N5

Mathematics

Lecturer Guide

Sparrow Consulting

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CONTENTS

1. Subject aims

1.1 General subject aims

Mathematics N5 aims to provide learners with the skills to identify and calculate mathematical problems in N5 and the content forms part of engineering calculation problems from industry.

Furthermore, Mathematics N5 will equip students with the relevant knowledge to enable them to integrate meaningfully into their trade subjects and also serve as the foundation for the Mathematics N6 syllabus in order to achieve a national diploma.

Upon completion of this subject, the student should be able to:
- apply the necessary knowledge of Mathematics to various engineering fields in their respective working environments;
- apply higher cognitive skills pertaining to application, analysis, synthesis and evaluation, and logical and critical thought processes;
- apply their understanding in the interpretation of real world problems;
- promote Mathematics as a tool to be used to trouble shoot in different fields of study; and
- calculate using certain theorems, the proofs of which are not examinable.

1.2 Specific subject aims

The specific aims of Mathematics N5 is to continue with the study of Differential and Integral Calculus and serve as a prerequisite for Mathematics N6.

Mathematics N5 strives to assist students to obtain trade-specific calculation knowledge.

Other specific aims of Mathematics N5 also include:
- Promote correct mathematical terminology.
- Promote and focus on word problems and the problem solving thereof, in order to prepare the students for their relevant careers.
- Use technology in Mathematics and apply Mathematics to further technology.

2. Admission requirements

For admission to N5 Mathematics, a student must have passed N4 Mathematics.

3. Duration of course

The duration of the subject is one trimester on full-time, part-time or distance-learning mode.

4. Evaluation

Candidates must be evaluated continually as follows:

4.1 ICASS Trimester Mark
- Assessment marks are valid for a period of one year and are referred to as ICASS Trimester marks.
- A minimum of 40% is required for a student to qualify for entry to the final examination.
- Two formal class tests for full-time and part-time students (or two assignments for distance-learning students only).

4.2 Calculation of trimester mark will be as follows:
- weight of test or assignment 1 = 30% of the syllabus; and
- weight of test or assignment 2 = 70% of the syllabus.

5. Examination

A final examination will be conducted in April, August and November of each year. The pass requirement is 40%.

The final examination will consist of 100% of the syllabus

The duration of the final examination will be 3 hours.

The final examination will be a closed book examination.

Minimum pass percentage will be 40%.

Assessments will be based on the cognitive domain of Bloom's Taxonomy, that is remember, understand, apply, analyse, evaluate, and create.

The division of these aspects are as follows:

Remember	Understand	Apply	Analyse	Evaluate	Create
20%	20%	20%	10%	20%	10%

6. General information

Problems should be based on real world scenarios allowing students to relate theory to practice.

Emphasis of correct mathematical terminology should be encouraged and promoted at all times.

A systematic approach to problem solving should be adhered to.

Students should be encouraged to understand rather than memorise the basic formulae applicable to N5 Mathematics.

Calculators may be used to do mathematical calculations.

Answers to all calculations must be approximated correctly to three decimal places, unless otherwise stated. Unless otherwise stated, approximations may not be done during calculations. The final answer must be approximated to the stipulated degree of accuracy.

The weight value of a module gives an indication of the time to be spent on teaching the module as well as the relative percentage of the total marks allocated to the module in the final examination (1 mark = 1,8 minutes).

7. Subject matter

Mathematics N5 strives to assist students to obtain trade-specific calculation knowledge. Students should be able to acquire in-depth knowledge of the following content:

Module	Weighted value
1. Limits and continuity	6
2. Differentiation	22
3. Applications of differentiation	20
4. Integration techniques	24
5. The definite integral	6
6. Areas and volumes	8
7. Second moment of inertia and moment of inertia (second moment of mass)	6
8. Differential equations	8
Total	**100**

