3$

Volume:

$$V_y = \pi\int_a^b f(y)^2\,dy$$

$$= \pi\int_{-1}^{3}\left(-y^2 + 2y + 3\right)^2 dy$$

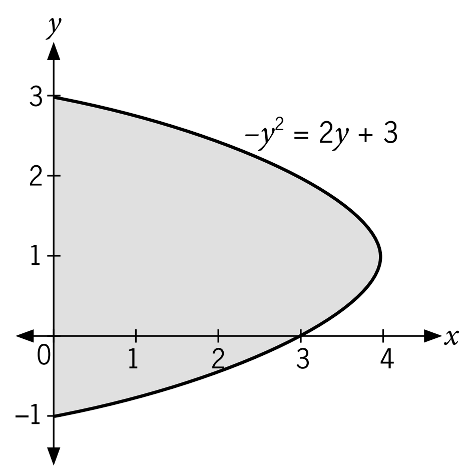
$$= \pi\int_{-1}^{3}\left(y^4 - 2y^3 - 3y^2 - 2y^3 + 4y^2 + 6y - 3y^2 + 6y + 9\right) dy$$

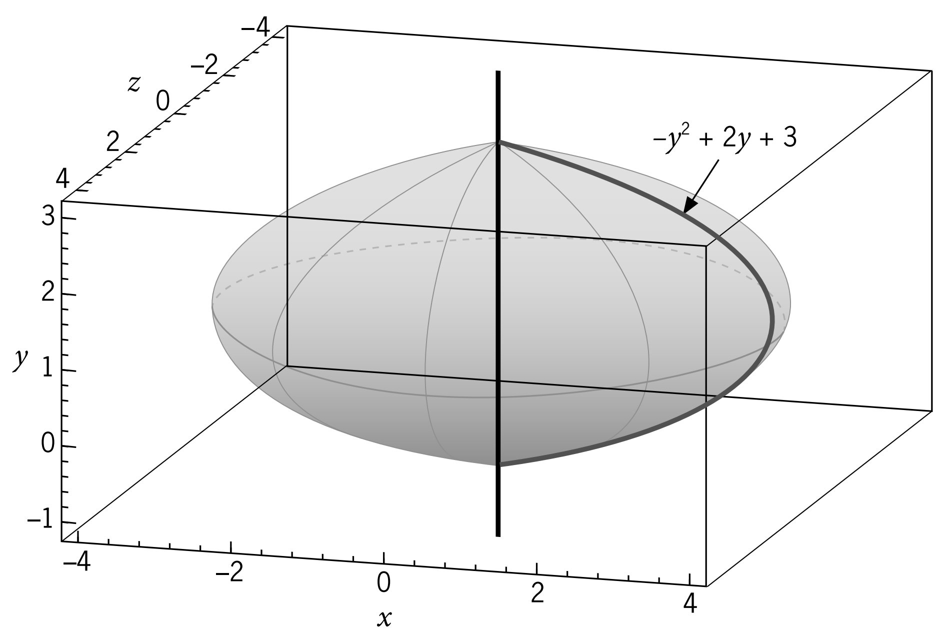
$$= \pi\int_{-1}^{3}\left(y^4 - 4y^3 - 2y^2 + 12y + 9\right) dy$$

$$= \pi\left(\frac{y^5}{5} - \frac{4y^4}{4} - \frac{2y^3}{3} + \frac{12y^2}{2} + 9y\right)\Bigg|_{-1}^{3}$$

$$= \pi\left[\left(\frac{243}{5} - 81 - 18 + 54 + 27\right) - \left(-\frac{1}{5} - 1 + \frac{2}{3} + 6 - 9\right)\right]$$

$$= \pi.\frac{512}{15} = 107{,}233 \text{ units}^3$$





(5)

2.2 Limits are given: $0 \leq x \leq 2$

$$V_x = \pi\int_a^b f(x)^2 \, dx$$

$$= \pi\int_0^2 (x + 1)^2 \, dx$$

$$= \pi\int_0^2 (x^2 + 2x + 1) \, dx$$

$$= \pi\left(\frac{x^3}{3} + x^2 + x\right)\Big|_0^2$$

$$= \pi\left(\frac{8}{3} + 4 + 2\right)$$

$$= \frac{26}{3}\pi \approx 27{,}227 \text{ units}^3$$ (5)

3. 3.1 Rewrite in terms of y:

$$x = \frac{1}{2}y + 1$$

$$\therefore y = 2x - 2$$

Limits:

$y = 0 \Rightarrow x = 1$

$y = 2 \Rightarrow x = 2$

Volume:

$$V_y = 2\pi\int_a^b x\, f(x) \, dx$$

$$= 2\pi\int_1^2 x(2x - 2) \, dx$$

$$= 4\pi\int_1^2 (x^2 - x) \, dx$$

$$= 4\pi\left(\frac{x^3}{3} - \frac{x^2}{2}\right)\Big|_1^2$$

$$= 4\pi\left(\frac{8}{3} - 2 - \frac{1}{3} + \frac{1}{2}\right)$$

$$= \frac{10}{3}\pi \approx 10{,}472 \text{ units}^3$$ (5)

3.2 Given: $y = \frac{1}{2}x^2 + x$

$$\frac{dy}{dx} = x + 1$$

$$\therefore dy = (x + 1) \, dx$$

Volume:

$V_x = 2\pi\int_a^b xy\, dy$

$= 2\pi\int_1^3 x\left(\frac{1}{2}x^2 + x\right)(x + 1)\, dx$

$= \pi\int_1^3 (x^4 + 3x^3 + 2x^2)\, dx$

$= \pi\left(\frac{x^5}{5} + \frac{3x^4}{4} + \frac{2x^3}{3}\right)\Big|_1^3$

$= \pi\left(\frac{243}{5} + \frac{243}{4} + \frac{54}{3} - \frac{1}{5} - \frac{3}{4} - \frac{2}{3}\right)$

$= 395 \text{ units}^3$ (5)

4. 4.1 Limits:

$\frac{1}{3}x^2 = 2x$

$\therefore x = 0$ and $x = 6$

Volume:

$V_x = \pi\int_c^d [f(x)^2 - gx)^2|\, dx$

$V_x = \pi\int_0^6 (2x)^2 - \left(\frac{1}{3}x^2\right)^2 dx$

$= \pi\left(\frac{4}{3}x^3 - \frac{1}{9.5}x^5\right)\Big|_0^6$

$= \pi\left(288 - \frac{7776}{45} - 0\right)$

$= 115{,}2.\pi \approx 361{,}911 \text{ units}^3$ (5)

4.2. Limits:

$y^2 + 4 = 8$

$\therefore y = \pm 2$

Volume:

$V_y = \pi\int_c^d [f(y)^2 - gy)^2|\, dy$

$= \pi\int_{-2}^2 8^2 - \left(4 + y^2\right)^2 dy$

$= \pi\int_{-2}^2 \left(64 - 16 - 8y^2 - y^4\right) dy$

$= \pi\left(48y - \frac{8}{3}y^3 - \frac{1}{5}y^5\right)\Big|_{-2}^2$

$= \pi\left(48.2 - \frac{8}{3}.8 - \frac{32}{5} + 48.2 - \frac{8}{3}.8 - \frac{32}{5}\right)$

$= \pi\left(192 - \frac{128}{3} - \frac{64}{5}\right) \approx 428{,}932 \text{ units}^3$ (5)

TOTAL: [50]

MODULE

6 *Centroids and centre of gravity*

After they have completed this module, students should be able to:

- calculate the centroid and centre of gravity of one or two given functions;
- calculate the distance from any of the reference axes to the centroid of the area:
  - between a given curve and an axis;
  - between two given curves; and
- calculate the distance from a reference axis to the centre of gravity of a solid of revolution generated when:
  - the area between a given curve and an axis is rotated about a reference axis;
  - the area between two given curves is rotated about a reference axis.

When answering questions, they should be able to:

- sketch functions on a given interval;
- calculate and sketch the points of intersection of two functions; and
- calculate the centroid and centre of gravity of one or two given functions.

Introduction

We have previously covered a few integration applications throughout the course. This module will add new ones such as centroid and centre of gravity. These concepts were briefly covered in N5 Mathematics but will now be explored in more detail. Students will learn how to perform calculations with the centroid and centre of gravity.

Students need the following pre-knowledge to successfully complete this module.

Pre-knowledge

Students should already know how to:

- Use different strategies to solve integrals.
- Find the roots and intersection points of functions.
- Sketch a function on a given interval.
- Sketch the points of intersection of areas and volumes under two functions.
- Calculate the areas and volumes between two functions.

Activity 6.1 **SB page 171**

1. 1.1 $\bar{x} = \dfrac{\int_{x_1}^{x_2} x.f(x)\,dx}{\int_{x_1}^{x_2} f(x)\,dx}$

$= \dfrac{\int_0^4 x.x^2\,dx}{\int_0^4 x^2\,dx}$

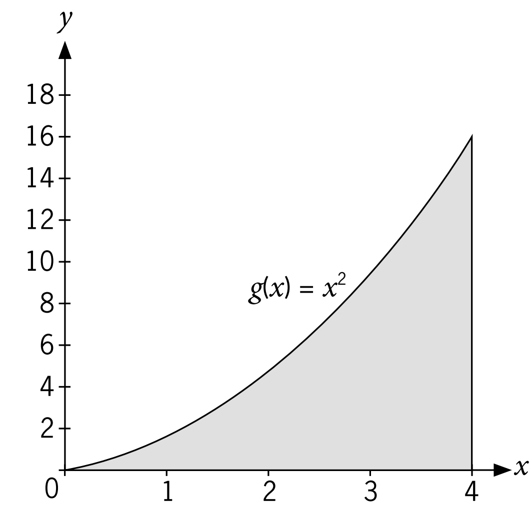
$= \dfrac{\frac{1}{4}x^4}{\frac{1}{3}x^3}\Bigg|_0^4 = \dfrac{3}{4}x\Big|_0^4 = 3$

$\bar{y} = \dfrac{\int_{x_1}^{x_2} \frac{1}{2}(f(x))^2\,dx}{\int_{x_1}^{x_2} f(x)\,dx}$

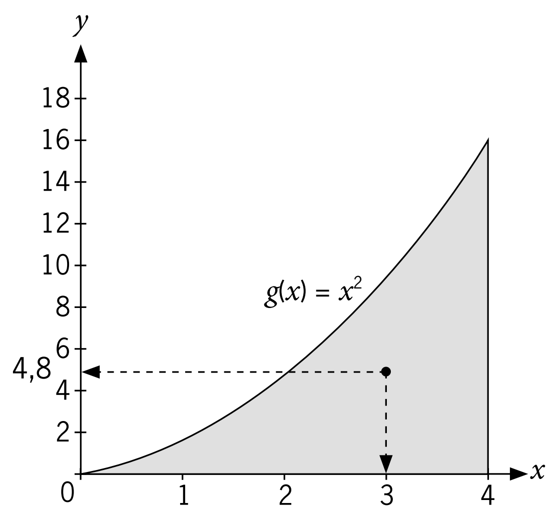
$= \dfrac{\int_0^4 \frac{1}{2}x^4\,dx}{\int_0^4 x^2\,dx}$

$= \dfrac{1}{2}.\dfrac{\frac{1}{5}x^5}{\frac{1}{3}x^3}\Bigg|_0^4 = \dfrac{3}{10}x^2\Big|_0^4 = \dfrac{48}{10} = 4,8$

$\therefore (\bar{x}; \bar{y}) = (3; 4,8)$



Shape bounded by a function and an axis



Centroid of the area beneath the function

1.2 For a circle:

$r^2 = x^2 + y^2 \therefore y^2 = r^2 - x^2$

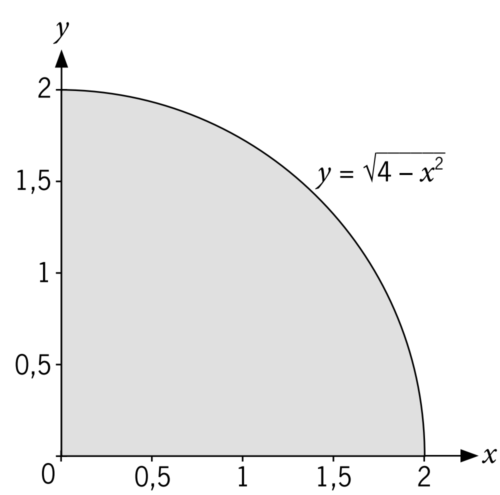
$= 2^2 - x^2$

$= 4 - x^2$

$y = \sqrt{4 - x^2} = f(x)$

$\underline{x} = \dfrac{\int_{x_1}^{x_2} x.f(x)\,dx}{\int_{x_1}^{x_2} f(x)\,dx}$

$= \dfrac{\int_0^2 x.\sqrt{4 - x^2}\,dx}{\int_0^2 \sqrt{4 - x^2}\,dx}$



A quarter circle

It is hard to calculate these integrals. However:

$\int_0^2 \sqrt{4 - x^2}\ dx = \int_{x_1}^{x_2} f(x)\ dx = \int_{x_1}^{x_2} dA = A$

But we know the area of a circle is:

$A = \pi r^2$

Therefore, for this $\frac{1}{4}$ circle with $r = 2$:

$\int_{x_1}^{x_2} dA = \frac{\pi.2^2}{4} = \pi$

We also know that due to symmetry for this shape: $\bar{x} = \bar{y}$

Therefore, let us try the $\bar{y}$ formula since it will get rid of the square root integration:

$$\bar{y} = \frac{\int_{x_1}^{x_2} \frac{1}{2}(f(x))^2\ dx}{\int_{x_1}^{x_2} f(x)\ dx}$$

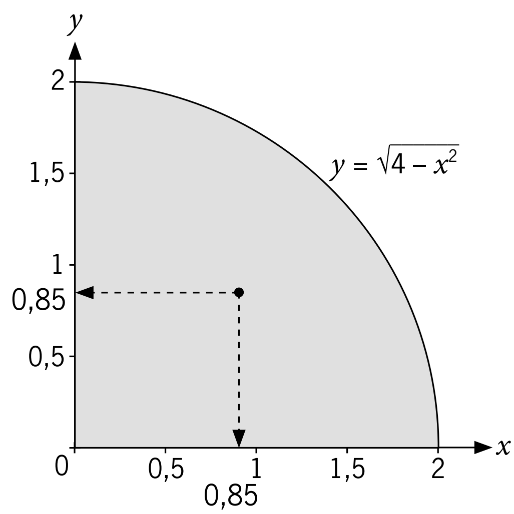
$$= \frac{1}{2}.\frac{\int_0^2 (4 - x^2)\ dx}{\pi}$$

$$= \frac{1}{2\pi}\int_0^2 (4 - x^2)\ dx$$

$$= \frac{1}{2\pi}\left(4x - \frac{1}{3}x^3\right)\Big|_0^2$$

$$= \frac{1}{2\pi}\left(8 - \frac{8}{3}\right) = \frac{8}{3\pi} \approx 0{,}849$$

$$\therefore (\bar{x}; \bar{y}) = \left(\frac{8}{3\pi}; \frac{8}{3\pi}\right)$$



Centroid of a quarter circle

1.3 (Remember that radians are used for the trigonometric calculations.)

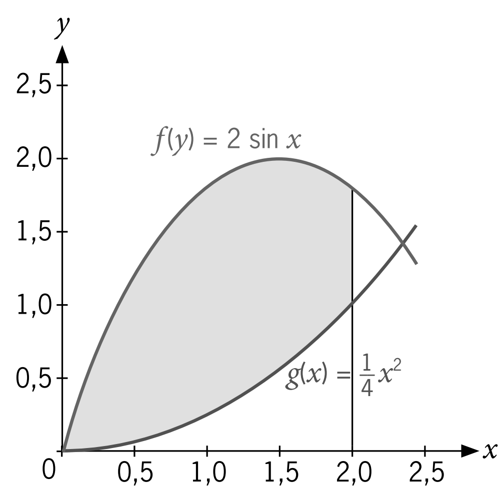
$$\bar{x} = \frac{\int_{x_1}^{x_2} x(f(x) - g(x))\ dx}{\int_{x_1}^{x_2} (f(x) - g(x))\ dx}$$

$$= \frac{\int_0^2 x\left(2\sin x - \frac{1}{4}x^2\right)\ dx}{\int_0^2 \left(2\sin x - \frac{1}{4}x^2\right)\ dx}$$

$$= \frac{-2x\cos x + 2\sin x - \frac{1}{4.4}x^4\Big|_0^2}{-2\cos x - \frac{1}{4.3}x^3\Big|_0^2}$$

$$= \frac{-4\cos 2 + 2\sin 2 - 1 + 0\cos 0 - 2\sin 0}{-2\cos 2 - \frac{2}{3} + 2\cos 0}$$

$$= \frac{1{,}66 + 1{,}82 - 1 + 0 - 0}{0{,}83 - 0{,}67 + 2} \approx 1{,}148$$



A shape bounded by two curves

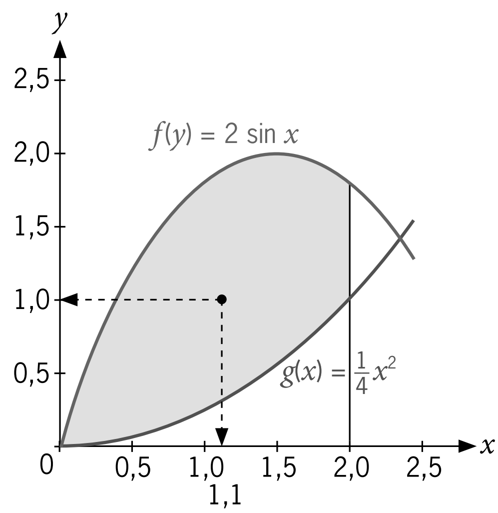
$$\bar{y} = \frac{\int_{x_1}^{x_2} \frac{1}{2}\left([f(x)]^2 - [g(x)]^2\right) dx}{\int_{x_1}^{x_2} (f(x) - g(x))\, dx}$$

$$= \frac{\int_0^2 \frac{1}{2}\left([2 \sin x]^2 - \left[\frac{1}{4}x^2\right]^2\right) dx}{\int_0^2 \left(2 \sin x - \frac{1}{4}x^2\right) dx}$$

$$= \frac{1}{2}.\frac{2x - \sin 2x - \frac{1}{80}x^5}{-2 \cos x - \frac{1}{12}x^3}\Bigg|_0^2$$

$$= \frac{1}{2}.\frac{4 - \sin 4 - \frac{2}{5} - \sin 0}{-2 \cos 2 - \frac{8}{12} + 2 \cos 0} \approx 1{,}006$$

Therefore the centroid of the shape is:
$(\bar{x}; \bar{y}) = (1{,}148; 1{,}006)$



Centroid of the area between two curves

2. 2.1 $\bar{x} = \frac{\int_{x_1}^{x_2} x.f(x)\, dx}{\int_{x_1}^{x_2} f(x)\, dx}$

$$= \frac{\int_0^3 x.\frac{1}{2}x\, dx}{\int_0^3 \frac{1}{2}x\, dx}$$

$$= \frac{\frac{1}{2}\int_0^3 x^2\, dx}{\frac{1}{2}\int_0^3 x\, dx}$$

$$= \frac{\left|\frac{1}{2}.\frac{1}{3}x^3\right|_0^3}{\left|\frac{1}{2}.\frac{1}{2}x^2\right|_0^3}$$

$$= \frac{\frac{1}{6}.3^3 - 0}{\frac{1}{4}.3^2 - 0}$$

$$= \frac{\frac{27}{6}}{\frac{9}{4}} = 2 \text{ units}$$

The distance is 2 units from the y-axis.

2.2 $\overline{x} = \dfrac{\int_{x_1}^{x_2} x(f(x) - g(x))\, dx}{\int_{x_1}^{x_2} (f(x) - g(x))\, dx}$

$= \dfrac{\int_0^4 x\left(2x - \frac{1}{2}x^2\right) dx}{\int_0^4 \left(2x - \frac{1}{2}x^2\right) dx}$

$= \dfrac{\int_0^4 \left(2x^2 - \frac{1}{2}x^3\right) dx}{\int_0^4 \left(2x - \frac{1}{2}x^2\right) dx}$

$= \dfrac{\left[2\frac{x^3}{3} - \frac{1}{2}\left(\frac{x^4}{4}\right)\right]_0^4}{\left[2\frac{x^2}{2} - \frac{1}{2}\left(\frac{x^3}{3}\right)\right]_0^4}$

$= \dfrac{\left(2\frac{4^3}{3} - \frac{1}{2}\left(\frac{4^4}{4}\right)\right) - (0 - 0)}{\left(2\frac{4^2}{2} - \frac{1}{2}\left(\frac{4^3}{3}\right)\right) - (0 - 0)}$

$= \dfrac{(6)}{(3)}$

$= 2$ units

The distance is 2 units from the y-axis.