8. Workschedule

Week	Module	Topic	Activities	Hours
1	**Module 1** Limits and continuity	1.1 Limits (Revision) 1.2 L'Hôpital's rule 1.3 Continuity	Activity 1.1 Activity 1.2 Activity 1.3 Activity 1.4 Activity 1.5 Summative assessment: Module 1	6 hours
1–3	**Module 2** Differentiation	2.1 What is differentiation? 2.2 Differentiation from first principles 2.3 Differentiation techniques	 Activity 2.1 Activity 2.2 Activity 2.3 Activity 2.4 Activity 2.5 Activity 2.6 Activity 2.7 Activity 2.8 Summative assessment: Module 2	22 hours
4–5	**Module 3** Applications of differentiation	3.1 Newton's method 3.2 Optimisation using maximum and minimum values 3.3 Rates of change 3.4 Related rates	Activity 3.1 Activity 3.2 Activity 3.3 Activity 3.4 Summative assessment: Module 3	20 hours

Week	Module	Topic	Activities	Hours
6–7	**Module 4** Integration techniques	4.1 Basic integration 4.2 Integration by inspection 4.3 Integration by means of algebraic substitution 4.4 Integration of trigonometric functions 4.5 Integration of algebraic fractions 4.6 Integration by using partial fractions 4.7 Integration by parts	Activity 4.1 Activity 4.2 Activity 4.3 Activity 4.4 Activity 4.5 Activity 4.6 Summative assessment: Module 4	24 hours
7–9	**Module 5** The definite integral	5.1 Basic definite integrals 5.2 Change of limits 5.3 Infinity as a limit 5.4 Laplace transform	Activity 5.1 Activity 5.2 Activity 5.3 Activity 5.4 Summative assessment: Module 5	6 hours
	Module 6 Areas and volumes	6.1 Area of a function 6.2 Volume of a solid of revolution	Activity 6.1 Activity 6.2 Activity 6.3 Activity 6.4 Activity 6.5 Activity 6.6 Summative assessment: Module 6	8 hours

Week	Module	Topic	Activities	Hours
	Module 7 Second moment of area and moment of inertia (second moment of mass)	7.1 Sketching lamina on a given interval with respect to an axis 7.2 Second moment of area 7.3 Moment of inertia	 Activity 7.1 Activity 7.2 Summative assessment: Module 7	6 hours
10	**Module 8** Differential equations	8.1 Distinguish between first order and second order differential equations 8.2 First order differential equations 8.3 Second order differential equations	 Activity 8.1 Activity 8.2 Summative assessment: Module 8	8 hours
TOTAL				**100 hours**

MODULE

1 *Limits and continuity*

After they have completed this module, students should be able to:

- apply L'Hôpital's rule to indeterminate functions by differentiating the numerator and denominator;
- state the conditions for continuity; and
- determine whether a function is continuous or discontinuous at a specific point.

Introduction

The Mathematics N5 curriculum is the study of calculus, which is a way of determining how quantities change. Many engineering problems depend on the use of calculus. For example, we may need to calculate how quickly an aeroplane accelerates, how a sphere's volume increases as it is inflated, or how quickly heat is transferred between two materials. Each of these examples is based on a specific quantity (acceleration, volume, or temperature) changing. This means they can be studied using calculus.

Calculus includes two concepts which work in opposite ways, namely differentiation and integration.

- Differentiation deals with finding the rate of change of a function.
- Integration deals with finding a function if we know the rate of change.

During this course students will learn about these two basic concepts underlying calculus.

This module focusses on limits and continuity. Calculus is based on determining very small changes in functions, which can be defined in terms of limits. To differentiate or integrate a function, the function must be continuous. Students will learn more about continuity in section 1.3.

Students need the following pre-knowledge to successfully complete this module.

Pre-knowledge

Students should already know how to evaluate limits.