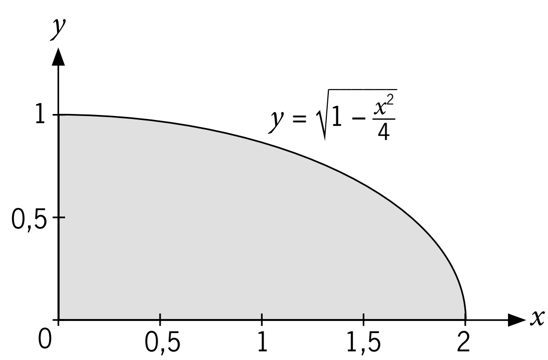
Activity 6.2 SB page 174

1. The y- and z-centres of gravity are at the origins of these two axes, so only the x centroid must be calculated.

Rewrite $\frac{x^2}{4} + y^2 = 1$ in terms of y:

$y = \sqrt{1 - \frac{x^2}{4}}$

We sketch this function and find the limits.



$\overline{x} = \dfrac{\int_{x_1}^{x_2} x\, dV}{\int_{x_1}^{x_2} dV}$

For the disk:

$dV = \pi y^2\, dx = \pi\left(1 - \frac{x^2}{4}\right) dx$

For $0 \le x \le 2$

$$\therefore \overline{x} = \frac{\int_0^2 x.\pi\left(1 - \frac{x^2}{4}\right) dx}{\int_0^2 \pi\left(1 - \frac{x^2}{4}\right) dx}$$

$$= \frac{\int_0^2 \left(x - \frac{x^3}{4}\right) dx}{\int_0^2 \left(1 - \frac{x^2}{4}\right) dx}$$

$$= \left.\frac{\frac{1}{2}x^2 - \frac{x^4}{16}}{x - \frac{x^3}{12}}\right|_0^2$$

$$= \frac{\frac{4}{2} - \frac{16}{16} - 0}{2 - \frac{8}{12} - 0}$$

$$= \frac{3}{4}$$

So, the centre of gravity is at $\left(\frac{3}{4}; 0; 0\right)$

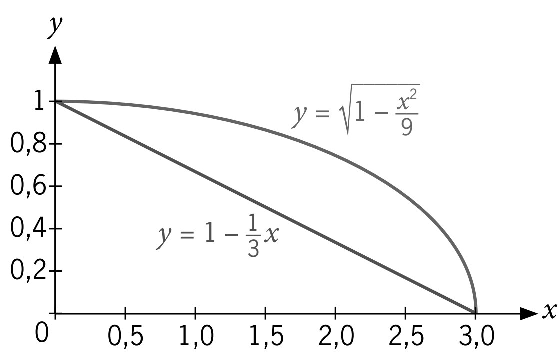
2. Both $\overline{y}$ and $\overline{z}$ are at the origins of their axes, therefore only $\overline{x}$ needs to be calculated.

$$\frac{x^2}{9} + y^2 = 1$$

$$\therefore y = f(x) = \sqrt{1 - \frac{x^2}{9}}$$

$$x + 3y = 3$$

$$\therefore y = g(x) = 1 - \frac{x}{3}$$



$$\overline{x} = \frac{\int_a^b x.\left[f(x)^2 - g(x)^2\right] dx}{\int_a^b \left[f(x)^2 - g(x)^2\right] dx}$$

$$= \frac{\int_0^3 x.\left[\left(1 - \frac{x^2}{9}\right) - \left(1 - \frac{x}{3}\right)^2\right] dx}{\int_0^3 \left[\left(1 - \frac{x^2}{9}\right) - \left(1 - \frac{x}{3}\right)^2\right] dx}$$

$$= \frac{\int_0^3 x.\left[1 - \frac{x^2}{9} - 1 + \frac{2x}{3} - \frac{x^2}{9}\right] dx}{\int_0^3 \left[1 - \frac{x^2}{9} - 1 + \frac{2x}{3} - \frac{x^2}{9}\right] dx}$$

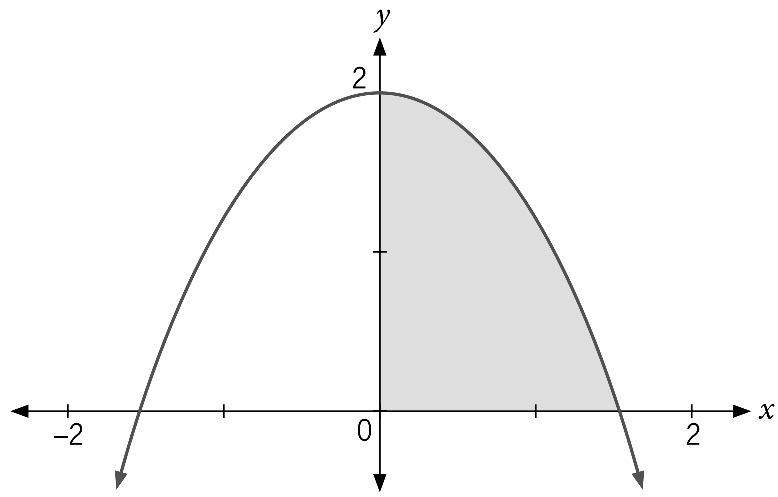
$$= \frac{\int_0^3 \left[-\frac{2x^3}{9} + \frac{2x^2}{3}\right] dx}{\int_0^3 \left[-\frac{2x^2}{9} + \frac{2x}{3}\right] dx}$$

$$= \left.\frac{-\frac{2x^4}{4.9} + \frac{2x^3}{3.3}}{-\frac{2x^3}{3.9} + \frac{2x^2}{2.3}}\right|_0^3$$

$$= \frac{-\frac{9}{2} + 6 - 0}{-2 + 3}$$

$= \frac{3}{2} = 1{,}5$ So, the centre of gravity is at (1,5; 0; 0)

3. Sketch the function to find the limits.



By inspection the limits are $y = 0$ and $y = 2$.

Rewrite $y = 2 - x^2$ in terms of x: $x = \sqrt{2 - y}$

$$\bar{y} = \frac{\int_{y_1}^{y_2} y \, dV}{\int_{y_1}^{y_2} dV}; \; dV = \pi x^2 \, dy$$

$$\bar{y} = \frac{\pi \int_a^b y x^2 \, dy}{\pi \int_a^b x^2 \, dy}$$

$$= \frac{\pi \int_0^2 y(\sqrt{2 - y})^2 \, dy}{\pi \int_0^2 (\sqrt{2 - y})^2 \, dy}$$

$$= \frac{\int_0^2 y(2 - y) \, dy}{\int_0^2 (2 - y) \, dy}$$

$$= \frac{\int_0^2 2y - y^2 \, dy}{\int_0^2 (2 - y) \, dy}$$

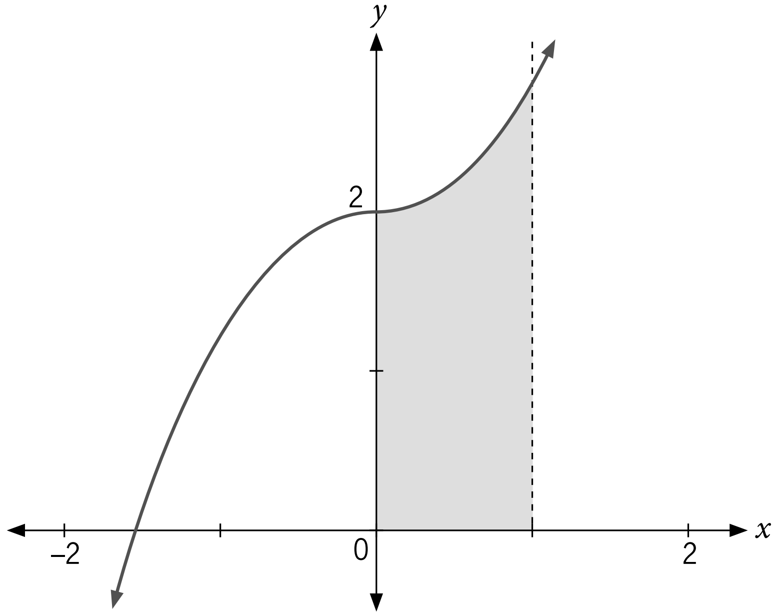
$$= \left[\frac{2y^2}{2} - \frac{y^3}{3}\right]_0^2 \div \left[2y - \frac{y^2}{2}\right]_0^2$$

$$= \left(4 - \frac{8}{3}\right) \div \left(4 - \frac{4}{2}\right)$$

$$= \left(\frac{4}{3}\right) \div (2) = \frac{2}{3}$$

$$\therefore \bar{y} = \frac{2}{3}$$

4. Sketch the functions.



Find the limits by inspection: $x = 0$ and $x = 1$. The shape is at the origin so $\bar{y} = 0$.

$$\bar{x} = \frac{\int_{x_1}^{x_2} x\, dV}{\int_{y_1}^{y_2} dV};\ dV = \pi y^2\, dx$$

$$\bar{x} = \frac{\pi\int_a^b xy^2\, dx}{\pi\int_a^b y^2\, dx}$$

$$= \frac{\pi\int_0^1 x(x^3+2)^2\, dx}{\pi\int_0^1 (x^3+2)^2\, dx}$$

$$= \frac{\int_0^1 x^7 + 4x^4 + 4x\, dx}{\int_0^1 x^6 + 4x^3 + 4\, dx}$$

$$= \left[\frac{x^8}{8} + \frac{4x^5}{5} + \frac{4x^2}{2}\right]_0^1 \div \left[\frac{x^7}{7} + \frac{4x^4}{4} + 4x\right]_0^1$$

$$= \left(\frac{1}{8} + \frac{4}{5} + \frac{4}{2}\right) \div \left(\frac{1}{7} + \frac{4}{4} + 4\right)$$

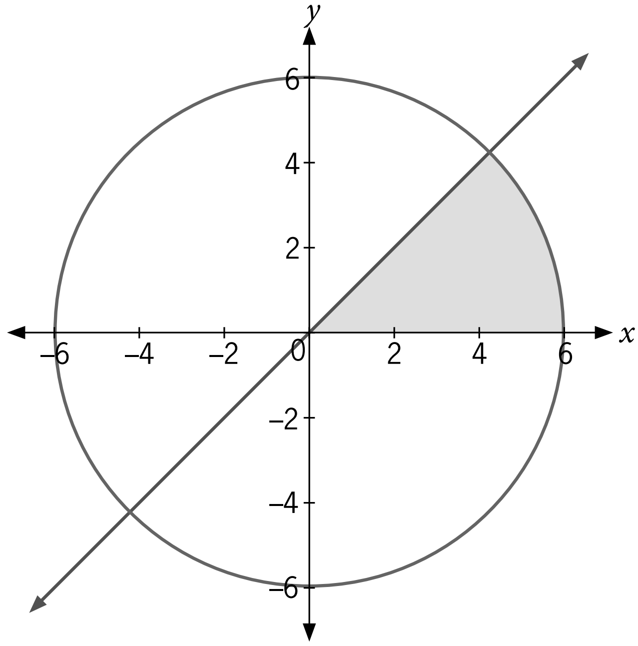
$$= \left(\frac{5 + 32 + 80}{40}\right) \div \left(\frac{1 + 35}{7}\right)$$

$$= \left(\frac{117}{40}\right) \times \left(\frac{7}{36}\right)$$

$$= \frac{91}{160}$$

The centre of gravity is at $\left(\frac{91}{160}; 0\right)$ or (0,569; 0).

5. Sketch the functions.



$$V_x = \pi\int_a^b y_1^2 - y_2^2\, dx$$

$$= \pi\int_0^6 (x)^2 - (36 - x^2)dx$$

$$= \pi\int_0^6 (2x^2 - 36)\, dx$$

$$= \pi\left[\frac{2x^3}{3} - 36x\right]_0^6$$

$$= \pi\left[\frac{2(6)^3}{3} - 36(6)\right]$$

$$= \pi[144 - 216]$$

$$= -72\pi \text{ units}^3$$

$$V_{m-y} = \pi\int_a^b x(y_1^2 - y_2^2)\, dx$$

$$= \pi\int_0^6 x(2x^2 - 36)\, dx$$

$$= \pi\int_0^6 2x^3 - 36x\, dx$$

$$= \pi\left[\frac{2x^4}{4} - \frac{36}{2}x^2\right]_0^6$$

$$= \pi\left[\frac{2(6)^4}{4} - \frac{36}{2}(6)^2\right]$$

$$= \pi[648 - 648]$$

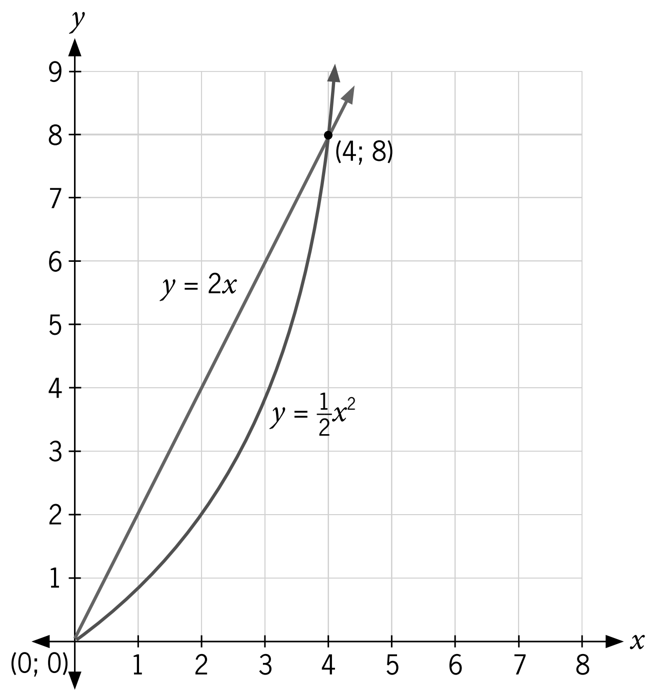
$$= 0 \text{ units}^4$$

$\therefore \overline{x} = 0$

As $\overline{y} = 0$ and $\overline{z} = 0$ we can see that the centre of gravity is at the origin.

Summative assessment: Module 6 — SB page 176

1. 1.1 The two curves plotted on the same axes along with their points of intercept are shown alongside.



Find x by letting the two equations equal each other.

$\frac{1}{2}x^2 = 2x$

$\frac{1}{2}x^2 - 2x = 0$

$x\left(\frac{1}{2}x - 2\right) = 0$

$\therefore x = 0$ or $x = 4$

$y = 2x$

$\therefore$ for $x = 0$, $y = 0$ and for $x = 4$, $y = 8$

The points of intercept are (0; 0) and (4; 8) (4)

1.2 To find the x-coordinate of the centroid:

$$\overline{x} = \frac{\int_{x_1}^{x_2} x(f(x) - g(x))\,dx}{\int_{x_1}^{x_2} (f(x) - g(x))\,dx}$$

$$= \frac{\int_0^4 x\left(2x - \frac{1}{2}x^2\right)dx}{\int_0^4 \left(2x - \frac{1}{2}x^2\right)dx}$$

$$= \frac{\int_0^4 \left(2x^2 - \frac{1}{2}x^3\right)dx}{\int_0^4 \left(2x - \frac{1}{2}x^2\right)dx}$$

$$= \frac{\left[2\frac{x^3}{3} - \frac{1}{2}\frac{x^4}{4}\right]_0^4}{\left[2\frac{x^2}{2} - \frac{1}{2}\frac{x^3}{3}\right]_0^4}$$

$$= 2$$

To find the *y*-coordinate of the centroid:

$$\bar{y} = \frac{\int_{x_1}^{x_2} \frac{1}{2}\left([f(x)]^2 - [g(x)]^2\right) dx}{\int_{x_1}^{x_2} (f(x) - g(x))\, dx}$$

$$= \frac{\int_0^4 \frac{1}{2}\left((2x)^2 - \left(\frac{1}{2}x^2\right)^2\right) dx}{\int_0^4 \left(2x - \frac{1}{2}x^2\right) dx}$$

$$= \frac{\int_0^4 \left(2x^2 - \frac{1}{8}x^4\right) dx}{\int_0^4 \left(2x - \frac{1}{2}x^2\right) dx}$$

$$= \frac{\left[2\frac{x^3}{3} - \frac{1}{8}\frac{x^5}{5}\right]_0^4}{\left[2\frac{x^2}{2} - \frac{1}{2}\frac{x^3}{3}\right]_0^4}$$

$$= \frac{\left(2\frac{4^3}{3} - \frac{1}{8}\left(\frac{4^5}{5}\right)\right) - (0 - 0)}{\left(2\frac{4^2}{2} - \frac{1}{2}\left(\frac{4^3}{3}\right)\right) - (0 - 0)}$$

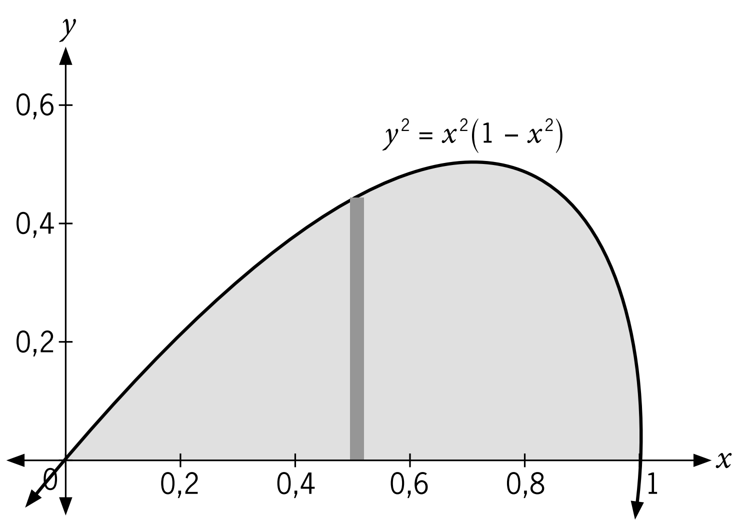
$$= \frac{(16)}{(5)}$$

$$= 3{,}2 \text{ units}$$

Therefore, the coordinates of the centroid are $(\bar{x}; \bar{y}) = (2; 3{,}2)$. (8)

2. 2.1 Find values for sketching:

x	0	0,25	0,5	0,75	1
y	0	0,243	0,433	0,496	0



(2)

2.2 By inspection the limits are $x = 0$ and $x = 1$.

$$V_x = \pi\int_a^b y^2\,dx$$
$$= \pi\int_0^1 x^2(1 - x^2)\,dx$$
$$= \pi\int_0^1 x^2 - x^4\,dx$$
$$= \pi\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1$$
$$= \pi\left(\frac{1}{3} - \frac{1}{5}\right)$$
$$= \frac{2}{15}\pi = 0{,}419 \text{ units}^3 \qquad (3)$$

2.3 The shape is at the origin and revolved about the x-axis, therefore we only need to find $\bar{x}$.

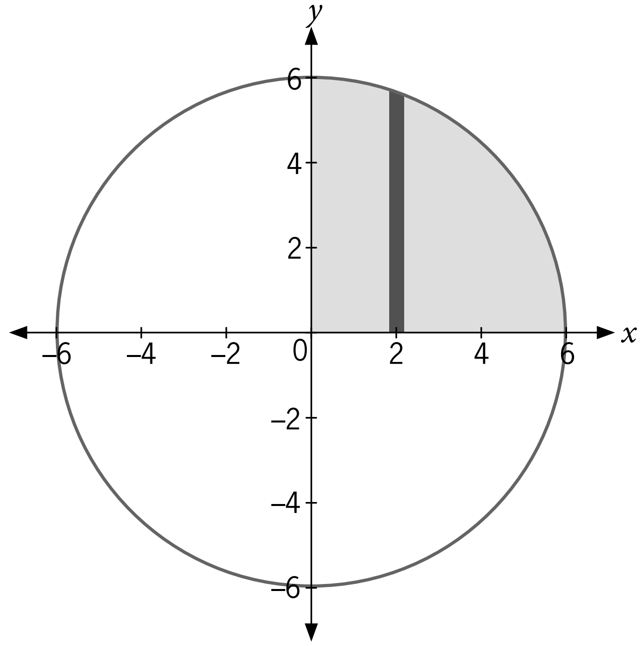
$$\bar{x} = \frac{\int_{x_1}^{x_2} x\,dV}{\int_{x_1}^{x_2} dV}$$

$$\int_{x_1}^{x_2} x\,dV = \pi\int_a^b x(y^2)\,dx$$
$$= \pi\int_0^1 x\left[x^2(1 - x^2)\right]\,dx$$
$$= \pi\int_0^1 x^3 - x^5\,dx$$
$$= \pi\left[\frac{x^4}{4} - \frac{x^6}{6}\right]_0^1$$
$$= \pi\left(\frac{1}{4} - \frac{1}{6}\right)$$
$$= \frac{1}{12}\pi = 0{,}262 \text{ units}^4$$

$$\therefore \bar{x} = \frac{\frac{1}{12}\pi}{\frac{2}{15}\pi} \text{ or } \frac{0{,}262}{0{,}419} = \frac{5}{8}$$

So the centroid is at $\left(\frac{5}{8}; 0; 0\right)$ (5)

3. 3.1 Sketch the graph and show the strip.