- By substitution:

  $\lim_{x \to 0} (x^2 - 4) = (0^2 - 4) = -4$

- By factoring, including limits that are indeterminate:

  $\lim_{x \to 4} \frac{x^2 - 16}{x - 4} = \lim_{x \to 4} \frac{(x - 4)(x + 4)}{x - 4} = \lim_{x \to 4} (x + 4) = 8$

- By using conjugates to eliminate square roots:

  $\lim_{x \to 0} \frac{\sqrt{x^2 + 9} - 3}{x^2}$

  $= \lim_{x \to 0} \frac{\sqrt{x^2 + 9} - 3}{x^2} . \frac{\sqrt{x^2 + 9} + 3}{\sqrt{x^2 + 9} + 3}$

  $= \lim_{x \to 0} \frac{1}{\sqrt{x^2 + 9} + 3} = \frac{1}{6}$

- By using the properties of limits for $\lim_{x \to a} f(x) = \mathrm{L}$ and $\lim_{x \to a} g(x) = \mathrm{M}$:

Table 1.1: Properties of limits

Property	Rule
Sum of functions	The limit of the sum of two functions is the sum of their limits. $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = \mathrm{L} + \mathrm{M}$
Difference of functions	The limit of the difference of two functions is the difference of their limits. $\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x) = \mathrm{L} - \mathrm{M}$
Product of functions	The limit of a product of two functions is the product of their limits. $\lim_{x \to a} [f(x) \times g(x)] = \lim_{x \to a} f(x) \times \lim_{x \to a} g(x) = \mathrm{L} \times \mathrm{M}$
Constant multiple	The limit of a constant times a function is the constant times the limit of the function. $\lim_{x \to a} [k \times f(x)] = k \times \lim_{x \to a} f(x) = k \times \mathrm{L}$
Rational function	The limit of a quotient of two functions is the quotient of their limits, provided the limit of the denominator is not zero. $\lim_{x \to a} \left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{\mathrm{L}}{\mathrm{M}},\ \mathrm{M} \neq 0$

Activity 1.1 SB page 8

1. $\lim\limits_{x \to 1} \frac{2}{x-2} = \frac{2}{1-2} = -2$

2. $\lim\limits_{x \to 2} (x + x^2) = 2 + 2^2 = 6$

3. $\lim\limits_{x \to \infty} (e^{-x} + 2) = 0 + 2 = 2$

4. $\lim\limits_{x \to 3} \frac{x-3}{x^2 - 2x - 3}$

   $= \lim\limits_{x \to 3} \frac{x-3}{(x-3)(x+1)} = \lim\limits_{x \to 3} \frac{1}{x+1}$

   $= \frac{1}{4}$

5. $\lim\limits_{x \to \infty} x - \sqrt{x^2 + 4x}$

   $= \lim\limits_{x \to \infty} \left(x - \sqrt{x^2 + 4x}\right) . \frac{x + \sqrt{x^2 + 4x}}{x + \sqrt{x^2 + 4x}}$

   $= \lim\limits_{x \to \infty} \frac{x^2 - x^2 + 4x}{x + \sqrt{x^2 + 4x}}$

   $= \lim\limits_{x \to \infty} \frac{4x}{x + \sqrt{x^2 + 4x}}$

   $= \lim\limits_{x \to \infty} \frac{4}{1 + \sqrt{1 + \frac{4}{x}}}$

   $= \frac{4}{1 + \sqrt{1 + 0}} = 2$

6. $\lim\limits_{x \to 4} \frac{x^2 - x + 20}{x - 4}$

   $= \lim\limits_{x \to 4} \frac{(x-4)(x+5)}{x-4}$

   $= \lim\limits_{x \to 4} (x + 5)$

   $= 9$

Activity 1.2 SB page 12

1. $\lim\limits_{x \to 0} \frac{\sin x}{x}$
   - $\left[\frac{0}{0}\right]$

   $= \lim\limits_{x \to 0} \frac{\cos x}{1}$
   - L'Hôpital

   $= \frac{\cos 0}{1}$
   - Substitute

   $= 1$

2. $\lim\limits_{x \to 1} \dfrac{\ln x}{2(x-1)}$ • $\left[\dfrac{0}{0}\right]$

$= \lim\limits_{x \to 1} \dfrac{\frac{1}{x}}{2}$ • L'Hôpital

$= \dfrac{1}{2(1)}$ • Substitute

$= \dfrac{1}{2}$

3. $\lim\limits_{x \to \frac{\pi}{2}} \dfrac{\frac{\pi}{2} - x}{\cos x}$ • $\left[\dfrac{0}{0}\right]$