(2)

3.2 $V_x = \pi\int_a^b y^2\,dx$

$= \pi\int_0^6 36 - x^2\,dx$

$= \pi\left[36x - \frac{x^3}{3}\right]_0^6$

$= \pi\left[36(6) - \frac{(6)^3}{3}\right]$

$= 144\pi$ or $452{,}390$ units^3 (4)

3.3 $V_{m-y} = \pi\int_a^b x(y^2)\,dx$

$= \pi\int_0^6 x(36 - x^2)\,dx$

$= \pi\int_0^6 36x - x^3\,dx$

$= \pi\left[\frac{36}{2}x^2 - \frac{x^4}{4}\right]_0^6$

$= 324\pi = 1\ 017{,}876$ units^4

$\therefore \bar{x} = \frac{324\pi}{144\pi} = \frac{9}{4} = 2{,}25$ (4)

TOTAL: [32]

MODULE

7 Second moment of area, moment of inertia and centre of fluid pressure

After they have completed this module, students should be able to:

- calculate the second moment of area, with respect to a reference axis, of:
  - an area between a given curve and an axis;
  - an area between two given curves;
- calculate the moment of inertia of a solid of revolution generated when the area between:
  - a given curve and an axis is rotated about an axis;
  - two given curves is rotated about an axis; and
- calculate the depth of the centre of fluid pressure on a vertical plane submerged in the fluid with respect to the surface fluid.

When answering questions, you should be able to:

- sketch functions on a given interval;
- calculate and sketch the points of intersection of two functions; and
- calculate the second moment of area, the moment of inertia and the centre of fluid pressure of one or two given functions.

Introduction

In Module 6 we calculated the first moment of area (centroid) and first moment of mass (centre of gravity). In this module we will calculate the second moment of area and second moment of mass (moment of inertia). The techniques in N5 Mathematics applied to rectangular or circular lamina. These techniques will now be expanded to include areas or solids of revolution formed between a curve and an axis, or between two curves. Students will also learn to determine the centre of pressure in a fluid. These three calculations are very similar.

Students need the following pre-knowledge to successfully complete this module.

Pre-knowledge

Students should already know how to use integration to calculate several related quantities including:

- The second moment of area and moment of inertia for rectangular and circular lamina.
- The first moment of area, or the first moment of mass for a solid of revolution, between a curve and an axis, or between two curves.

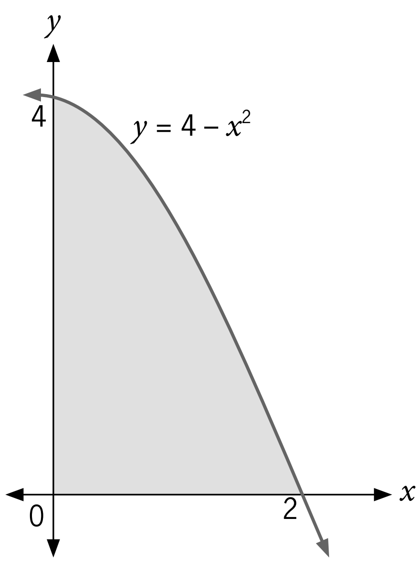
  $A_m = \int r^2\, dA$

  $V_m = \int r\, dV$
- The volume of a solid of revolution, particularly using the shell method:

  $V = \int dV = 2\pi \int r\, dA$

Activity 7.1 — SB page 200

1.



1.1 About the x-axis, $I_x = \int_c^d y^2\, dA$ and $dA = x\, dy$, with $c = 0$ and $d = 4$.

Substituting the y's:

$dy = -2x\, dx$. When $c = 0$ then $a = 2$. When $d = 4$ then $b = 0$.

$\therefore I_x = \int_c^d y^2 x\, dy$

$= \int_a^b (4 - x^2)^2\, x(-2x\, dx)$

$= -2\int_2^0 (16x^2 - 8x^4 + x^6)\, dx$

$= -2\left[\frac{16}{3}x^3 - \frac{8}{5}x^5 + \frac{1}{7}x^7\right]_2^0$

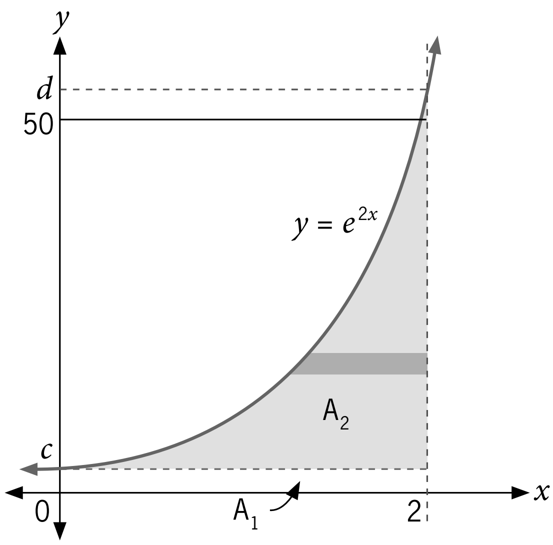
$= 19{,}505 \text{ m}^4$

1.2 About the y-axis, $I_y = \int_a^b x^2\, dA$ and $dA = y\, dx$, with $a = 0$ and $b = 2$.

Therefore

$$I_y = \int_0^2 x^2 y\, dx$$
$$= \int_0^2 x^2(4 - x^2)\, dx$$
$$= \left[\frac{4}{3}x^3 - \frac{1}{5}x^5\right]_0^2$$
$$= 4{,}267 \text{ m}^4$$

2.



About the x-axis, $c = e^{2(0)} = 1$ and $d = e^{2(2)} = e^4$.
Therefore, there are two sections: $I_x = I_{x_1} + I_{x_2}$.
For A_1: the area $dA = 2\, dy$, and:

$$I_{x_1} = \int_0^c y^2\, dA = 2\int_0^1 y^2\, dy = 2\left[\frac{1}{3}y^3\right]_0^1 = 0{,}667 \text{ units}^4$$

For A_2: the area $dA = (2 - x)\, dy$, and:

$$I_{x_2} = \int_c^d y^2\, dA = \int_c^d y^2(2 - x)\, dy$$

Substituting the y's:
$dy = 2e^{2x}\, dx$. When $c = 1$ then $a = 0$. When $d = e^4$ then $b = 2$.

$$I_{x_2} = \int_a^b \left(e^{2x}\right)^2(2 - x)(2e^{2x}\, dx)$$
$$= \int_0^2 \left(4e^{6x} - 2xe^{6x}\right) dx$$
$$= \left[\frac{4}{6}e^{6x}\right]_0^2 - 2\int_0^2 xe^{6x}\, dx$$

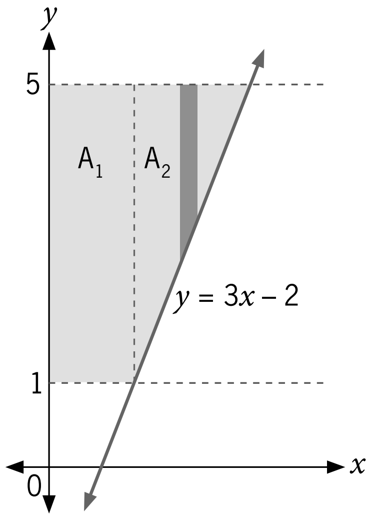
Integrate by parts, where $f = x$ and $g' = e^{6x}$, then $f' = 1$ and $g = \frac{1}{6}e^{6x}$:

$$I_{x_2} = \left[\frac{4}{6}e^{6x}\right]_0^2 - 2\left(\left[x.\frac{1}{6}e^{6x}\right]_0^2 - \int_0^2 \frac{1}{6}e^{6x}.1\, dx\right)$$
$$= \left[\frac{4}{6}e^{6x} - \frac{2}{6}xe^{6x} + \frac{2}{36}e^{6x}\right]_0^2$$
$$= \frac{1}{3}\left[e^{6x}\left(\frac{13}{6} - x\right)\right]_0^2$$

$= 9\ 041{,}933 - 0{,}722$
$= 9\ 041{,}211 \text{ units}^4$

Therefore,
$I_x = I_{x_1} + I_{x_2} = 9\ 041{,}878 \text{ units}^4$

3.



About the y-axis, there are two sections: $I_y = I_{y1} + I_{y2}$.

For A_1: $I_{y1} = \int_0^a x^2\, dA$ where $dA = (5 - 1)\, dx$. When $c = 1$ then $a = \frac{c+2}{3} = 1$:

$$I_{y1} = 4\int_0^1 x^2\, dx$$
$$= 4\left[\frac{1}{3}x^3\right]_0^1$$
$$= 1{,}333 \text{ units}^4$$

For A_2: $I_{y2} = \int_a^b x^2\, dA$ where $dA = (5 - y)\, dx$. When $d = 5$ then $b = \frac{d+2}{3} = \frac{7}{3}$:

$$I_{y2} = \int_a^b x^2(5 - y)\, dx$$
$$= \int_1^{\frac{7}{3}} x^2(5 - (3x - 2))\, dx$$
$$= \int_1^{\frac{7}{3}} (7x^2 - 3x^3)\, dx$$
$$= \left[\frac{7}{3}x^3 - \frac{3}{4}x^4\right]_1^{\frac{7}{3}}$$
$$= 7{,}410 - 1{,}583$$
$$= 5{,}827 \text{ units}^4$$

Therefore,
$I_y = I_{y1} + I_{y2} = 7{,}160 \text{ units}^4$

4. 4.1 Given $y = 2x^2$ and $x = \frac{y}{4}$. Rewrite in terms of y: $y = 4x$

Set the equations equal to find the intersections: $2x^2 = 4x$

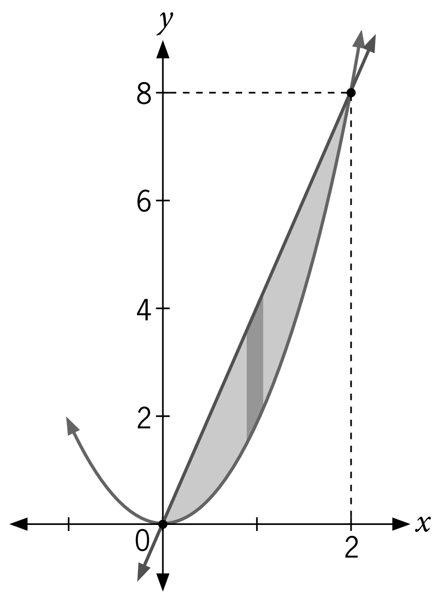
Solve for x: $2x^2 - 4x = 0$

$2x(x - 2) = 0$

$\therefore x = 0$ or $x = 2$ and $y = 0$ or $y = 8$

The points of intersection are (0; 0) and (2; 8).

Rough sketch:



4.2 $dA = (y_2 - y_1)\,dx$

$A = \int_a^b (y_2 - y_1)\,dx$

$= \int_0^2 (4x - 2x^2)\,dx$

$= \left[\frac{4x^2}{2} - \frac{2x^3}{3}\right]_0^2$

$= \frac{4(2)^2}{2} - \frac{2(2)^3}{3}$

$= 8 - \frac{16}{3}$

$= \frac{8}{3}$

$\therefore A = 2{,}667$ units^2

4.3 $I_y = \int_a^b x^2(y_2 - y_1)\, dx$

$= \int_0^2 x^2(4x - 2x^2)\, dx$

$= \int_0^2 4x^3 - 2x^4\, dx$

$= \left[\frac{4x^4}{4} - \frac{2x^5}{5}\right]_0^2$

$= (2)^4 - \frac{2(2)^5}{5}$

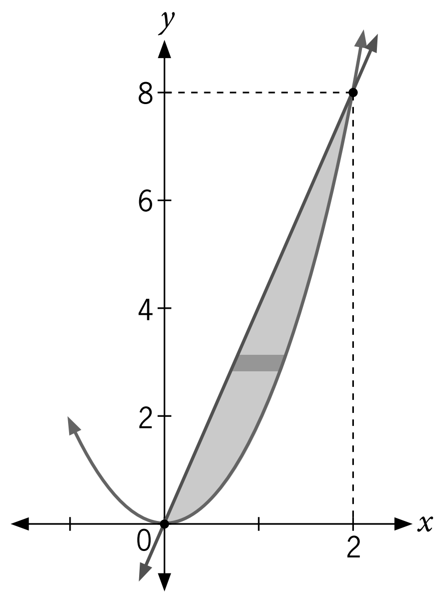
$= 16 - \frac{64}{5}$

$= \frac{16}{5}$

$\therefore I_y = 3{,}2 \text{ units}^4$

5. The points of intersection are (0; 0) and (2; 8).

Rough sketch:



Given $y = 2x^2$ and $x = \frac{y}{4}$. Rewrite in terms of x: $x = \sqrt{\frac{y}{2}}$

$I_x = \int_c^d y^2(x_1 - x_2)\, dy$

$= \int_0^8 y^2\left(\sqrt{\frac{y}{2}} - \frac{y}{4}\right) dy$

$= \int_0^8 \frac{y^{\frac{5}{2}}}{\sqrt{2}} - \frac{y^3}{4}\, dy$

$= \left[\frac{2y^{\frac{7}{2}}}{(7)\sqrt{2}} - \frac{y^4}{16}\right]_0^8$

$= \frac{\sqrt{2}(8)^{\frac{7}{2}}}{7} - \frac{(8)^4}{16}$

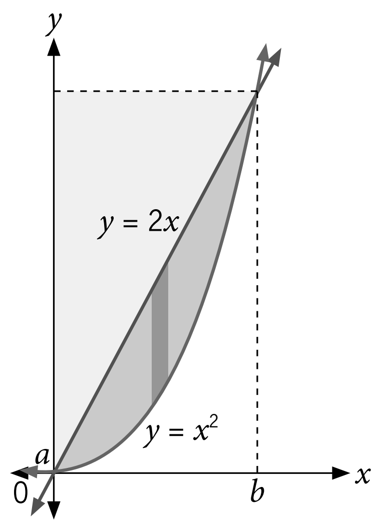
$= 292{,}571 - 256$

$= 36{,}571$

$\therefore I_x = 36{,}571 \text{ units}^4$

Activity 7.2 SB page 206

1. The reference axis is y, so $I = \rho\int_a^b x^2\, dV$. Let f represent the curve $y = 2x$, and g the curve $y = x^2$.



For the solid of revolution:

$dV = 2\pi\, dA = 2\pi(y_f - y_g)\, dx$

The boundary values are the points of intersection, where $f(x) = g(x)$:

$2x = x^2$

$x(x - 2) = 0$

$\therefore x = 0;\ x = 2$

Calculate the volume of the solid of revolution, $V = \int dV$:

$\therefore V = \int_a^b 2\pi(y_f - y_g)\, dx$

$= 2\pi\int_0^2 (2x - x^2)\, dx$

$= 2\pi\left[x^2 - \frac{1}{3}x^3\right]_0^2$

$= 2\pi\left(\frac{4}{3}\right) \text{ units}^3$

Calculate the moment of inertia in terms of density, ρ:

$I = \rho\int_a^b x^2\, dV$

$= 2\pi\rho\int_a^b x^2(y_f - y_g)\, dx$

$= 2\pi\rho\int_0^2 x^2(2x - x^2)\,dx$

$= 2\pi\rho\int_0^2 (2x^3 - x^4)\,dx$

$= 2\pi\rho\left[\frac{1}{2}x^4 - \frac{1}{5}x^5\right]_0^2$

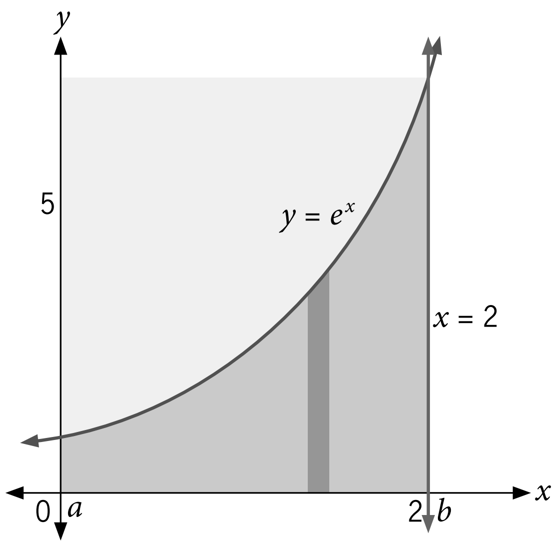
$= 2\pi\rho\left(\frac{8}{5}\right)$ units^4

Make the substitution $\rho = \frac{m}{V}$ to express the moment of inertia in terms of mass, m.

$I = \dfrac{2\pi m\left(\frac{8}{5}\right)}{2\pi\left(\frac{4}{3}\right)} = \frac{6}{5}m$

$= 1{,}2m$ kg.m^2

2. The reference axis is y, so $I = \rho\int_a^b x^2\,dV$.



The boundary values are $a = 0$ and $b = 2$. For the solid of revolution:
$dV = 2\pi\,dA = 2\pi y\,dx$

Calculate the volume of the solid of revolution, $V = \int dV$:

$\therefore V = \int_a^b 2\pi y\,dx$

$= 2\pi\int_0^2 e^x\,dx$

$= 2\pi\left[e^x\right]_0^2$

$= 2\pi(e^2 - 1)$ units^3

Calculate the moment of inertia in terms of density, ρ:

$I = \rho\int_a^b x^2\,dV$

$= 2\pi\rho\int_a^b x^2 y\,dx$

$= 2\pi\rho\int_0^2 x^2 e^x\,dx$

Using integration by parts, with $f = x^2$ and $g' = e^x$, then $f' = 2x$ and $g = e^x$.

$I = 2\pi\rho\left(\left[x^2 e^x\right]_0^2 - \int_0^2 2xe^x\,dx\right)$

Using integration by parts again, with $f = 2x$ and $g' = e^x$, then $f' = 2$ and $g = e^x$.

$$I = 2\pi\rho\left([x^2 e^x]_0^2 - [2xe^x]_0^2 + \int_0^2 2e^x\, dx\right)$$

$$= 2\pi\rho\,[x^2 e^x - 2xe^x + 2e^x]_0^2$$

$$= 2\pi\rho\,[(x^2 - 2x + 2)\,e^x]_0^2$$

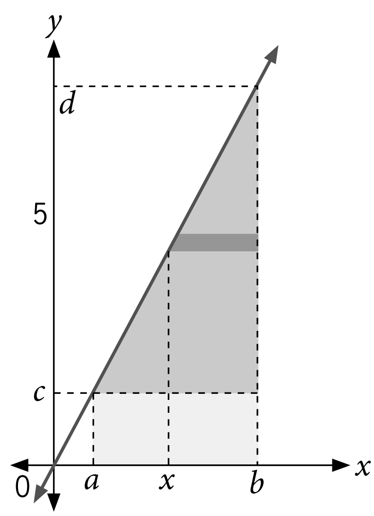
$$= 2\pi\rho(2e^2 - 2) \text{ units}^4$$

Make the substitution $\rho = \frac{m}{V}$ to express the moment of inertia in terms of mass, m.

$$I = \frac{2\pi m(2e^2 - 2)}{2\pi(e^2 - 1)}$$

$$= 2m \text{ kg.m}^2$$

3. The reference axis is x, so $I = \rho\int_c^d y^2\, dV$.



The boundary values are $c = 1$ and $d = 8$. For the solid of revolution:

$dV = 2\pi\, dA = 2\pi x\, dy$

To substitute the y's:

$dy = 2\, dx$

When $c = 1$ then $a = \frac{(1)}{2}$. When $d = 8$ then $b = \frac{(8)}{2} = 4$.