$= \lim\limits_{x \to \frac{\pi}{2}} \dfrac{-1}{-\sin x}$ • L'Hôpital

$= \dfrac{1}{\sin\frac{\pi}{2}}$ • Substitute

$= 1$

4. $\lim\limits_{x \to 0} \dfrac{2\sin x - \sin 2x}{x - \sin x}$

$= \dfrac{2\sin 0 - \sin(2.0)}{0 - \sin 0} = \left[\dfrac{0}{0}\right]$ • Indeterminate

$= \lim\limits_{x \to 0} \dfrac{2\cos x - 2\cos 2x}{1 - \cos x}$ • L'Hôpital

$= \dfrac{2\cos 0 - 2\cos(2.0)}{1 - \cos 0} = \dfrac{2-2}{1-1} = \left[\dfrac{0}{0}\right]$ • Substitute but still indeterminate

$= \lim\limits_{x \to 0} \dfrac{-2\sin x + 4\sin 2x}{\sin x}$ • L'Hôpital

$= \dfrac{-2\sin 0 + 4\sin 2.0}{\sin 0} = \dfrac{-2.0 + 4.0}{0} = \left[\dfrac{0}{0}\right]$ • Substitute but still indeterminate

$= \lim\limits_{x \to 0} \dfrac{-2\cos x + 8\cos 2x}{\cos x}$ • L'Hôpital

$= \dfrac{-2\cos 0 + 8\cos 2.0}{\cos 0} = \dfrac{-2.1 + 8.1}{1} = \left[\dfrac{6}{1}\right]$ • Substitute

$= 6$

5. $\lim\limits_{x \to 1} \dfrac{x^2 - 1}{\ln x}$

$= \lim\limits_{x \to 1} \dfrac{2x}{\frac{1}{x}}$

$= \lim\limits_{x \to 1} 2x^2$

$= 2$

6. $\lim_{x\to 0}\frac{x - \cos x}{3 \sin x}$

$= \lim_{x\to 0}\frac{1 + \sin x}{3 \cos x}$

$= \frac{1}{3}$

Activity 1.3 SB page 14

1. $\lim_{x\to\infty}\frac{x^2 - 5x}{x^3}$	• $\left[\frac{\infty}{\infty}\right]$
$= \lim_{x\to\infty}\frac{x - 5}{x^2}$	• Simplify $\left[\frac{\infty}{\infty}\right]$
$= \lim_{x\to\infty}\frac{1}{2x}$	• L'Hôpital
$= 0$	• Substitute
2. $\lim_{x\to\infty}\frac{8x^2}{e^x}$	• $\left[\frac{\infty}{\infty}\right]$
$= \lim_{x\to\infty}\frac{16x}{e^x}$	• L'Hôpital $\left[\frac{\infty}{\infty}\right]$
$= \lim_{x\to\infty}\frac{16}{e^x}$	• L'Hôpital
$= 0$	• Substitute
3. $\lim_{x\to\infty}\frac{2x^2 - 6}{x^2 + 4}$	• $\left[\frac{\infty}{\infty}\right]$
$= \lim_{x\to\infty}\frac{4x}{2x}$	• L'Hôpital
$= \lim_{x\to\infty} 2$	• Simplify
$= 2$	• Substitute
4. $\lim_{x\to\infty}\frac{e^{2x}}{2x^2}$	• $\left[\frac{\infty}{\infty}\right]$
$= \lim_{x\to\infty}\frac{2e^{2x}}{4x}$	• L'Hôpital $\left[\frac{\infty}{\infty}\right]$
$= \lim_{x\to\infty}\frac{4e^{2x}}{4}$	• L'Hôpital
$= \lim_{x\to\infty} e^{2x}$	• Simplify
$= \infty$	• Substitute