Calculate the volume of the solid of revolution, $V = \int dV$:

$$\therefore V = \int_c^d 2\pi x\, dy$$

$$= 2\pi\int_a^b 2x\, dx$$

$$= 2\pi\,[x^2]_{\frac{1}{2}}^4$$

$$= 2\pi\left(\frac{63}{4}\right) \text{ units}^3$$

Calculate the moment of inertia in terms of density, ρ:

$$I = \rho\int_c^d y^2\,dV$$

$$= 2\pi\rho\int_c^d y^2 x\,dy$$

$$= 2\pi\rho\int_a^b (2x)^2 x(2\,dx)$$

$$= 2\pi\rho\int_{\frac{1}{2}}^4 4x^3\,dx$$

$$= 2\pi\rho\left[x^4\right]_{\frac{1}{2}}^4$$

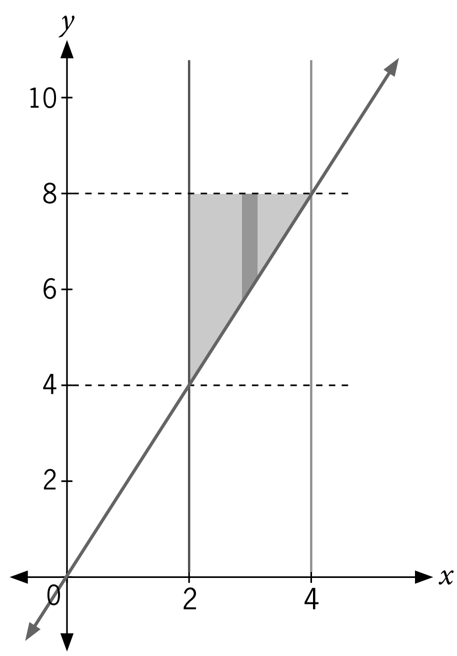
$$= 2\pi\rho\left(\frac{4\,095}{16}\right)\text{ units}^4$$

Make the substitution $\rho = \frac{m}{V}$ to express the moment of inertia in terms of mass, m.

$$I = \frac{2\pi m\left(\frac{4\,095}{16}\right)}{2\pi\left(\frac{63}{4}\right)}$$

$$= \frac{65}{4}m\text{ kg.m}^2$$

4. Rough sketch



The reference axis is y so $I = \rho\int_a^b x^2\,dV$

The boundary values are $a = 2$ and $b = 4$.

$$V = 2\pi\int_a^b y\,dx$$

$$= 2\pi\int_2^4 2x\,dx$$

$$= 2\pi\left[\frac{2x^2}{2}\right]_2^4$$

$$= 2\pi\left[(4)^2 - (2)^2\right]$$

$= 2\pi[12]$

$= 24\pi$ or 75,398 units^3

$I = \rho\int_a^b x^2\, dV$

$= 2\pi\rho\int_a^b x^2 y\, dx$

$= 2\pi\rho\int_2^4 x^2 2x\, dx$

$= 2\pi\rho\int_2^4 2x^3\, dx$

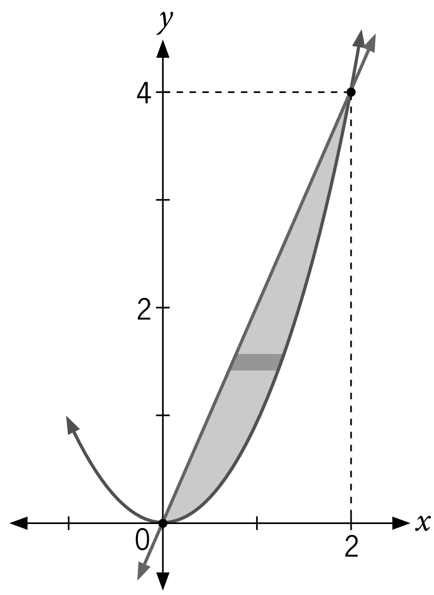
$= 2\pi\rho\left[\frac{2x^4}{4}\right]_2^4$

$= \pi\rho\left(\frac{2(4)^4}{2} - \frac{2(2)^4}{2}\right)$

$= \pi\rho\ (256 - 16)$

$= 240\pi\rho$ units^4

5. Rough sketch:



The reference axis is x so $I = \rho\int_c^d y^2\, dV$

The boundary values are $c = 0$ and $d = 4$.

Rewrite in terms of x: $y = 2x \therefore x = \frac{y}{2}$ and $y = x^2 \therefore x = \sqrt{y}$

$$I = \rho\int_c^d y^2\, dV$$

$$= 2\pi\rho\int_c^d y^2\left(y^{\frac{1}{2}} - \frac{y}{2}\right) dy$$

$$= 2\pi\rho\int_0^4 y^{\frac{5}{2}} - \frac{y^3}{2}\, dy$$

$$= 2\pi\rho\left[\frac{2y^{\frac{7}{2}}}{7} - \frac{y^4}{8}\right]_0^4$$

$$= 2\pi\rho\left(\frac{2(4)^{\frac{7}{2}}}{7} - \frac{(4)^4}{8}\right)$$

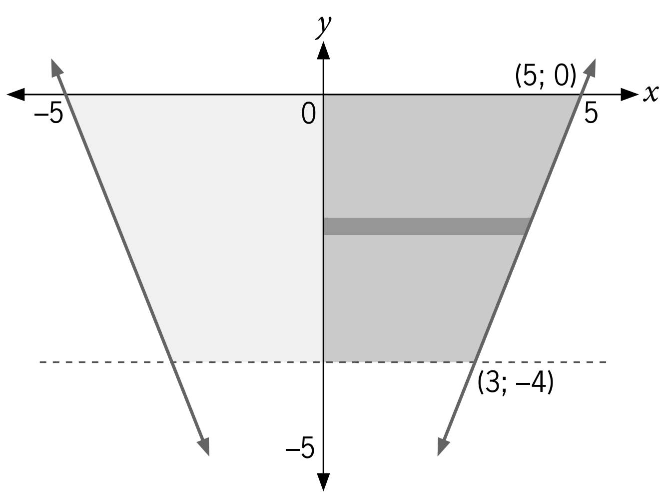
$$= 2\pi\rho\left(\frac{256}{7} - \frac{256}{8}\right)$$

$$= 2\pi\rho\left(\frac{32}{7}\right)$$

$$= \left(\frac{64}{7}\right)\pi\rho \text{ units}^4$$

Activity 7.3 SB page 211

1. The equations bounding the trapezium must be determined.



The boundaries at the surface and bottom are $y = 0$ and $y = -4$.
The coordinates at the righthand boundaries are (5; 0) and (3; −4).
Determine the equation of the function line:

$m = \frac{-4-0}{3-5} = 2$ and:

$y - (0) = 2[x - (5)]$

$\therefore y = 2x - 10$

The left-hand boundary is symmetrical to the right-hand boundary about the y-axis.

You can calculate the equation of the left-side line: (−5; 0) and (−3; −4) means that $m = -2$ and $y = -2x - 10$. Alternatively, use the boundary $x = 0$ and double the calculated values of the right-side.

Using the latter approach, the parallel strip is: $dA = 2x\,dy$

Therefore, the first moment of area is:

$$A_{m,x} = \int_c^d y(2x\,dy)$$

Substitute the x, where $2x = y + 10$:

$$A_{m,x} = \int_{-4}^{0} y(y + 10)\,dy = \left[\frac{1}{3}y^3 + 5y^2\right]_{-4}^{0} = -58{,}667 \text{ m}^3$$

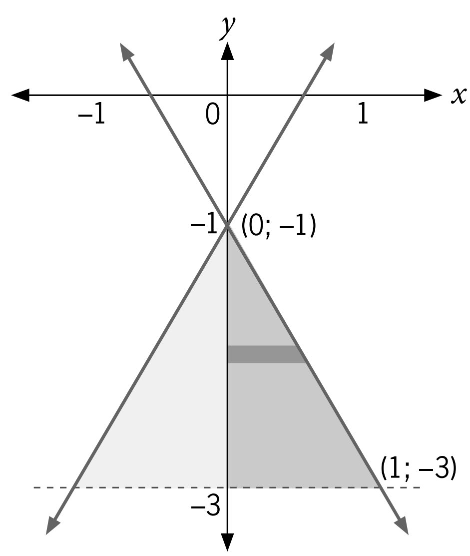
Similarly, the second moment of area is:

$$I_x = \int_c^d y^2(2x\,dy)$$
$$= \int_{-4}^{0} y^2(y + 10)\,dy$$
$$= \left[\frac{1}{4}y^4 + \frac{10}{3}y^3\right]_{-4}^{0}$$
$$= 149{,}333 \text{ m}^4$$

The depth of the centre of fluid pressure is:

$$\overline{\overline{y}} = \frac{I_x}{A_{m,x}} = \frac{149{,}333}{-58{,}667} = -2{,}545 \text{ m} \approx 2{,}545 \text{ m below the surface}$$

2. The equations bounding the triangle must be determined.



The boundary at the bottom is $y = -3$.
The coordinates at the right-hand boundaries are (0; −1) and (1; −3).
Determine the equation of the function line:

$$m = \frac{(-3) - (-1)}{(1) - (0)} = -2 \text{ and:}$$

$$y - (-1) = -2[x - (0)]$$
$$\therefore y = -2x - 1$$

The left-hand boundary is symmetrical to the right-hand boundary about the y-axis. You can calculate the equation: (0; −1) and (−1; −3) means that $m = 2$ and $y = 2x - 1$. Alternatively, use the boundary $x = 0$ and double the calculated values of the right-side.

Using the latter approach, the parallel strip is: $dA = 2x\,dy$

The boundary values are $c = -3$ and $d = -1$.

Therefore, the first moment of area is:

$$A_{m,x} = \int_c^d y(2x\,dy)$$

Substitute the x, where $x = -\frac{y+1}{2}$:

$$A_{m,x} = 2\int_{-3}^{-1} y\left(-\frac{y+1}{2}\right) dy$$

$$= -\int_{-3}^{-1} (y^2 + y)\,dy$$

$$= -\left[\frac{1}{3}y^3 + \frac{1}{2}y^2\right]_{-3}^{-1}$$

$$= -\left[\left(\frac{1}{6}\right) - \left(-\frac{9}{2}\right)\right]$$

$$= -4{,}667 \text{ m}^3$$

Similarly, the second moment of area is:

$$I_x = \int_c^d y^2(2x\,dy)$$

$$= 2\int_{-3}^{-1} y^2\left(-\frac{y+1}{2}\right) dy$$

$$= -\int_{-3}^{-1} (y^3 + y^2)\,dy$$

$$= -\left[\frac{1}{4}y^4 + \frac{1}{3}y^3\right]_{-3}^{-1}$$

$$= -\left[\left(-\frac{1}{12}\right) - \left(\frac{45}{4}\right)\right]$$

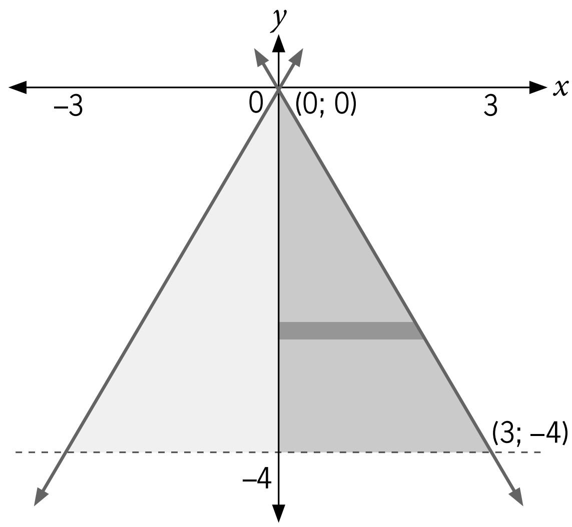
$$= 11{,}333 \text{ m}^4$$

The depth of the centre of fluid pressure is:

$$\bar{\bar{y}} = \frac{I_x}{A_{m,x}}$$

$$= \frac{11{,}333}{-4{,}667} = -2{,}429 \text{ m} \approx 2{,}429 \text{ m below the surface}$$

3. 3.1 The equations bounding the triangle must be determined.



The boundary at the bottom is $y = -4$.

The coordinates at the right-hand boundaries are (0; 0) and (3; −4).

Determine the equation of the function line:

$m = \frac{(-4) - (0)}{(3) - (0)} = -\frac{4}{3}$ and:

$\therefore y = -\frac{4}{3}x$

The left-hand boundary is symmetrical to the right-hand boundary about the y-axis. You can calculate the equation: (0; 0) and (−3; −4) means that $m = \frac{4}{3}$ and $y = \frac{4}{3}x$. Alternatively, use the boundary $x = 0$ and double the calculated values of the right-side.

Using the latter approach, the parallel strip is: $dA = 2x\,dy$

The boundary values are $c = -4$ and $d = 0$.

Therefore, the first moment of area is:

$A_{m,x} = \int_c^d y(2x\,dy)$

Substitute the x, where $x = -\frac{3}{4}y$:

$$A_{m,x} = 2\int_{-4}^{0} y\left(-\frac{3}{4}y\right) dy$$

$$= -\frac{3}{2}\int_{-4}^{0} y^2\,dy$$

$$= -\frac{3}{2}\left[\frac{1}{3}y^3\right]_{-4}^{0}$$

$$= -\frac{1}{2}[64]$$

$$= -32 \text{ m}^3$$

Similarly, the second moment of area is:

$$I_x = \int_c^d y^2(2x\,dy)$$

$$= 2\int_{-4}^{0} y^2\left(-\frac{3}{4}y\right) dy$$

$$= -\frac{3}{2}\int_{-4}^{0} y^3\,dy$$

$$= -\frac{3}{2}\left[\frac{1}{4}y^4\right]_{-4}^{0}$$

$$= -\frac{3}{8}[-256]$$

$$= 96 \text{ m}^4$$

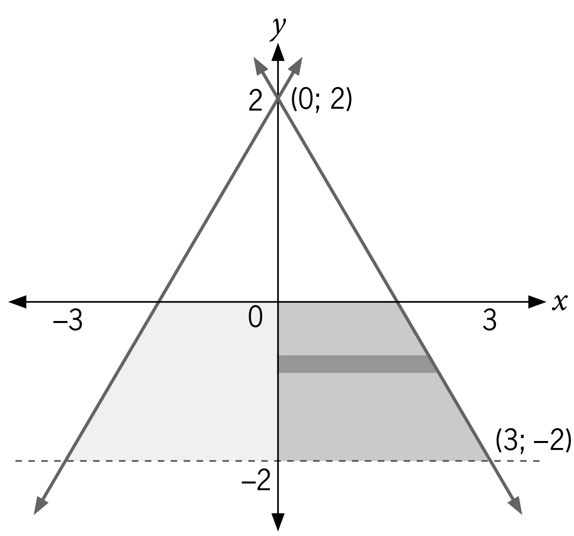
The depth of the centre of fluid pressure is:

$$\bar{\bar{y}} = \frac{I_x}{A_{m,x}} = \frac{96}{-32} = -3 \text{ m}$$

From the surface, the depth of the centre of fluid pressure is 3 m below the surface.

The total depth is 4 m, so centre of fluid pressure is (4 − 3) = 1 m above the base.

3.2 The tank is filled halfway to the top, so to a depth of 2 m. The equations bounding the trapezium must be determined.



Remember that the depth of the centre of fluid pressure is always measured from the surface of the fluid, which is taken as the x-axis. Therefore, the boundary at the surface is $y = 0$ and at the bottom is $y = -2$.

The coordinates at the right-hand boundaries are (0; 2) and (3; −2).

We need to determine the equation of the function line again because we moved the x-axis:

Therefore $m = \frac{(-2) - (2)}{(3) - (0)} = -\frac{4}{3}$ and:

$$y - (2) = -\frac{4}{3}[x - (0)]$$

$$y = -\frac{4}{3}x + 2$$

Note that the gradient is unchanged from the first part of this question. All that changed was that the tank was "shifted up" by 2 m, if the fluid surface remains at 0 m.

The left-hand boundary is symmetrical to the right-hand boundary about the y-axis. You can calculate the equation: (0; 2) and (−3; −2) means that $m = \frac{4}{3}$ and $y = \frac{4}{3}x + 2$. Alternatively, use the boundary $x = 0$ and double the calculated values of the right-side.

Using the latter approach, the parallel strip is: $dA = 2x\,dy$

The boundary values are $c = -2$ and $d = 0$.

Therefore, the first moment of area is:

$$A_{m,x} = \int_c^d y(2x\,dy)$$

Substitute the x, where $x = \frac{3}{4}(2 - y)$:

$$\begin{aligned} A_{m,x} &= 2\int_{-2}^{0} y\left(\frac{3}{4}(2 - y)\right) dy \\ &= \frac{3}{2}\int_{-2}^{0} \left(2y - y^2\right) dy \\ &= \frac{3}{2}\left[y^2 - \frac{1}{3}y^3\right]_{-2}^{0} \\ &= \frac{3}{2}\left[-\frac{20}{3}\right] \\ &= -10 \text{ m}^3 \end{aligned}$$

Similarly, the second moment of area is:

$$\begin{aligned} I_x &= \int_c^d y^2(2x\,dy) \\ &= 2\int_{-2}^{0} y^2\left(\frac{3}{4}(2 - y)\right) dy \\ &= \frac{3}{2}\int_{-2}^{0} \left(2y^2 - y^3\right) dy \\ &= \frac{3}{2}\left[\frac{2}{3}y^3 - \frac{1}{4}y^4\right]_{-2}^{0} \\ &= \frac{3}{2}\left[\frac{28}{3}\right] \\ &= 14 \text{ m}^4 \end{aligned}$$

The depth of the centre of fluid pressure is:

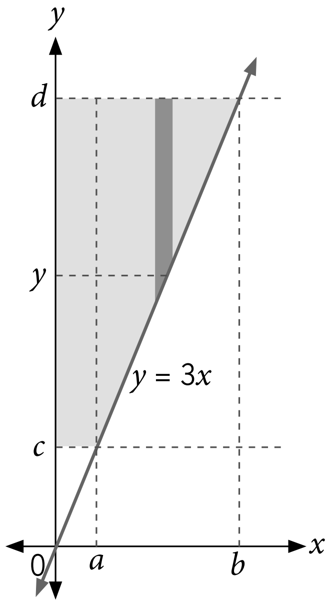
$$\bar{\bar{y}} = \frac{I_x}{A_{m,x}} = \frac{14}{-10} = -1{,}4 \text{ m}$$

From the surface, the depth of the centre of fluid pressure is 1,4 m below the surface.

The total depth is 2 m, so it is (2 – 1,4) = 0,6 m above the base.

Summative assessment: Module 7 SB page 212

1.