5. $\lim_{x\to\infty}\frac{e^{x+3}}{\ln(x + 3)}$

$= \lim_{x\to\infty}\frac{e^{x+3}}{\frac{1}{x+3}}$

$= \lim_{x\to\infty}(x + 3)e^{x+3}$

$= \infty$

6. $\lim\limits_{x\to\infty} \frac{3x^2}{5x - x^2}$

$= \lim\limits_{x\to\infty} \frac{6x}{5 - 2x}$

$= \lim\limits_{x\to\infty} \frac{6}{-2}$

$= -3$

Activity 1.4 SB page 15

1. $\lim\limits_{x\to 0^+} x^2 \ln x$ • $[0.\infty]$

$= \lim\limits_{x\to 0^+} \frac{\ln x}{x^{-2}}$ • Rearrange $\left[\frac{\infty}{\infty}\right]$

$= \lim\limits_{x\to 0^+} \frac{\frac{1}{x}}{-2x^{-3}}$ • L'Hôpital

$= \lim\limits_{x\to 0^+} \frac{x^2}{-2}$ • Simplify

$= 0$ • Substitute

2. $\lim\limits_{x\to 0^+} \left(\frac{1}{x} - \frac{1}{\sin x}\right)$ • $[\infty - \infty]$

$= \lim\limits_{x\to 0^+} \frac{\sin x - x}{x.\sin x}$ • Rearrange $\left[\frac{0}{0}\right]$

$= \lim\limits_{x\to 0^+} \frac{\cos x - 1}{1.\sin x + x.\cos x}$ • L'Hôpital $\left[\frac{0}{0}\right]$

$= \lim\limits_{x\to 0^+} \frac{-\sin x}{\cos x + (1.\cos x - x.\sin x)}$ • L'Hôpital

$= \frac{-\sin 0}{2\cos 0 - (0)\sin 0}$ • Substitute

$= 0$

3. $\lim\limits_{x\to 0} x.\cot x$ • $[0.\infty]$

$= \lim\limits_{x\to 0} \frac{x}{\tan x}$ • Rearrange $\left[\frac{0}{0}\right]$

$= \lim\limits_{x\to 0} \frac{1}{\sec^2 x}$ • L'Hôpital

$= \lim\limits_{x\to 0} \cos^2 x$ • Simplify

$= \cos^2 0$ • Substitute

$= 1$

4. $\lim_{x \to 1} \left(\frac{x}{x-1} - \frac{1}{\ln x} \right)$ • $[\infty - \infty]$

$= \lim_{x \to 1} \frac{x \ln x - (x-1)}{(x-1)\ln x}$ • Rearrange $\left[\frac{0}{0}\right]$

$= \lim_{x \to 1} \frac{\left(\ln x + x.\frac{1}{x}\right) - 1}{\ln x + (x-1).\frac{1}{x}}$ • L'Hôpital

$= \lim_{x \to 1} \frac{\ln x}{\ln x + 1 - \frac{1}{x}}$ • Simplify $\left[\frac{0}{0}\right]$

$= \lim_{x \to 1} \frac{\frac{1}{x}}{\frac{1}{x} - \left(-\frac{1}{x^2}\right)}$ • L'Hôpital