(3)

About the y-axis, there are two sections: $I_y = I_{y1} + I_{y2}$. (1)

When $c = 1{,}5$ then $a = \frac{1{,}5}{3} = 0{,}5$. When $d = 6$ then $b = \frac{6}{3} = 2$. (2)

For section 1: $I_{y1} = \int_0^a x^2\, dA$ where $dA = (d - c)\, dx = 4{,}5\, dx$: (1)

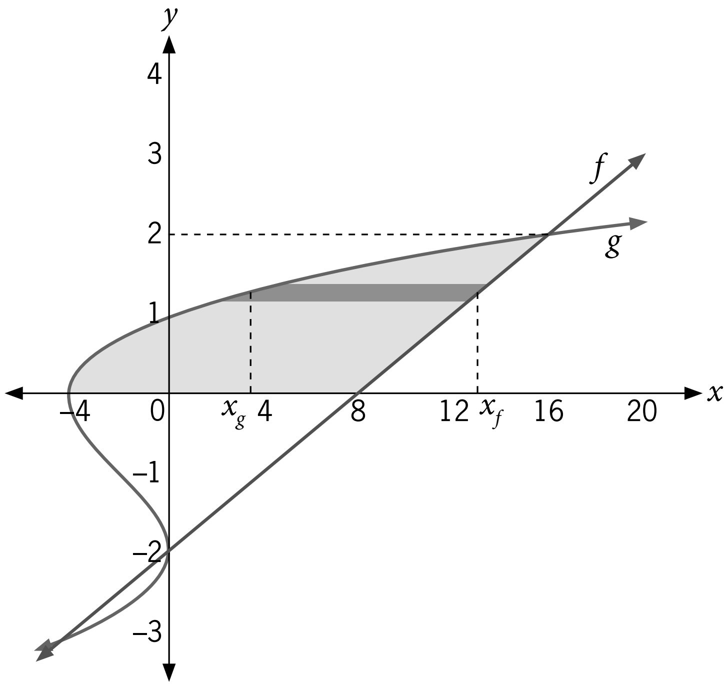
$$I_{y1} = \int_0^{0,5} 4{,}5x^2\, dx$$
$$= \left[1{,}5x^3\right]_0^{0,5}$$
$$= 0{,}1875 \text{ units}^4$$ (2)

For section 2: $I_{y2} = \int_a^b x^2\, dA$ where $dA = (d - y)\, dx$: (1)

$$I_{y2} = \int_a^b x^2(6 - y)\, dx$$
$$= \int_{0,5}^{2} x^2(6 - 3x)\, dx$$
$$= \left[2x^3 - \tfrac{3}{4}x^4\right]_{0,5}^{2}$$
$$= 4 - 0{,}203$$
$$= 3{,}797 \text{ units}^4$$ (3)

Therefore,
$I_y = I_{y1} + I_{y2} = 3{,}984 \text{ units}^4$

2.



(4)

About the x-axis: $dA = \left(x_f - x_g\right)\, dy$ (1)

The area is above the x-axis, so the lower boundary value is $c = 0$. (1)

The upper boundary value, d, is one of the points of intersection, where $f(y) = g(y)$:

$$4\left(y+2\right) = (y-1)(y+2)^2 4y + 8$$
$$= y^3 + 3y^2 - 40$$
$$= y^3 + 3y^2 - 4y - 120$$
$$= (y-2)(y+2)(y+3)$$

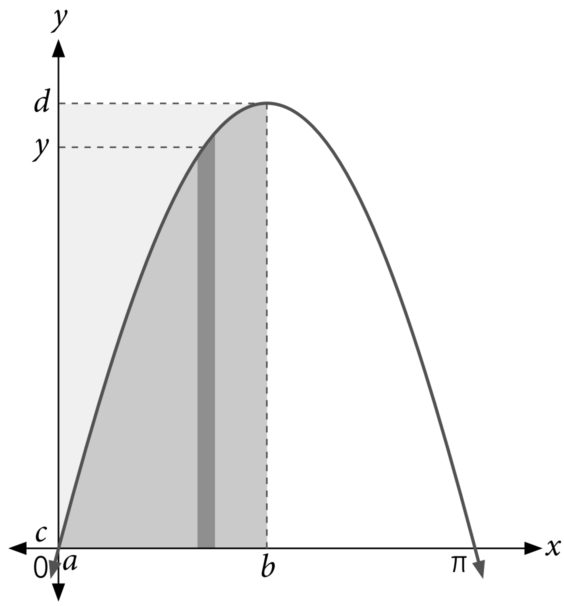
$\therefore y = -3;\ y = -2;\ y = 2$

$d = 2$ (1)

Calculate the second moment of area: (3)

$$I_x = \int_c^d y^2\, dA$$
$$= \int_c^d y^2\left(x_f - x_g\right)\, dy$$
$$= \int_0^2 y^2\left[4(y+2) - (y-1)(y+2)^2\right]\, dy$$
$$= \int_0^2 \left(-y^5 - 3y^4 + 4y^3 + 12y^2\right)\, dy$$
$$= \left[-\frac{1}{6}y^6 - \frac{3}{5}y^5 + y^4 + 4y^3\right]_0^2$$
$$= 18{,}133 \text{ units}^4$$

3. The reference axis is x, so $\text{I} = \rho\int_c^d y^2\, d\text{V}$.



(3)

The boundary value when $a = 0$ is $c = 0$; and when $b = \frac{\pi}{2}$ is $d = 1$. (2)

For the solid of revolution:

$d\text{V} = 2\pi\, d\text{A} = 2\pi y\, dx$ (1)

Calculate the volume of the solid of revolution, $\text{V} = \int d\text{V}$:

$\text{V} = \int_a^b 2\pi y\, dx$

$= 2\pi\int_0^{\frac{\pi}{2}} \sin x\, dx$

$= 2\pi\,[-\cos x]_0^{\frac{\pi}{2}}$

$= 2\pi(1)$ units^3 (3)

Calculate the moment of inertia in terms of density, ρ:

$\text{I} = \rho\int_a^b x^2\, d\text{V}$

$= 2\pi\rho\int_0^{\frac{\pi}{2}} x^2 y\, dx$

$= 2\pi\rho\int_0^{\frac{\pi}{2}} x^2 \sin x\, dx$ (6)

Using integration by parts, with $f = x^2$ and $g' = \sin x$, then $f' = 2x$ and $g = -\cos x$.

$\text{I} = 2\pi\rho\left([-x^2\cos x]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}}(-2x\cos x)\, dx\right)$

$= 2\pi\rho\left(0 + \int_0^{\frac{\pi}{2}} 2x\cos x\, dx\right)$

Using integration by parts again, with $f = 2x$ and $g' = \cos x$, then $f' = 2$ and $g = \sin x$.

$\text{I} = 2\pi\rho\left([2x\sin x]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} 2\sin x\, dx\right)$

$= 2\pi\rho\,[2x\sin x + \cos x]_0^{\frac{\pi}{2}}$

$= 2\pi\rho(\pi - 1)$ units^4

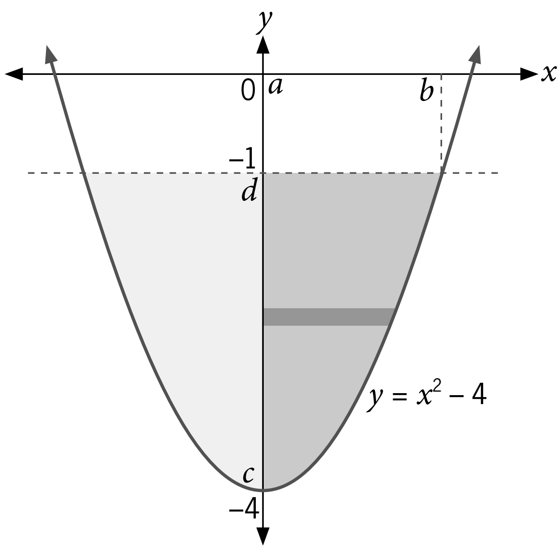
Make the substitution $\rho = \frac{m}{V}$ to express the moment of inertia in terms of mass, m. (2)

$$I = \frac{2\pi m(\pi - 1)}{2\pi(1)}$$

$$= (\pi - 1)m \text{ units}^2$$

$$= 2{,}142m \text{ kg.m}^2$$

4.



(3)

At the turning point, when $c = -4$, then $a = 0$. (1)

The upper edge of the gate is where $y = d = -1$.

The left-hand boundary is symmetrical to the right-hand boundary about the y-axis. You can calculate the integral in two sections, $x \in (-b; 0)$ and $x \in (0; b)$. Alternatively, use the boundary $x = 0$ and double the calculated values of the right-side shape.

Since $b^2 = d + 4$ and using the latter approach, $b = \sqrt{3}$. (1)

The area of the parallel strip is: (1)

$dA = 2x\,dy$

Therefore, the first moment of area is: (3)

$$A_{m,x} = \int_c^d y(2x\,dy)$$

Substitute the y, where $dy = 2x\,dx$: (1)

$$A_{m,x} = 2\int_a^b (x^2 - 4)x(2x\,dx)$$

$$= 4\int_0^{\sqrt{3}} (x^4 - 4x^2)\,dx$$

$$= 4\left[\frac{1}{5}x^5 - \frac{4}{3}x^3\right]_0^{\sqrt{3}}$$

$$= -15{,}242 \text{ units}^3$$

Similarly, the second moment of area is: (3)

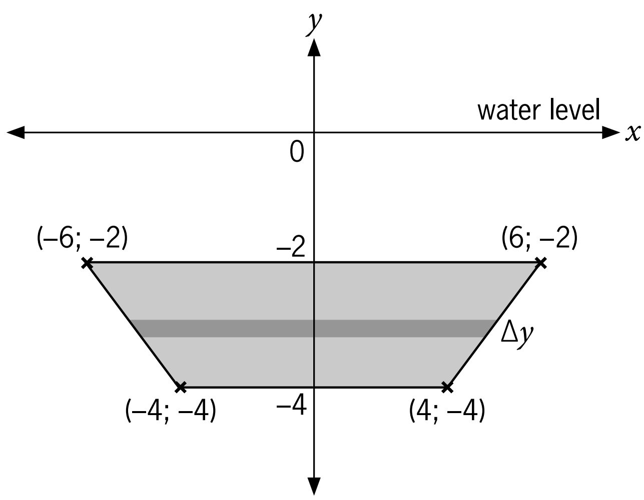
$$\begin{aligned} I_x &= \int_c^d y^2(2x\,dy) \\ &= 2\int_a^b (x^2-4)^2 x(2x\,dx) \\ &= 4\int_0^{\sqrt{3}} (x^6 - 8x^4 + 16x^2)\,dx \\ &= 4\left[\tfrac{1}{7}x^7 - \tfrac{8}{5}x^5 + \tfrac{16}{3}x^3\right]_0^{\sqrt{3}} \\ &= 37{,}808 \text{ units}^4 \end{aligned}$$

The depth of the centre of fluid pressure is:

$$\bar{\bar{y}} = \frac{I_x}{A_{m,x}}$$

$$= \frac{37{,}808}{-15{,}242} = -2{,}481 \text{ units} \approx 2{,}481 \text{ m below the surface}$$ (2)

5. 5.1



(2)

5.2 $m = \frac{-2+4}{6-4} = 1$

$y + 2 = (x - 6)$

$\therefore y = x - 8$ and $x = y + 8$

First moment of area:

$$\begin{aligned} \int_a^b r\,dA &= \int_{-4}^{-2} 2y(y+8)\,dy \\ &= 2\int_{-4}^{-2} y^2 + 8y\,dy \\ &= 2\left[\frac{y^3}{3} + 4y^2\right]_{-4}^{-2} \\ &= 2\left[\frac{(-2)^3}{3} + 4(-2)^2 - \left(\frac{-64}{3} + 4(16)\right)\right] \\ &= -58{,}667 \text{ m}^3 \end{aligned}$$ (4)

5.3 Second moment of area:

$$\int_a^b r^2 \, dA = \int_{-4}^{-2} y^2 \, 2(y + 8) \, dy$$

$$= 2\int_{-4}^{-2} y^3 + 8y^2 \, dy$$

$$= 2\left[\frac{y^4}{4} + \frac{8y^3}{3}\right]_{-4}^{-2}$$

$$= 2\left[\frac{(-2)^4}{4} - \frac{8(-2)^3}{3} - \left(\frac{(-4)^4}{4} + \frac{8(-4)^3}{3}\right)\right]$$

$$= 66 \text{ m}^4$$

Depth of the centre of pressure:

$$\bar{y} = \frac{66 \text{ m}^4}{-58{,}667 \text{ m}^3} = -1{,}125 \text{ m}$$ (4)

TOTAL: [55]

MODULE

8 Combinations of differentiation and integration

After they have completed this module, students should be able to:

- calculate the arc length of a given curve between two given points by applying differentiation and integration for:
  - non-parametric equations;
  - parametric equations; and
- calculate the area of the surface of revolution generated when the arc of a curve between two points rotates through a full revolution about an axis for:
  - non-parametric equations;
  - parametric equations.

Introduction

This course covers a variety of calculus techniques often used in engineering. In this final module, differentiation and integration techniques are combined to determine complicated dimensions. Students will learn how to calculate the length of a curve, and the area of a rotated surface. This is just a small glimpse into the many interesting and practical combinations and applications of calculus.

Students need the following pre-knowledge to successfully complete this module.

Pre-knowledge

Students should already know:

- Many advanced integration techniques, such as:
  - Inspection
  - Algebraic substitution
  - Integration by parts
  - Trigonometric substitution
  - Trigonometric identities
- How to differentiate parametric equations.

Activity 8.1 **SB page 218**

1. 1.1 $y = \ln x$

Determining the $\frac{dy}{dx}$:

$\frac{dy}{dx} = \frac{1}{x}$

The arc length is as follows:

$S = \int\sqrt{1 + \left(\frac{1}{x}\right)^2}\ dx$

$= \int\sqrt{\frac{x^2 + 1}{x^2}}\ dx$

$\therefore S = \int\frac{\sqrt{x^2 + 1}}{x}\ dx$

1.2 $y = \cos x$

Determining the $\frac{dy}{dx}$:

$\frac{dy}{dx} = -\sin x$

The arc length is as follows:

$\therefore S = \int\sqrt{1 + (-\sin x)^2}\ dx$

1.3 $4y = x^2 \therefore y = \frac{x^2}{4}$

Determining the $\frac{dy}{dx}$:

$\frac{dy}{dx} = \frac{x}{2}$

The arc length is as follows:

$S = \int\sqrt{1 + \left(\frac{x}{2}\right)^2}\ dx$

$= \int\sqrt{1 + \frac{x^2}{4}}\ dx$

$\therefore S = \int\frac{\sqrt{4 + x^2}}{2}\ dx$

1.4 $y = 5x - 7$

Determining the $\frac{dy}{dx}$:

$\frac{dy}{dx} = 5$

The arc length is as follows:

$S = \int\sqrt{1 + (5)^2}\ dx$

$\therefore S = \int\sqrt{26}\ dx$

2. 2.1 $y^2 = (x + 3)^3$ for $-1 \leq x \leq 2$

$y = \sqrt{(x + 3)^3}$

Determining the $\frac{dy}{dx}$:

$\frac{dy}{dx} = \frac{3}{2}\sqrt{(x + 3)}$

The arc length is as follows:

$S = \int_{-1}^{2} \sqrt{1 + \left(\frac{3}{2}\sqrt{(x + 3)}\right)^2}\, dx$

$= \int_{-1}^{2} \sqrt{\frac{9}{4}x + \frac{31}{4}}\, dx$

Let $u = \frac{9}{4}x + \frac{31}{4}$ such that $du = \frac{9}{4}\, dx$

So, limits become: $u = \frac{11}{2}$ when $x = -1$ and $u = \frac{49}{4}$ when $x = 2$

$S = \frac{4}{9}\int_{\frac{11}{2}}^{\frac{49}{4}} \sqrt{u}\, du$

$S = \frac{4}{9}\left(\frac{2}{3}u^{\frac{3}{2}}\right)_{\frac{11}{2}}^{\frac{49}{4}}$

Substituting boundary values $\frac{49}{4}$ and $\frac{11}{2}$:

S = 8,882 units

2.2 $y = \sqrt{9 - x^2}$ for $0 \leq x \leq 1$

Determining the $\frac{dy}{dx}$:

$\frac{dy}{dx} = \frac{-x}{\sqrt{9 - x^2}}$

The arc length is as follows:

$S = \int_{0}^{1} \sqrt{1 + \left(\frac{-x}{\sqrt{9 - x^2}}\right)^2}\, dx$

$= \int_{0}^{1} \sqrt{1 + \left(\frac{x^2}{9 - x^2}\right)}\, dx$

$= \int_{0}^{1} \sqrt{\left(\frac{9}{9 - x^2}\right)}\, dx$

$= \int_{0}^{1} 3\sqrt{\left(\frac{1}{3^2 - x^2}\right)}\, dx$

Using trigonometric identities:

$S = \left[3 \sin^{-1}\frac{x}{3}\right]_{0}^{1}$

Substituting boundary values 1 and 0:

S = 1,0195 units

2.3 $y = 16 - x^2$ for $0 \leq x \leq 3$

$\frac{dy}{dx} = -2x$

The arc length is as follows:

$$S = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$

$$= \int_0^3 \sqrt{1 + 4x^2}\ dx$$

$$= \int_0^3 \sqrt{4\left(\frac{1}{4} + x^2\right)}\ dx$$

$$= 2\int_0^3 \sqrt{\frac{1}{4} + x^2}\ dx$$

$$= 2\left[\frac{x}{2}\sqrt{\frac{1}{4} + x^2} + \left(\frac{\frac{1}{4}}{2}\right)\ln\left(x + \sqrt{\frac{1}{4} + x^2}\right)\right]_0^3$$

$$=2\left[\frac{3}{2}\sqrt{\frac{1}{4} + 3^2} + \frac{\frac{1}{4}}{2}\ln\left(3 + \sqrt{\frac{1}{4} + 3^2}\right) - \left(0 + \frac{1}{8}\ln\left(0 + \frac{1}{2}\right)\right)\right]$$

$$= 9{,}747 \text{ units}$$

2.4 $y = 2x^2 - 6$ for $0 \leq x \leq 2$

$\frac{dy}{dx} = 4x$

The arc length is as follows:

$$S = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$

$$= \int_0^2 \sqrt{1 + 16x^2}\ dx$$

$$= \int_0^2 \sqrt{16\left(\frac{1}{16} + x^2\right)}\ dx$$

$$= 4\int_0^2 \sqrt{\frac{1}{16} + x^2}\ dx$$

$$= 4\left[\frac{x}{2}\sqrt{\frac{1}{16} + x^2} + \left(\frac{\frac{1}{16}}{2}\right)\ln\left(x + \sqrt{\frac{1}{16} + x^2}\right)\right]_0^2$$

$$=4\left[\frac{2}{2}\sqrt{\frac{1}{16} + 2^2} + \frac{1}{32}\ln\left(2 + \sqrt{\frac{1}{16} + 2^2}\right)\right] - \left[0 + \frac{1}{32}\ln\left(0 + \frac{1}{4}\right)\right]$$

$$= 8{,}409 \text{ units}$$

2.5 $y = \frac{x^3}{3}$ for $2 \leq x \leq 4$

$\frac{dy}{dx} = x$

The arc length is as follows:

$$S = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_2^4 \sqrt{1 + x^2}\, dx$$

$$= \int_0^2 \sqrt{16\left(\frac{1}{16} + x^2\right)}\, dx$$

$$= \left[\frac{x}{2}\sqrt{1 + x^2} + \left(\frac{1}{2}\right) \ln\left(x + \sqrt{1 + x^2}\right)\right]_2^4$$

$$= \left[\frac{4}{2}\sqrt{1 + 4^2} + \frac{1}{2}\ln\left(4 + \sqrt{1 + 4^2}\right)\right] - \left[\frac{2}{2}\sqrt{1 + 2^2} + \frac{1}{2}\ln\left(2 + \sqrt{1 + 2^2}\right)\right]\Bigg|$$

$$= \left[2\sqrt{17} + \frac{1}{2}\ln(4 + \sqrt{17})\right] - \left[\sqrt{5} + \frac{1}{2}\ln(2 + \sqrt{5})\right]\Bigg|$$

$= 6{,}336$ units

Activity 8.2 SB page 221

1. $x = t$ and $y = 2t$ for $t = 2$ and $t = 4$

Differentiate both functions:

$\frac{dx}{dt} = 1$ and $\frac{dy}{dt} = 2$

$$S = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_2^4 \sqrt{(1)^2 + (2)^2}\, dt$$

$$= \int_2^4 \sqrt{5}\, dt$$

$$\therefore S = [t\sqrt{5}]_2^4$$

Substituting boundary values:
$S = 2\sqrt{5}$ units

2. $x = 1$ and $y = t^2$ for $t = 1$ and $t = 2$
Differentiate both functions:

$\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 2t$

$$S = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_1^2 \sqrt{(0)^2 + (2t)^2}\, dt$$

$$= \int_1^2 2t\, dt$$

$\therefore \text{S} = [t^2]_1^2$

Substituting boundary values:

S = 3 units

3. $x = t^2$ and $y = t^3$ for $t = 0$ and $t = 2$

Differentiate both functions:

$\frac{dx}{dt} = 2t$ and $\frac{dy}{dt} = 3t^2$

$$\text{S} = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^2 \sqrt{(2t)^2 + (3t^2)^2}\, dt$$

$$= \int_0^2 t\sqrt{4 + 9t^2}\, dt$$

Let $4 + 9t^2 = u$ such that $du = 18t\, dt$

Amend boundary values: when $t = 0$ then $u = 4$; and when $t = 2$ then $u = 40$

Now, $\text{S} = \int_4^{40} \frac{1}{18}\sqrt{u}\, du$

$$\therefore \text{S} = \left[\frac{1}{27}u^{\frac{3}{2}}\right]_4^{40}$$

Substituting boundary values:

$$\text{S} = \frac{1}{27}\left[40^{\frac{3}{2}} - 4^{\frac{3}{2}}\right]$$

S = 9,073 units

4. $x = \sin t$ and $y = \cos t$ for $t = 0$ and $t = \pi$

Differentiate both functions:

$\frac{dx}{dt} = \cos t$ and $\frac{dy}{dt} = -\sin t$

$$\text{S} = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^\pi \sqrt{(\cos t)^2 + (-\sin t)^2}\, dt$$

$$= \int_0^\pi 1\, dt$$

$\therefore \text{S} = [t]_0^\pi$

Substituting boundary values:

S = π units ≈ 3,14 units

5. $x = e^t$ and $y = e^3$ for $t = 0$ and $t = 2$

   Differentiate both functions:

   $\frac{dx}{dt} = e^t$ and $\frac{dy}{dt} = 0$

   $S = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$

   $= \int_0^2 \sqrt{(e^t)^2 + (0)^2}\ dt$

   $\therefore S = [e^t]_0^2$

   Substituting boundary values:
   $S = (e^2 - 1)$ units $\approx 6{,}389$ units

6. $x = 3(\cos t + t \sin t)$ and $y = 3(\sin t - t \cos t)$ for $t = 0$ and $t = \pi$

   Differentiate both functions:

   $\frac{dx}{dt} = -3 \sin t + 3t \cos t + 3 \sin t = 3t \cos t$

   $\frac{dy}{dt} = 3 \cos t + 3t \sin t - 3 \cos t = 3t \sin t$

   $S = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$

   $= \int_0^\pi \sqrt{(3t \cos t)^2 + (3t \sin t)^2}\ dt$

   $= \int_0^\pi \sqrt{9t^2\left[(\cos t)^2 + (\sin t)^2\right]}\ dt$

   $= \int_0^\pi \sqrt{9t^2}\ dt$

   $= 3\int_0^\pi t\ dt$

   $= 3\left[\frac{t^2}{2}\right]_0^\pi$

   $= \frac{3}{2}\pi^2$ units

7. $x = e^\theta \sin \theta + 5$ and $y = e^\theta \cos \theta + 8$ between $\theta = 0$ and $\theta = \frac{\pi}{3}$

   Differentiate both functions:

   $x = e^\theta \sin \theta + 5$

   $\frac{dx}{d\theta} = e^\theta \cos \theta + e^\theta \sin \theta = e^\theta(\cos \theta + \sin \theta)$

   $\left[\frac{dx}{d\theta}\right]^2 = e^{2\theta}(\cos \theta + \sin \theta)^2$

   $y = e^\theta \cos \theta + 8$

$$\frac{dy}{d\theta} = -e^{\theta} \sin \theta + e^{\theta} \cos \theta = e^{\theta}(\cos \theta - \sin \theta)$$

$$\left[\frac{dy}{d\theta}\right]^2 = e^{2\theta}(\cos \theta - \sin \theta)^2$$

$$\left[\frac{dx}{d\theta}\right]^2 + \left[\frac{dy}{d\theta}\right]^2 = e^{2\theta}\left[\left(\cos \theta + \sin \theta\right)^2 + \left(\cos \theta - \sin \theta\right)^2\right]$$

$$= e^{2\theta}[\cos^2\theta + \sin^2\theta + 2 \sin \theta \cos \theta + \cos^2\theta + \sin^2\theta - 2 \sin \theta \cos \theta]$$

$$= 2e^{2\theta}$$

$$S = \int_{\theta_1}^{\theta_2} \sqrt{\left[\frac{dx}{d\theta}\right]^2 + \left[\frac{dy}{d\theta}\right]^2}\, d\theta$$

$$= \int_0^{\frac{\pi}{3}} \sqrt{2e^{2\theta}}\, d\theta$$

$$= \sqrt{2} \int_0^{\frac{\pi}{3}} e^{\theta}\, d\theta$$

$$= \sqrt{2}\left[e^{\theta}\right]_0^{\frac{\pi}{3}}$$

$$= \sqrt{2}\left[e^{\frac{\pi}{3}} - e^0\right]$$

$$= 2{,}616 \text{ units}$$

Activity 8.3 SB page 225

1. 1.1 $y = x^3$ for $0 \leq x \leq 2$

$$\frac{dy}{dx} = 3x^2$$

Using appropriate formula:

$$A_x = 2\pi \int_a^b y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= 2\pi \int_0^2 x^3 \sqrt{1 + \left(3x^2\right)^2}\, dx$$

$$= 2\pi \int_0^2 x^3 \sqrt{1 + 9x^4}\, dx$$

Let $u = 1 + 9x^4$ such that $du = 36x^3\, dx$.

When $x = a = 0$, then $u = 1$. When $x = b = 2$, then $u = 145$.

$$A_x = \frac{2\pi}{36} \int_1^{145} \sqrt{u}\, du$$

$$= \frac{\pi}{18}\left[\frac{2}{3} u^{\frac{3}{2}}\right]_1^{145}$$

$$= 203{,}044 \text{ units}^2$$

1.2 $y = \frac{1}{2}x$ for $1 \leq x \leq 3$

$$\frac{dy}{dx} = \frac{1}{2}$$

$$A_x = 2\pi\int_a^b y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$

$$= 2\pi\int_1^3 \frac{1}{2}x\sqrt{1 + \left(\frac{1}{2}\right)^2}\ dx$$

$$= \pi\sqrt{\frac{5}{4}}\int_1^3 x\ dx$$

$$= \frac{\pi\sqrt{5}}{2}\left[\frac{x^2}{2}\right]_1^3$$

$$= \frac{\pi\sqrt{5}}{2}\left|\frac{(9)}{2} - \frac{(1)}{2}\right|$$

$$= 2\sqrt{5}\,\pi \text{ units}^2 \approx 14{,}05 \text{ units}^2$$

1.3 $y = \frac{1}{9}x^2$

$$\frac{dy}{dx} = \frac{2}{9}x;\ \left[\frac{dy}{dx}\right]^2 = \left(\frac{2}{9}x\right)^2$$

$$1 + \left[\frac{dy}{dx}\right]^2 = 1 + \left(\frac{2}{9}x\right)^2 = 1 + \frac{4x^2}{81} = \frac{81 + 4x^2}{81}$$

$$A_x = 2\pi\int_0^6 x\frac{\sqrt{81 + 4x^2}}{9}\ dx$$

Let $u = 81 + 4x^2$ such that $du = 8x\ dx$

When $x = a = 0$ then $u = 81$. When $x = b = 6$ then $u = 225$.

$$A_x = \frac{2}{9}\pi\int_{81}^{225} yu^{\frac{1}{2}}\frac{du}{8y}$$

$$= \left(\frac{2}{9}\right)\left(\frac{1}{8}\right)\pi\int_{81}^{225} u^{\frac{1}{2}}\ du$$

$$= \frac{1}{36}\pi\left[\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right]_{81}^{225}$$

$$= \left(\frac{1}{36}\right)\left(\frac{2}{3}\right)\pi\left[u^{\frac{3}{2}}\right]_{81}^{225}$$

$$= \frac{1}{54}\pi\left[225^{\frac{3}{2}} - 81^{\frac{3}{2}}\right]$$

$$= 49\pi \text{ or } 153{,}938 \text{ units}^2$$

1.4 $x = \frac{1}{9}y^2$

$\frac{dx}{dy} = \frac{2}{9}y$; $\left[\frac{dx}{dy}\right]^2 = \left(\frac{2}{9}y\right)^2$

$1 + \left[\frac{dx}{dy}\right]^2 = 1 + \left(\frac{2}{9}y\right)^2 = 1 + \frac{4y^2}{81} = \frac{81 + 4y^2}{81}$

$A_x = 2\pi\int_0^6 y\frac{\sqrt{81 + 4y^2}}{9}\,dy$

Let $u = 81 + 4y^2$ such that $du = 8y\,dy$

When $y = c = 0$ then $u = 81$. When $y = d = 6$ then $u = 225$.

$A_x = \frac{2}{9}\pi\int_{81}^{225} yu^{\frac{1}{2}}\frac{du}{8y}$

$= \left(\frac{2}{9}\right)\left(\frac{1}{8}\right)\pi\int_{81}^{225} u^{\frac{1}{2}}\,du$

$= \frac{1}{36}\pi\left[\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right]_{81}^{225}$

$= \left(\frac{1}{36}\right)\left(\frac{2}{3}\right)\pi\left[u^{\frac{3}{2}}\right]_{81}^{225}$

$= \frac{1}{54}\pi\left[225^{\frac{3}{2}} - 81^{\frac{3}{2}}\right]$

$= 49\pi$ or 153,938 units^2

2. 2.1 $y = \cos 2t$ and $x = \sin 2t$ for $0 \leq t \leq \pi$

Differentiating both functions:

$\frac{dy}{dt} = -2\sin 2t$ and $\frac{dx}{dt} = 2\cos 2t$

$A = 2\pi\int_a^b y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$

$= 2\pi\int_0^{\pi}\cos 2t\sqrt{(2\cos 2t)^2 + (-2\sin 2t)^2}\,dt$

$= 4\pi\int_0^{\pi}\cos 2t\sqrt{(\cos 2t)^2 + (\sin 2t)^2}\,dt$

Use the identity $\cos^2 t + \sin^2 t \equiv 1$:

$A = 4\pi\int_0^{\pi}\cos 2t\,dt$

$= 4\pi\left[\frac{1}{2}\sin 2t\right]_0^{\pi}$

Substituting limits:

$A = 0$ units^2

2.2 $x = t^2$ and $y = 3t$ for $0 \leq t \leq 2$

Differentiating both functions

$\frac{dx}{dt} = 2t$ and $\frac{dy}{dt} = 3$

$$A = 2\pi \int_a^b y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= 2\pi \int_0^2 3t\sqrt{(2t)^2 + (3)^2}\, dt$$

$$= 2\pi \int_0^2 3t\sqrt{4t^2 + 9}\, dt$$

Let $4t^2 + 9 = u$ such that $du = 8t\ dt$.

When $t = a = 0$, then $u = 9$. When $t = b = 2$, then $u = 25$.

$$A = \frac{3\pi}{4}\int_9^{25}\sqrt{u}\, du$$

$$= 153{,}932 \text{ units}^2$$

2.3 $x = 2t + 7$ and $y = 4t$ for $0 \leq t \leq 2$

Differentiating both functions:

$\frac{dy}{dt} = 4$ and $\frac{dx}{dt} = 2$

$$A = 2\pi \int_a^b y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= 2\pi \int_0^2 4t\sqrt{(2)^2 + (4)^2}\, dt$$

$$= 2\pi \int_0^2 8\sqrt{5}t\, dt$$

$$= 16\sqrt{5}\,\pi \int_0^2 t\, dt$$

$$= 16\sqrt{5}\,\pi \left[\frac{t^2}{2}\right]_0^2$$

Substituting limits:

$$A = 16\sqrt{5}\,\pi(2 - (0))$$

$$= 32\sqrt{5}\,\pi \text{ units}^2$$

2.4 $x = 2\cos t$ and $y = 2\sin t$ for $0 \leq t \leq \pi$

Differentiating both functions:

$\frac{dy}{dt} = 2\cos t$ and $\frac{dx}{dt} = -2\sin t$

$$A = 2\pi \int_a^b y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= 2\pi \int_0^{\pi} 2\sin t\sqrt{(-2\sin t)^2 + (2\cos t)^2}\, dt$$

$$= 2\pi \int_0^{\pi} 2\sin t\sqrt{4(\sin^2 t + \cos^2 t)}\, dt$$

Use the identity $\cos^2 t + \sin^2 t \equiv 1$:

$A = 8\pi \int_0^{\pi} \sin t \, dt$

$= 8\pi[-\cos t]_0^{\pi}$

Substituting limits:

$A = 8\pi(1 - (-1))$

$= 16\pi \text{ units}^2$

Summative assessment: Module 8 SB page 226

1. Arc length of $y = \ln(\sec x)$ between the points $0 \leq x \leq \frac{\pi}{4}$

Use: $S = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$

$\frac{dy}{dx} = \frac{1}{\sec x} \cdot \frac{d}{dx}(\sec x)$

$\frac{d}{dx}(\sec x) = \frac{d}{dx}\left(\frac{1}{\cos x}\right)$

$= -1.(\cos x)^{-2}.(-\sin x)$

$= \frac{\sin x}{\cos^2 x}$

$\therefore \frac{dy}{dx} = \frac{1 \sin x}{\sec x \cos^2 x}$

$\Rightarrow \frac{dy}{dx} = \tan x$

$\therefore S = \int_0^{\frac{\pi}{4}} \sqrt{1 + \tan^2 x} \, dx$

$= \int_0^{\frac{\pi}{4}} \sqrt{\sec^2 x} \, dx$

$= \ln|\sec x + \tan x|_0^{\frac{\pi}{4}}$

$= \ln\left|\sec\frac{\pi}{4} + \tan\frac{\pi}{4}\right| - \ln|\sec 0 + \tan 0| = \ln|\sqrt{2} + 1| - \ln|1 + 0|$

$= \ln(\sqrt{2} + 1)$ units $\approx 0{,}881$ units (6)

2. Arc length given by: $x = \sin 2t$; $y = \cos 2t$ for $0 \leq t \leq 2\pi$

Use: $S = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt$

$\frac{dx}{dt} = 2 \cos 2t$

$\frac{dy}{dt} = -2 \sin 2t$

$\therefore S = \int_0^{2\pi} \sqrt{(2\cos 2t)^2 + (2\sin 2t)^2}\ dt$

$= \int_0^{2\pi} 2\sqrt{\cos^2 2t + \sin^2 2t}\ dt$

$= \int_0^{2\pi} 2\ dt$

$= [2t|_0^{2\pi}$

$= 4\pi$ units $\approx 12{,}566$ units (6)

3. Area of surface formed by rotating $y = \sqrt{1 + 4x}$ about the x-axis on the interval $1 \leq x \leq 5$.

   Use: $A = 2\pi\int_a^b y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$

   $\frac{dy}{dx} = \frac{2}{\sqrt{1 + 4x}}$

   $S = 2\pi\int_1^5 \sqrt{1 + 4x}.\sqrt{1 + \frac{4}{1 + 4x}}\ dx$

   $= 2\pi\int_1^5 \sqrt{5 + 4x}\ dx$

   Let $u = 5 + 4x$

   $\frac{du}{dx} = 4 \Rightarrow dx = \frac{du}{4}$

   When $x = a = 1$, then $u = 9$. When $x = b = 5$, then $u = 25$.

   $\therefore 2\pi\int_1^5 \sqrt{5 + 4x}\ dx$

   $= \frac{2\pi}{4}\int_9^{25} \sqrt{u}\ du$

   $= \frac{\pi}{2}\left[u^{\frac{3}{2}}\left(\frac{2}{3}\right)\right|_9^{25}$

   $= \frac{\pi}{3}\left[u^{\frac{3}{2}}\right]_9^{25}$

   $= \frac{98}{3}\pi$ units$^2 \approx 102{,}625$ units^2 (6)

4. Area of surface formed by rotating $y = \sqrt{a^2 - x^2}$ about the x-axis on the interval $0 \leq y \leq \frac{a}{2}$.

   Use: $A = 2\pi\int_a^b y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$

   $\frac{dy}{dx} = \frac{-x}{\sqrt{a^2 - x^2}}$

$$A = 2\pi\int_0^{\frac{a}{2}}\sqrt{a^2 - x^2}\sqrt{1 + \left(\frac{-x}{\sqrt{a^2 - x^2}}\right)^2}\, dx$$

$$= 2\pi\int_0^{\frac{a}{2}}\sqrt{(a^2 - x^2)\left(1 + \frac{x^2}{a^2 - x^2}\right)}\, dx$$

$$= 2\pi\int_0^{\frac{a}{2}} a\, dx$$

$$= 2\pi\,[ax]_0^{\frac{a}{2}}$$

$$= \pi a^2 \text{ units}^2$$ (6)

5. Area of surface formed by rotating $x = a\cos^3\theta$ and $y = a\sin^3\theta$ where $0 \leq \theta \leq \frac{\pi}{2}$

Use: $A = 2\pi\int_a^b y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$\frac{dx}{d\theta} = -3a\cos^2\theta\sin\theta$

$\frac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$

$$A = 2\pi\int_0^{\frac{\pi}{2}}(a\sin^3\theta)\sqrt{(-3a\cos^2\theta\sin\theta)^2 + (3a\sin^2\theta\cos\theta)^2}\, d\theta$$

$$= 2\pi\int_0^{\frac{\pi}{2}}(a\sin^3\theta)\sqrt{9a^2\cos^4\theta\sin^2\theta + 9a^2\sin^4\theta\cos^2\theta}\, d\theta$$

$$= 2\pi\int_0^{\frac{\pi}{2}}(a\sin^3\theta)\sqrt{9a^2\cos^2\theta\sin^2\theta(\cos^2\theta + \sin^2\theta)}\, d\theta$$

$$= 2\pi\int_0^{\frac{\pi}{2}}(a\sin^3\theta).3\,a\cos\theta\sin\theta\, d\theta$$

$$= 6\pi\int_0^{\frac{\pi}{2}} a^2\cos\theta\sin^4\theta\, d\theta$$

Use the substitution $u = \sin\theta$

$\frac{du}{d\theta} = \cos\theta \Rightarrow d\theta = \frac{du}{\cos\theta}$

When $x = a = 0$, then $u = 0$. When $x = b = \frac{\pi}{2}$, then $u = 1$.

$$\therefore 6\pi\int_0^{\frac{\pi}{2}} a^2\cos\theta\sin^4\theta\, d\theta$$

$$= 6\pi\int_0^1 a^2u^4\, du$$

$$= 6\pi a^2\left[\frac{1}{5}u^5\right|_0^1$$

$$= \frac{6}{5}\pi a^2 \text{ units}^2$$ (6)

TOTAL: [30]

Exemplar examination paper

Time: 3 hours **Marks: 100**

INSTRUCTIONS AND INFORMATION

1. Answer all the questions.
2. Read all the questions carefully.
3. Number the answers according to the numbering system used in this question paper.
4. Show all intermediate steps and simplify where possible.
5. All answers must be rounded off to THREE decimals.
6. Questions may be answered in any order, but subsections of questions must be kept together.
7. Sketches must be large, neat and fully labelled.
8. Start each question on a new page.
9. Only use a black or a blue pen.
10. Write neatly and legibly.

QUESTION 1

1.1 Given: $w = 2x^2 + 4xy + 2y^2$

1.1.1 Prove that $2x\frac{\partial w}{\partial x} + 2y\frac{\partial w}{\partial y} = 4w$ (4)

1.1.2 Determine $\frac{\partial^2 w}{\partial x^2}$. (1)

1.2 If $y = pq^3$ find the percentage change in y when p increases by 3% and q decreases by 2%. (5)

[10]

QUESTION 2

Determine $\int y\,dx$ if:

2.1 $y = \ln\left(\frac{1}{x}\right)$ (1)

2.2 $y = t - x^2 - 6x$ (4)

2.3 $y = \cos^3 x - \cos^5 x$ (4)

2.4 $y = e^{-5x}\cos 5x$ (5)

2.5 $y = \tan^4 3x$ (4)

[18]

QUESTION 3

Use partial fractions to calculate the following integrals.