$= \lim_{x \to 1} \frac{x}{x+1}$ • Simplify

$= \frac{(1)}{(1)+1}$ • Substitute

$= \frac{1}{2}$

5. $\lim_{x \to 0} (\cot x - \operatorname{cosec} x)$

$= \lim_{x \to 0} \frac{\cos x - 1}{\sin x}$

$= \lim_{x \to 0} \frac{-\sin x}{\cos x}$

$= \lim_{x \to 0} (-\tan x)$

$= 0$

6. $\lim_{x \to \infty} 7xe^{-4x}$

$= \lim_{x \to \infty} \frac{7x}{e^{4x}}$

$= \lim_{x \to \infty} \frac{7}{4e^{4x}}$

$= 0$

Activity 1.5 — SB page 18

1. 1.1 $f(x)$ is discontinuous where the denominator $x = 0$

   1.2 $g(x)$ is discontinuous where the denominator $x - 2 = 0$

   $\therefore x = 2$

   1.3 $h(x)$ is discontinuous where the denominator $x^2 - 1 = 0$

   $\therefore x^2 = 1$

   $x = -1$ and $x = 1$

1.4 $j(x)$ is discontinuous where tan x has vertical asymptotes

$\therefore x = \frac{\pi}{2} + \pi k$ where $k \in \mathbb{Z}$

1.5 $k(x)$ is discontinuous where the denominator $\cos x - 1 = 0$

$\therefore \cos x = 1$

$x = 2\pi k$ where $k \in \mathbb{Z}$

1.6 $l(x)$ is discontinuous where the denominator $x^2 + 5 = 0$.

$\therefore x^2 = -5$

However, there is no real value of x that satisfies this equation, so:
The function $l(x)$ is continuous for all x.

2. 2.1 $f(x) = \frac{3x - x^2}{2 - x}$

$f(2) = \frac{2}{0}$

undefined, discontinuous

2.2 $g(x) = \text{cosec}\left(\frac{x}{2}\right)$

$g(\pi) = 1$

continuous

2.3 $h(x) = \frac{x^2 + 3}{\sqrt{x + 3}}$

$h(-3) = \frac{12}{0}$

undefined, discontinuous

2.4 $j(x) = \sqrt{x - 5}$

$j(4) = \sqrt{-1}$

undefined, discontinuous

Summative assessment: Module 1 — SB page 20

1. If the denominator is equal to 0 the function is undefined and has no limit. Therefore, it is discontinuous at:

1.1 $x = 3$ (1)

1.2 $x = 1$ or $x = -1$ (2)

1.3 $\tan x = 0$ at $\frac{\pi}{2}, \frac{\pi}{2} + \pi, \frac{\pi}{2} + 2\pi, \frac{\pi}{2} + 3\pi$, etc.

Therefore, tan x is undefined where $x = \frac{\pi}{2} + k\pi, k \in \mathbb{Z}$ (2)

1.4 $x^3 + 1 = 0 \therefore x = -1$ (1)

2. $\lim_{x\to\infty}\left(\frac{5x-2}{7x+3}\right)$ • $\left[\frac{\infty}{\infty}\right]$

$\lim_{x\to\infty}\left(\frac{5}{7}\right)=\frac{5}{7}$ • L'Hôpital (2)

3. $\lim_{x\to -1}\left(\frac{x^3+1}{x^2-1}\right)$ • $\left[\frac{0}{0}\right]$

$=\lim_{x\to -1}\left(\frac{3x}{2}\right)$ • L'Hôpital

$=-\frac{3}{2}$ • Substitute (2)

4. $\lim_{x\to 0}\left(\frac{x^2-\sin x}{2x}\right)$ • $\left[\frac{\infty}{\infty}\right]$

$\lim_{x\to 0}\left(\frac{2x-\cos x}{2}\right)=-\frac{1}{2}$ (2)

5. $\lim_{x\to\infty}\left(\frac{3t^2}{1+t^2}\right)$ • $\left[\frac{\infty}{\infty}\right]$

$\lim_{x\to\infty}\left(\frac{6}{2}\right)=3$ (2)

6. $\lim_{x\to\infty}\left(\frac{4^x}{x^2+x-1}\right)$ • $\left[\frac{\infty}{\infty}\right]$

$\lim_{x\to\infty}\left(\frac{4^x\ln 4}{2x+1}\right)$ • $\left[\frac{\infty}{\infty}\right]$

$\lim_{x\to\infty}\left(\frac{4^x\ln 4.\ln 4}{2}\right)=\infty$ (3)

7. $\lim_{x\to\infty}\left(\frac{1}{x}\ln x\right)$ • $[0.\infty]$

$\lim_{x\to\infty}\left(\frac{\ln x}{x}\right)$ • $\left[\frac{\infty}{\infty}\right]$