3.1 $\int x^3 + \frac{5 - 5x}{6x^2 + x - 1}\, dx$ (6)

3.2 $\int \frac{2x^3 + 6x^2 - 12}{x(x+3)(x^2 + 3x + 4)}\, dx$ (8)

[14]

QUESTION 4

4.1 Find the particular solution of $\frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2}y = 3x - \sin x + \cos x$ if $y(1) = 2$. (6)

4.2 Determine the general solution of $6\frac{d^2y}{dx^2} - \frac{dy}{dx} - y = 2x^2$ (6)

[12]

QUESTION 5

5.1 5.1.1 Sketch the graphs of $4x^2 + y^2 = 16$ and $x^2 + y^2 = 16$. Show the area bounded by the graphs in the first quadrant. Show a representative strip parallel to the x-axis. (3)

5.1.2 Calculate the area described in QUESTION 5.1.1. (5)

5.2 5.2.1 Find the points of intersection of two functions $y = \frac{5}{x}$ and $y = 6 - x$. Make a neat sketch of the curves and show the area bounded by the curves in the first quadrant. Show the representative strip that you would use to calculate the volume generated (using the SHELL method) if the area is rotated about the y-axis. (3)

5.2.2 Use the SHELL method to calculate the volume generated if the area between $y = \frac{5}{x}$ and $y = 6 - x$ described in QUESTION 5.2.1 is rotated about the y-axis. (5)

[16]

QUESTION 6

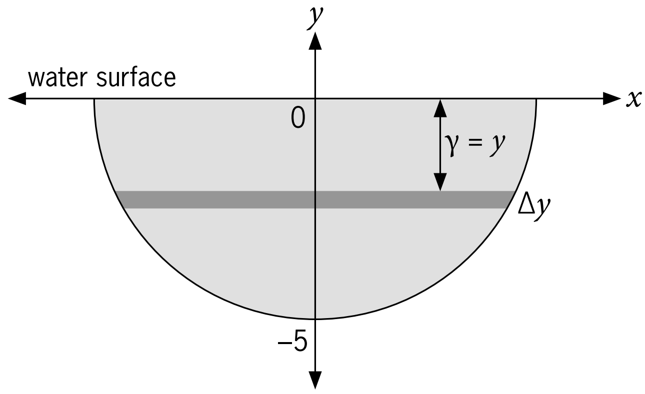
6.1 Given that the area between functions $x^2 + 4y^2 = 16$ and $x^2 + y^2 = 16$ in the first quadrant is 2π units^2. Calculate the distance of the centroid of the area from the y-axis. (5)

6.2 Calculate the x-coordinate of the centre of gravity for the solid generated about the x-axis by the area under $x^2 + y^2 = 16$ in the first quadrant. (7)

[12]

QUESTION 7

A flat plate in the shape of a semicircle is placed under water across a pipe. The straight side of the semicircle is at the water level. The plate has a radius of 5 m.



7.1 Calculate the area moment of the plate about the water level. (5)

7.2 Calculate the depth of the centre of pressure on the plate if the second moment of area of the plate about the water level is given as 245,437 m^4. (1)

[6]

QUESTION 8

8.1 Calculate the length of the curve $3y = x^3$ between (0; 0) and (3; 9). (6)

8.2 Calculate the surface area generated, when the curve $y = 3\sqrt{x}$ from $y = 0$ to $y = 4$ rotates about the x-axis. (6)

[12]

TOTAL: 100

Formula sheet

Any applicable formula may also be used.

TRIGONOMETRY

$\sin^2 x + \cos^2 x = 1$

$1 + \tan^2 x = \sec^2 x$

$1 + \cot^2 x = \operatorname{cosec}^2 x$

$\sin 2A = 2 \sin A \cos A$

$\cos 2A = \cos^2 A - \sin^2 A$

$\tan 2A = \dfrac{2 \tan A}{1 - \tan^2 A}$

$\sin^2 A = \frac{1}{2} - \frac{1}{2} \cos 2A$

$\cos^2 A = \frac{1}{2} + \frac{1}{2} \cos 2A$

$\sin (A \pm B) = \sin A \cos B \pm \sin B \cos A$

$\cos (A \pm B) = \cos A \cos B \mp \sin A \sin B$

$\tan (A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

$\sin A \cos B = \frac{1}{2} [\sin (A + B) + \sin (A - B)]$

$\cos A \sin B = \frac{1}{2} [\sin (A + B) - \sin (A - B)]$

$\cos A \cos B = \frac{1}{2} [\cos (A + B) + \cos (A - B)]$

$\sin A \sin B = \frac{1}{2} [\cos (A - B) - \cos (A + B)]$

$\tan x = \dfrac{\sin x}{\cos x}$; $\sin x = \dfrac{1}{\operatorname{cosec} x}$; $\cos x = \dfrac{1}{\sec x}$

$f(x)$	$\frac{d}{dx} f(x)$	$\int f(x)dx$
x^n	nx^{n-1}	$\dfrac{x^{n+1}}{n+1} + C \quad (n \neq -1)$
ax^n	$a \frac{d}{dx} x^n$	$a \int x^n \, dx$
e^{ax+b}	$e^{ax+b} . \frac{d}{dx}(ax + b)$	$\dfrac{e^{ax+b}}{\frac{d}{dx}(ax + b)} + C$
a^{dx+e}	$a^{dx+e} . \ln a . \frac{d}{dx}(dx + e)$	$\dfrac{a^{dx+e}}{\ln a . \frac{d}{dx}(dx + e)} + C$

$f(x)$	$\frac{d}{dx}f(x)$	$\int f(x)dx$
$\ln(ax)$	$\frac{1}{ax}.\frac{d}{dx}ax$	$x\ln ax - x + C$
$e^{f(x)}$	$e^{f(x)}\frac{d}{dx}f(x)$	–
$a^{f(x)}$	$a^{f(x)}.\ln a.\frac{d}{dx}f(x)$	–
$\ln f(x)$	$\frac{1}{f(x)}.\frac{d}{dx}f(x)$	–
$\sin ax$	$a\cos ax$	$-\frac{\cos ax}{a} + C$
$\cos ax$	$-a\sin ax$	$\frac{\sin ax}{a} + C$
$\tan ax$	$a\sec^2 ax$	$\frac{1}{a}\ln[\sec(ax)] + C$
$\cot ax$	$-a\,\text{cosec}^2\,ax$	$\frac{1}{a}\ln[\sin(ax)] + C$
$\sec ax$	$a\sec ax\tan ax$	$\frac{1}{a}\ln[\sec ax + \tan ax] + C$
$\text{cosec}\,ax$	$-a\,\text{cosec}\,ax\cot ax$	$\frac{1}{a}\ln\left[\tan\left(\frac{ax}{2}\right)\right] + C$
$\sin f(x)$	$\cos f(x).f'(x)$	–
$\cos f(x)$	$-\sin f(x).f'(x)$	–
$\tan f(x)$	$\sec^2 f(x).f'(x)$	–
$\cot f(x)$	$-\text{cosec}^2 f(x).f'(x)$	–
$\sec f(x)$	$\sec f(x)\tan f(x).f'(x)$	–
$\text{cosec}\,f(x)$	$-\text{cosec}\,f(x)\cot f(x).f'(x$	–
$\sin^{-1} f(x)$	$\frac{f'(x)}{\sqrt{1-[f(x)]^2}}$	–
$\cos^{-1} f(x)$	$\frac{-f'(x)}{\sqrt{1-[f(x)]^2}}$	–
$\tan^{-1} f(x)$	$\frac{f'(x)}{[f(x)]^2+1}$	–
$\cot^{-1} f(x)$	$\frac{-f'(x)}{[f(x)]^2+1}$	–
$\sec^{-1} f(x)$	$\frac{f'(x)}{f(x)\sqrt{[f(x)]^2-1}}$	–
$\text{cosec}^{-1} f(x)$	$\frac{-f'(x)}{f(x)\sqrt{[f(x)]^2-1}}$	–

$f(x)$	$\frac{d}{dx}f(x)$	$\int f(x)dx$
$\sin^2(ax)$	–	$\frac{x}{2} - \frac{\sin(2ax)}{4a} + C$
$\cos^2(ax)$	–	$\frac{x}{2} + \frac{\sin(2ax)}{4a} + C$
$\tan^2(ax)$	–	$\frac{1}{a}\tan(ax) - x + C$
$\cot^2(ax)$	–	$-\frac{1}{a}\cot(ax) - x + C$

$$\int f(x)\, g'(x)\, dx = f(x)\, g(x) - \int f'(x)\, g(x)\, dx$$

$$\int [f(x)]^n f'(x)\, dx = \frac{[f(x)]^{n+1}}{n+1} + C \quad (n \neq -1)$$

$$\int \frac{f'(x)}{f(x)}\, dx = \ln f(x) + C$$

$$\int \frac{dx}{\sqrt{a^2 - b^2x^2}} = \frac{1}{b}\sin^{-1}\frac{bx}{a} + C$$

$$\int \frac{dx}{a^2 + b^2x^2} = \frac{1}{ab}\tan^{-1}\frac{bx}{a} + C$$

$$\int \sqrt{a^2 - b^2x^2}\, dx = \frac{a^2}{2b}\sin^{-1}\frac{bx}{a} + \frac{x}{2}\sqrt{a^2 - b^2x^2} + C$$

$$\int \frac{dx}{a^2 - b^2x^2} = \frac{1}{2ab}\ln\left(\frac{a + bx}{a - bx}\right) + C$$

$$\int \sqrt{x^2 \pm b^2}\, dx = \frac{x}{2}\sqrt{x^2 \pm b^2} \pm \frac{b^2}{2}\ln\left[x + \sqrt{x^2 \pm b^2}\right] + C$$

$$\int \frac{dx}{\sqrt{b^2x^2 \pm a^2}} = \frac{1}{b}\ln\left[bx + \sqrt{b^2x^2 \pm a^2}\right] + C$$

APPLICATIONS OF INTEGRATION

AREAS

$A_x = \int_a^b y\, dx$; $A_x = \int_a^b (y_1 - y_2)\, dx$

$A_y = \int_a^b x\, dy$; $A_y = \int_a^b (x_1 - x_2)\, dy$

VOLUMES

$V_x = \pi\int_a^b y^2\, dx;\ V_x = \pi\int_a^b \left(y_1^2 - y_2^2\right) dx;\ V_x = 2\pi\int_a^b xy\, dy$

$V_y = \pi\int_a^b x^2\, dy;\ V_y = \pi\int_a^b \left(x_1^2 - x_2^2\right) dy;\ V_y = 2\pi\int_a^b xy\, dx$

AREA MOMENTS

$A_{m-x} = rdA \qquad A_{m-y} = rdA$

CENTROID

$\bar{x} = \frac{A_{m-y}}{A} = \frac{\int_a^b rdA}{A};\ \bar{y} = \frac{A_{m-x}}{A} = \frac{\int_a^b rdA}{A}$

SECOND MOMENT OF AREA

$I_x = \int_a^b r^2\, dA;\ I_y = \int_a^b r^2\, dA$

VOLUME MOMENTS

$V_{m-x} = \int_a^b rdV;\ V_{m-y} = \int_a^b rdV$

CENTRE OF GRAVITY

$\bar{x} = \frac{v_{m-y}}{V} = \frac{\int_a^b rdV}{V};\ \bar{y} = \frac{v_{m-x}}{V} = \frac{\int_a^b rdV}{V}$

MOMENTS OF INERTIA

Mass = Density × volume

$M = rV$

DEFINITION: $I = mr^2$

GENERAL: $I = \int_a^b r^2\, dm = \rho\int_a^b r^2\, dV$

CIRCULAR LAMINA

$I_z = \frac{1}{2} mr^2$

$I = \frac{1}{2}\int_a^b r^2\, dm = \frac{1}{2}\rho\int_a^b r^2\, dV$

$I_x = \frac{1}{2}\rho\pi\int_a^b y^4\, dx \qquad I_y = \frac{1}{2}\rho\pi\int_a^b x^4\, dy$

CENTRE OF FLUID PRESSURE

$$\bar{\bar{y}} = \frac{\int_a^b r^2 dA}{\int_a^b rdA}$$

$$\frac{f(x)}{(ax+b)^n} = \frac{A}{ax+b} + \frac{B}{(ax+b)^2} + \frac{C}{(ax+b)^3} + \dots \frac{Z}{(ax+b)^n}$$

$$\frac{f(x)}{(ax+b)^3(cx+d)^3} = \frac{A}{ax+b} + \frac{B}{(ax+b)^2} + \frac{C}{(ax+b)^3} + \frac{D}{(cx+d)} + \frac{E}{(cx+d)^2} + \frac{F}{(cx+d)^3}$$

$$\frac{f(x)}{(ax^2+bx+c)(dx+e)^n} = \frac{Ax+F}{ax^2+bx+c} + \frac{B}{dx+e} + \frac{C}{(dx+e)^2} + \ldots + \frac{Z}{(dx+e)^n}$$

$$A_x = \int_a^b 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$A_x = \int_d^c 2\pi y\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

$$A_y = \int_a^b 2\pi x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$A_y = \int_d^c 2\pi x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

$$A_x = \int_{u1}^{u2} 2\pi y\sqrt{\left(\frac{dx}{du}\right)^2+\left(\frac{dy}{du}\right)^2}\,du$$

$$A_y = \int_{u1}^{u2} 2\pi x\sqrt{\left(\frac{dx}{du}\right)^2+\left(\frac{dy}{du}\right)^2}\,du$$

$$S = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$S = \int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

$$S = \int_{u1}^{u2} \sqrt{\left(\frac{dx}{du}\right)^2+\left(\frac{dy}{du}\right)^2}\,du$$

$$\frac{dy}{dx} + Py = Q \quad \therefore ye^{\int P dx} = \int Qe^{\int P dx}\,dx$$

$$y = Ae^{r_1 x} + Be^{r_2 x}\ r_1 \neq r_2$$

$$y = e^{rx}(A + Bx) r_1 = r_2$$

$$y = e^{ax}[A\cos bx + B\sin bx] r = a \pm ib$$

$$\frac{d^2y}{dx^2} = \frac{d}{d\theta}\left(\frac{dy}{dx}\right)\frac{d\theta}{dx}$$

Exemplar examination paper memorandum

QUESTION 1

1.1 1.1.1 $\frac{\partial w}{\partial x} = 4x + 4y$✓ $\frac{\partial w}{\partial y} = 4x + 4y$✓

$$\therefore 2x\frac{\partial w}{\partial x} + 2y\frac{\partial w}{\partial y} = 2x(4x + 4y) + 2y(4x + 4y)✓$$
$$= (4x + 4y)(2x + 2y)$$
$$= 4(x + y)2(x + y)$$
$$= 8(x + y)^2$$
$$= 8\left(x^2 + 2xy + y^2\right)$$
$$= 4\left(2x^2 + 4xy + 2y^2\right)$$
$$= 4w✓$$ (4)

1.1.2 $\frac{\partial w}{\partial x} = 4x + 4y$

$\therefore \frac{\partial^2 w}{\partial x^2} = 4$✓ (1)

1.2 $\Delta y = \frac{\partial y}{\partial p}\Delta p + \frac{\partial y}{\partial q}\Delta q$✓

$\frac{\partial y}{\partial p} = q^3$✓ $\frac{\partial y}{\partial q} = 3pq^2$✓ $\Delta p = 0{,}03p$ $\Delta q = -0{,}02q$

$$\Delta y = q^3(0{,}03p) + 3pq^2(-0{,}02q)✓$$
$$= 0{,}03pq^3 - 0{,}06pq^3$$
$$= -0{,}03pq^3$$

Since $y = pq^3$, $\Delta y = -0{,}03y$

$\therefore$ The percentage decrease in y is 3%.✓ (5)

[10]

QUESTION 2

2.1 $\int \ln\left(\frac{1}{x}\right) dx = -\int \ln x \, dx$

$$= -(x \ln x - x) + C \text{ or } x - x\ln x + C✓$$

Alternative:

Using $f(x) = \ln\frac{1}{x}$; $g'(x) = 1$; $f'(x) = -\frac{1}{x}$; $g(x) = x$

$$\int \ln\left(\frac{1}{x}\right) dx = x\ln\left(\frac{1}{x}\right) - \int \frac{1}{\frac{1}{x}}\left(-\frac{1}{x^2}\right)x\,dx$$
$$= x\ln\left(\frac{1}{x}\right) - \int -\frac{1}{x}x\,dx$$
$$= x\ln\left(\frac{1}{x}\right) + \int 1\,dx$$
$$= x\ln\left(\frac{1}{x}\right) + x + C$$

$= x \ln x^{-1} + x + C$
$= x(-\ln x) + x + C$
$= x - x \ln x + C$✓ (1)

2.2 $t - x^2 - 6x = -[x^2 + 6x - t]$
$= -[x^2 + 6x + 9 - t - 9]$
$= -[(x + 3)^2 - t - 9]$
$= [(9 + t) - (x + 3)^2]$ ✓

$\int \sqrt{t - x^2 - 6x}\, dx = \int \sqrt{(9 + t) - (x + 3)^2}\, dx$✓

$= \frac{9 + t}{2} \sin^{-1} \frac{(x + 3)}{\sqrt{9 + t}} + \frac{x + 3}{2} \sqrt{[9 + t - (x + 3)^2]} + C$✓✓

or

$= \frac{9 + t}{2} \sin^{-1} \frac{x + 3}{\sqrt{9 + t}} + \frac{x + 3}{2} \sqrt{t - x^2 - 6x} + C$✓✓ (4)

2.3 $\int \cos^3 x - \cos^5 x\, dx$
$= \int \cos^3 x(1 - \cos^2 x) dx$
$= \int \cos^3 x \sin^2 x\, dx$✓
$= \int \cos x \cos^2 x \sin^2 x\, dx$
$= \int \cos x(1 - \sin^2 x) \sin^2 x\, dx$
$= \int \sin^2 x \cos x - \sin^4 x \cos x\, dx$✓
$= \frac{1}{3} \sin^3 x - \frac{1}{5} \sin^5 x + C$✓ (4)

2.4 Let: $f(x) = e^{-5x}$; $g'(x) = \cos 5x$; $f'(x) = -5e^{-5x}$; $g(x) = \frac{\sin 5x}{5}$

$\int e^{-5x} \cos 5x\, dx = e^{-5x} \frac{\sin 5x}{5} - \int -5e^{-5x} \frac{\sin 5x}{5} dx$✓✓
$= e^{-5x} \frac{\sin 5x}{5} + \int e^{-5x} \sin 5x\, dx$
$= e^{-5x} \frac{\sin 5x}{5} + e^{-5x}.-\frac{\cos 5x}{5} - \int -5e^{-5x}.-\frac{\cos 5x}{5} dx$✓
$= \frac{1}{5} e^{-5x} \sin 5x - \frac{1}{5} e^{-5x} \cos 5x - \int e^{-5x} \cos 5x\, dx$

$2\int e^{-5x} \cos 5x\, dx = \frac{1}{5} e^{-5x} \sin 5x - \frac{1}{5} e^{-5x} \cos 5x$✓

$\int e^{-5x} \cos 5x\, dx = \frac{1}{10} e^{-5x} \sin 5x - \frac{1}{10} e^{-5x} \cos 5x + C$✓

or

$= \frac{1}{10} e^{-5x} [\sin 5x - \cos 5x] + C$✓ (5)

2.5 $\int \tan^4 3x\, dx$

$= \int \tan^2 3x \tan^2 3x\, dx$

$= \int \tan^2 3x(\sec^2 3x - 1)\, dx$✓

$= \int (\tan^2 3x \sec^2 3x - \tan^2 3x)\, dx$

$= \int (\tan^2 3x \sec^2 3x\, dx - \int \tan^2 3x\, dx$✓

$= \frac{1}{3}\frac{\tan^2 3x}{3} - \frac{1}{3}\tan 3x + x + C$✓✓ (4)

[18]

QUESTION 3

3.1 $6x^2 + x - 1 = (3x - 1)(2x + 1)$

$\frac{5 - 5x}{6x^2 + x - 1} = \frac{A}{3x - 1} + \frac{B}{2x + 1}$✓

$5 - 5x = A(2x + 1) + B(3x - 1)$

When $x = -\frac{1}{2}$ then $B = -3$✓

When $x = \frac{1}{3}$ then $A = 2$✓

$\int x^3 + \frac{5 - 5x}{6x^2 + x - 1}\, dx = \int x^3 + \frac{2}{3x - 1} + \frac{-3}{2x + 1}\, dx$ ✓

$= \frac{x^4}{4} + \frac{2}{3}\ln|3x - 1| - \frac{3}{2}\ln|2x + 1| + C$✓✓ (6)

3.2 $\frac{2x^3 + 6x^2 - 12}{x(x + 3)(x^2 + 3x + 4)} = \frac{A}{x} + \frac{B}{x + 3} + \frac{Cx + D}{x^2 + 3x + 4}$✓

$2x^3 + 6x^2 - 12 = A(x + 3)(x^2 + 3x + 4) + Bx(x^2 + 3x + 4) + (Cx + D)x(x + 3)$

If $x = -3$: $-54 + 54 - 12 = B(-3)(9 - 9 + 4) - 12 = -12B \quad \therefore B = 1$✓

If $x = 0$: $-12 = A.3.4 \quad \therefore A = -1$✓

Equate x^3: $A + B + C = 2 \Rightarrow -1 + 1 + C = 2 \quad \therefore C = 2$✓

If $x = 1$: $2 + 6 - 12 = A(1 + 3)(1 + 3 + 4) + B.1.(1 + 3 + 4) + (C.1 + D).1.(1 + 3)$

$-4 = 32A + 8B + 4C + 4D \ldots (1)$

Substituting $A = -1$; $B = 1$; $C = 2$: $-4 = 32(-1) + 8.1 + 4.2 + 4D \quad \therefore D = 3$✓

$\therefore \int \frac{2x^3 + 6x^2 - 12}{x(x + 3)(x^2 + 3x + 4)}\, dx = \int \frac{-1}{x}\, dx + \int \frac{1}{x + 3}\, dx + \int \frac{2x + 3}{x^2 + 3x + 4}\, dx$✓

$= -\ln x + \ln|x + 3| + \ln|x^2 + 3x + 4| + C$✓✓ (8)

[14]

QUESTION 4

4.1 $\frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2}y = 3x - \sin x + \cos x$

$\frac{dy}{dx} - \frac{1}{x}y = x(3x - \sin x + \cos x)$✓

$e^{\int P\,dx} = e^{\int -\frac{1}{x}dx}$

$= e^{-\ln x}$

$= e^{\ln x^{-1}}$

$= x^{-1} = \frac{1}{x}$

$\int Qe^{\int P\,dx}\,dx = \int x(3x - \sin x + \cos x)\frac{1}{x}dx$✓

$= \int 3x - \sin x + \cos x\,dx$

$= \frac{3}{2}x^2 + \cos x + \sin x$

$\frac{y}{x} = \frac{3}{2}x^2 + \cos x + \sin x + C$✓

At $y(1) = 2$ the solution is:

$\frac{2}{1} = \frac{3}{2}(1)^2 + \cos(1) + \sin(1) + C$

$\therefore C = -0{,}882$

$\therefore \frac{3}{2}x^2 + \cos x + \sin x - 0{,}882$

The particular solution is $\frac{y}{x} = \frac{3}{2}x^2 + \cos x + \sin x - 0{,}882$✓✓ (6)

4.2 $6\frac{d^2y}{dx^2} - \frac{dy}{dx} - y = 2x^2$

$6r^2 - r - 1 = 0$

$(3r + 1)(2r - 1) = 0$

$r_1 = -\frac{1}{3};\ r_2 = \frac{1}{2}$✓

$y_c = Ae^{-\frac{1}{3}x} + Be^{\frac{1}{2}x}$✓

$y = Cx^2 + Dx + E$✓

$\frac{dy}{dx} = 2Cx + D$

$\frac{d^2y}{dx^2} = 2C$

$6(2C) - (2Cx + D) - \left(Cx^2 + Dx + E\right) = 2x^2$

$12C - 2Cx - D - Cx^2 - Dx - E = 2x^2$

x^2: $-C = 2 \quad \therefore C = -2$

x: $-2C - D = 0 \quad \therefore D = 4$

$12C - D - E = 0 \quad \therefore E = -28$

$y_p = -2x^2 + 4x - 28$✓

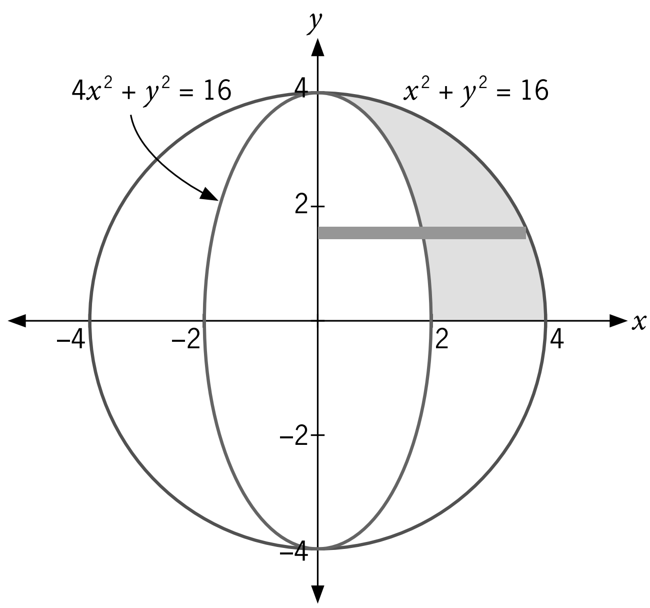
$y = y_c + y_p$

$\therefore y = Ae^{-\frac{1}{3}x} + Be^{\frac{1}{2}x} - 2x^2 + 4x - 28$✓✓ (6)

[12]

QUESTION 5

5.1 5.1.1



Marks awarded for: labelling circle and ellipse; *x*- or *y*-intercept of circle; *y*-intercept of ellipse.✓

Correct shading of area✓

Correct representative strip✓ (3)

5.1.2 $4x^2 + y^2 = 16$ and $x^2 + y^2 = 16$

$\therefore x = \frac{1}{2}\sqrt{16 - y^2}$ and $x = \sqrt{16 - y^2}$✓

$\text{Area} = \int_a^b x_1 - x_2 \, dy$

$= \int_0^4 \sqrt{16 - y^2} - \frac{1}{2}\sqrt{16 - y^2} \, dy$✓

$= \frac{1}{2}\int_0^4 \sqrt{16 - y^2} \, dy$

$= \frac{1}{2}\left[\frac{16}{2} \sin^{-1}\frac{y}{4} + \frac{y}{2}\sqrt{16 - y^2}\right]_0^4$✓

$= \frac{1}{2}\left[8 \sin^{-1}\frac{4}{4} + \frac{4}{2}\sqrt{16 - 4^2} - \left\{8 \sin^{-1}\frac{0}{4} + \frac{0}{2}\sqrt{16 - 0^2}\right\}\right]$

$= \frac{1}{2}[8 \sin^{-1} 1] = 6{,}283 = 2\pi \text{ units}^2$✓✓

(5)

5.2 5.2.1 Find the points of intersection:

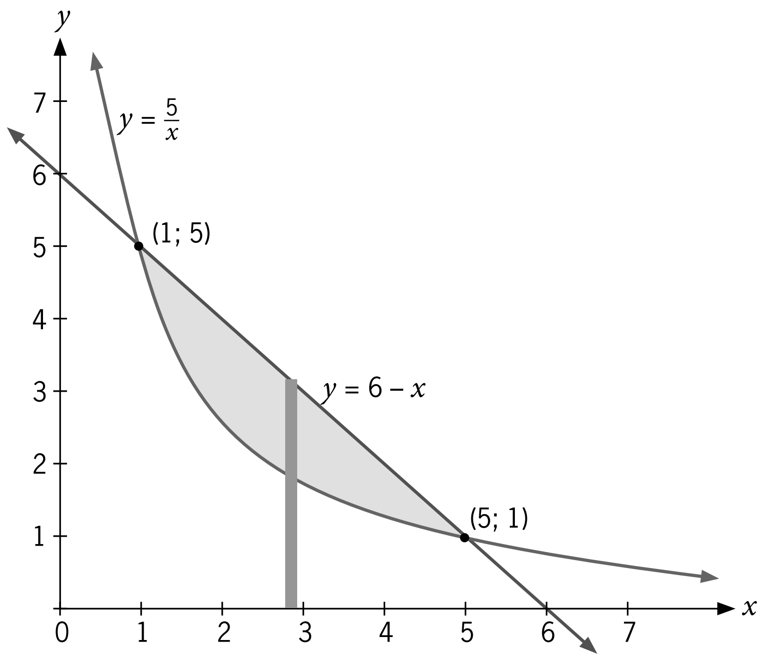
$\frac{5}{x} = 6 - x$

$x^2 - 6x + 5 = 0$

$(x - 5)(x - 1) = 0$

$\therefore x = 5$ or $x = 1$; $y = 1$ or $y = 5$

Points of intersection: (5; 1); (1; 5)



Marks awarded for: labelling line and curve; points of intersection.✓

Correct shading of area✓

Correct representative strip✓ (3)

5.2.2 $V_y = 2\pi\int_1^5 x[f(x) - g(x)]\ dx$

$= 2\pi\int_1^5 x\left[\left(6 - x\right) - \left(\frac{5}{x}\right)\right]\ dx$✓

$= 2\pi\int_1^5 -x^2 + 6x - 5\ dx$✓

$= 2\pi\left[-\frac{x^3}{3} + 6\frac{x^2}{2} - 5x\right]_1^5$✓

$= 2\pi\left[\left(-\frac{125}{3} + 75 - 25\right) - \left(-\frac{1}{3} + 3 - 5\right)\right]$

$= \frac{64}{3}\pi = 21{,}333\pi = 67{,}021$ units^3✓✓ (5)

[16]

QUESTION 6

6.1 $\bar{x} = \dfrac{A_{m-y}}{A}$

$A_{m-y} = \int_a^b r\,dA$

$= \int_0^4 x\left[\sqrt{16 - x^2} - \frac{1}{2}\sqrt{16 - x^2}\right] dx$ ✓

$= \frac{1}{2}\int_0^4 x\sqrt{16 - x^2}\, dx$

$= \frac{1}{2}\left(-\frac{1}{2}\right)\int_0^4 -2x(16 - x^2)^{\frac{1}{2}}\, dx$

$= -\frac{1}{4}\left[\dfrac{(16 - x^2)^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^4$ ✓

$= -\frac{1}{6}\left[(16 - x^2)\right]_0^4$

$= -\frac{1}{6}\left[(16 - 4^2)^{\frac{3}{2}} - (16 - 0^2)^{\frac{3}{2}}\right]$

$= \frac{32}{3}$ units^2 or 10,667 units^2

$\bar{x} = \dfrac{A_{m-y}}{A}$ ✓

$\therefore \bar{x} = \frac{32}{3} \div 2\pi = \frac{16}{3}\pi = 1{,}698$ units

The centroid is 1,698 units from the y-axis. ✓✓ (5)

6.2 $x^2 + y^2 = 16$ so $y^2 = 16 - x^2$

$\bar{x} = \dfrac{v_{m-y}}{V}$

$V_{m-y} = \pi\int_a^b x\left(y_1^2 - y_2^2\right) dx$ ✓

$= \pi\int_0^4 x(16 - x^2)\, dx$

$= \pi\int_0^4 16x - x^3\, dx$

$= \pi\left[\frac{16}{2}x^2 - \frac{x^4}{4}\right]_0^4$ ✓

$= 64\pi = 201{,}062$ units^4 ✓

$V_x = \pi\int_a^b y_1^2 - y_2^2\, dx$ ✓

$= \pi\int_0^4 16 - x^2\, dx$

$= \pi\left[16x - \frac{x^3}{3}\right]_0^4$

$$= \pi\left[16(4) - \frac{(4)^3}{3}\right]$$

$$= \frac{128}{3}\pi = 134{,}041 \text{ units}^3 ✓$$

$$\therefore \bar{x} = \frac{201{,}062}{134{,}041} = \frac{3}{2} = 1{,}5 \text{ units} ✓✓$$ (7)

[12]

QUESTION 7

7.1 The curve is given by $x^2 + y^2 = 25 \Rightarrow x = \sqrt{25 - y^2}$

First moment of area:

$$\int_a^b r\, dA = \int_a^b y\, 2x\, dy ✓$$

$$= \int_{-5}^{0} y\, 2\sqrt{25 - y^2}\, dy$$

$$= -\int_{-5}^{0} -y\, 2(25 - y^2)^{\frac{1}{2}}\, dy ✓$$

$$= -\left[\frac{(25 - y^2)^{\frac{3}{2}}}{\frac{3}{2}}\right]_{-5}^{0} ✓$$

$$= -\frac{2}{3}\left[(25 - y^2)^{\frac{3}{2}}\right]_{-5}^{0}$$

$$= -\frac{2}{3}\left[(25)^{\frac{3}{2}} - \{25 - (-5)^2\}^{\frac{3}{2}}\right] ✓$$

$$= -\frac{2}{3}(25)^{\frac{3}{2}} = -83\frac{1}{3} = -83{,}333 \text{ m}^3 ✓$$ (5)

7.2 $\bar{\bar{y}} = \frac{245{,}437 \text{ m}^4}{-83{,}333 \text{ m}^3} = -2{,}945 \text{ m} ✓$ (1)

[6]

QUESTION 8

8.1 $3y = x^3 \quad \therefore \frac{dy}{dx} = x$

$$\left[\frac{dy}{dx}\right]^2 = x^2;\ 1 + \left[\frac{dy}{dx}\right]^2 = 1 + x^2 ✓$$

$$S = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dy ✓$$

$$= \int_0^3 \sqrt{1 + x^2}\, dy ✓$$

$$= \left[\frac{x}{2}\sqrt{1 + x^2} + \frac{1}{2}\ln\left(x + \sqrt{1 + x^2}\right)\right]_0^3 ✓$$

$$\left[\frac{3}{2}\sqrt{1+3^2}+\frac{1}{2}\ln\left(3+\sqrt{1+3^2}\right)-\left\{0+\frac{1}{2}\ln\left(0+\sqrt{1+0}\right)\right\}\right]✓$$

$$=\left[\frac{3\sqrt{10}}{2}+\frac{\ln\left(3+\sqrt{10}\right)}{2}-\frac{\ln(1)}{2}\right]$$

$$=\left[\frac{3\sqrt{10}}{2}+\frac{\ln\left(3+\sqrt{10}\right)}{2}\right]$$

$= 2{,}034$ units✓ (6)

8.2 $y = 3\sqrt{x}; \quad \therefore x = \frac{1}{9}y^2 \therefore \frac{dx}{dy} = \frac{2}{9}y$

$$\left[\frac{dx}{dy}\right]^2 = \left(\frac{2}{9}y\right)^2;$$

$$1+\left[\frac{dx}{dy}\right]^2 = 1+\left(\frac{2}{9}y\right)^2 ✓$$

$$= 1+\frac{4y^2}{81}$$

$$= \frac{81+4y^2}{81}$$

$$\therefore A_x = 2\pi\int_0^4 y\frac{\sqrt{81+4y^2}}{9}\,dy✓$$

$$= \frac{2}{9}\pi\int_{81}^{145} yu^{\frac{1}{2}}\frac{du}{8y}✓$$

$$= \left(\frac{2}{9}\right)\left(\frac{1}{8}\right)\pi\int_{81}^{145} u^{\frac{1}{2}}\,du✓$$

$$= \frac{1}{36}\pi\left[\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right]_{81}^{145} ✓$$

$$= \left(\frac{1}{36}\right)\left(\frac{2}{3}\right)\pi\left[u^{\frac{3}{2}}\right]_{81}^{145}$$

$$= \frac{1}{54}\pi\left[145^{\frac{3}{2}} - 81^{\frac{3}{2}}\right]$$

$= 18{,}834\pi = 59{,}168$ units^2✓ (6)

[12]

TOTAL: 100

Glossary

A

Antiderivative – reversing the process of differentiation, the indefinite integral

Arbitrary – not specific, based on random choice rather than a specific system

Arc length (curve length) – the distance between two points along a curve

Auxiliary equation – a polynomial equation obtained by making an algebraic substitution of the derivatives in a homogeneous differential equation

C

Candidate – a reasonable choice, a good guess

Cartesian plane – a coordinate system in two-dimensions defined by a horizontal x-axis and a vertical y-axis

Centre of mass (centre of gravity) – the unique point at the centre of a distribution of mass in space; the geometric centre of a line, area or volume

Centre of pressure – the vertical depth at which the net force due to fluid pressure acts (in m)

Centroid – the centre point of a geometric object of uniform density; the gravitational centre of a line, area or volume

Complementary function – the solution to a homogeneous differential equation; also known as the characteristic equation

Complex number – the sum of a real and an imaginary number: $n = a + bi$; $n \in C$

Curve length (arc length) – the distance between two points along a curve

D

Density – the quantity of mass per unit volume

Differential equation – an equation that describes the relationship between a function and its derivatives

Discs (also disk) – thin round objects

F

First derivative – differentiating a function once with respect to a variable

First order differential equation – differential equation that contains only first derivatives, $\frac{dy}{dx}$

Fluid pressure – the force applied per unit area due to the weight of a fluid (in Pascal (Pa))

G

General solution – the sum of the complementary and particular functions resulting in a function (or set of functions) that satisfies the differential equation

Geometry – the relationship of points, lines, angles, surfaces and solids

H

Homogeneous differential equation– involves only functions and derivatives of *y*, and equals 0

I

Imaginary number – the square root of a negative number, the product of a real number *b* and the imaginary unit *i*, where $i^2 = -1$

Inertia – the resistance to movement or change in movement

Infinitesimal – an extremely small value, approaching zero

Initial conditions – starting values of the variables: x_0; y_0; y'_0

Integrand – the function to integrate

Irreducible factor – a factor that cannot be factorised (reduced) further; the algebraic equivalent of a prime number

L

Lamina – a two-dimensional surface in a plane which has both mass and surface density

Linear differential equation – a differential equation where the function and its derivatives are added (or subtracted) together and the dependent variable is of the first degree

Linear factor – a first degree polynomial, $ax + b$

Lowest common denominator – the smallest expression divisible by all the terms in question

M

Mixed partial derivative – a higher order partial derivative with respect to two or more variables

Moment of inertia – an indication of the resistance of an object to rotation; also called the second moment of mass (in $kg.m^2$)

N

Non-homogeneous differential equation – involves functions of *x* and constants, in addition to functions and derivatives of *y*

O

Order (of differential equations) – the order of the highest derivative in the differential equation

P

Parabola – a symmetrical and roughly U-shaped curve that is described by a quadratic function

Parameter – an independent variable, say θ, of which more than one dependent variables, say *x* and *y*, are functions

Parametric equations –two or more equations that involve the same independent variable or parameter to express a function

Partial derivatives – the partial derivative of a function of several variables is the derivative with respect to one of the variables, while the other variables are held constant

Partial fraction decomposition – a method to express a rational fraction as the sum of several rational fractions with simpler denominators

Particular function (solution) – the function that is obtained when particular values are assigned to the arbitrary constants in the general solution of a differential equation

Q

Quadratic factor – a polynomial with a degree of 2, such as $ax^2 + bx + c$

R

Rational fraction – a fraction where the numerator and denominator are both polynomials

Real number – any rational or irrational number n, where $n \in \mathrm{R}$

Recursive – repeated

Reference axis – the axis about which the second moment of area (or other quantity) is calculated

Rigid – resistant to change, does not bend or twist easily

Root – a solution of an algebraic equation; the x-intercept

S

Second derivative – the derivative of a derivative

Second moment of area – an indication of the resistance of a given shape to bending or torsion; measured in m^4

Second order differential equation – differential equation that includes a second derivative, $\frac{d^2y}{dx^2}$

Solids of revolution –a solid form obtained by rotating a plane curve around some straight line that lies on the same plane

Steel profiles – products such as beams, T-sections, U-sections, angles, bars made of steel

Surface of revolution –surface formed when a curve (function) is rotated about a line (axis)

T

Theorem – a truth established by means of accepted truths

Torsion – twisting of a body that has some resistance, as the result of an applied force

Trigonometric identity – equality that involves trigonometric functions where both sides of the equality are defined for all values of the variables

U

Unique solution – the general solution where all the coefficients have numerical values; sometimes called the particular solution

V

Vertices (singular: vertex) – point(s) where two or more straight lines meet