$\lim_{x\to\infty}\left(\frac{\frac{1}{x}}{1}\right)=0$ (3)

8. $\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{\sin x}\right)$

$=\lim_{x\to 0}\left(\frac{\sin x-x}{x\sin x}\right)$

$=\lim_{x\to 0}\frac{\cos x-1}{\sin x+x\cos x}$

$=\lim_{x\to 0}\frac{-\sin x}{\cos x+\cos x-x\sin x}$

$=\frac{-0}{1+1-0.0}$

$=0$ (4)

9. $\lim_{x\to\infty}\left(\frac{1}{\ln x} - \frac{x}{\ln x}\right)$

$= \lim_{x\to\infty}\frac{1}{\ln x} - \lim_{x\to\infty}\frac{x}{\ln x}$

$= 0 - \lim_{x\to\infty}\frac{x}{\ln x}$

$= -\lim_{x\to\infty}\frac{1}{\frac{1}{x}}$

$= -\lim_{x\to\infty} x$

$= -\infty$ (3)

10. 10.1 No, the form is not indeterminate: 1 – 0 = 1

10.2 Yes, the form is indeterminate: $\left[\frac{\infty}{\infty}\right]$

10.3 No, the form is not indeterminate: 0.0 = 0

10.4 No, the form is not indeterminate: $\frac{-5}{0} = -\infty$ (4)

TOTAL: [31]

MODULE

2 Differentiation

After they have completed this module, students should be able to:

- use first principles to differentiate functions of the form:
  - $f(x) = ax^n$ using the binomial theorem, where $n \in \mathbb{N}$, $n > 3$;
  - $f(x) = \sin x$;
  - $f(x) = \cos x$;
  - $f(x) = \frac{ax + b}{cx + d}$;
- use differentiation techniques to:
  - derive the derivatives of $\tan x$, $\cot x$, $\sec x$ and $\operatorname{cosec} x$, using $\sin x$ and $\cos x$ with the rules of differentiation;
  - differentiate composite functions with the chain rule using substitution in combination with the quotient and product rules;
  - differentiate implicit functions by differentiating both sides of the equation with respect to x and solving for $\frac{dy}{dx}$;
  - differentiate composite functions, by using logarithmic laws to rewrite the functions;
  - differentiate functions with a variable in both the radix and the index, by first taking logarithms on both sides to transform the function to a product; and
  - determine the derivatives of the inverse trigonometric functions, by applying differential coefficients and sketching the graphs of the six trigonometric functions.

Introduction

Differentiation is a technique used to analyse how rapidly functions change at any point. This module focuses on several differentiation techniques.

We will start by revising the concept of differentiation and work through the process of differentiating from first principles before exploring differentiation techniques.

Students need the following pre-knowledge to successfully complete this module.

Pre-knowledge

Students should already know how to:

- Differentiate many common algebraic functions using standard forms.

Table 2.1: Standard forms of differentiation

Function (y)	Derivative ($[dy/dx]$)		Function (y)	Derivative ($[dy/dx]$)
$f(x)$	$f'(x)$		k	0
$k.f(x)$	$k.f'(x)$		x^n	nx^{n-1}
e^x	e^x		a^x	$a^x \ln a$
$\ln(x)$	$\frac{1}{x}$		$\log_a x$	$\frac{1}{x \ln a}$
$\sin x$	$\cos x$		$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$		$\cot x$	$-\text{cosec}^2 x$
$\sec x$	$\sec x \tan x$		$\text{cosec}\, x$	$-\text{cosec}\, x \cot x$

- Differentiate third degree polynomials such as $ax^3 + bx^2 + cx + d$ from first principles.
- Expand binomial expressions by applying the binomial theorem.
- Use trigonometric identities to determine the sum of two angles:
  - $\sin x + h \equiv \sin x \cos h + \cos x \sin h$
  - $\cos x + h \equiv \cos x \cos h - \sin x \sin h$
- Use the product rule, when differentiating two functions multiplied together. "Second term multiplied by the derivative of the first, *plus* the first term multiplied by the derivative of the second."

  $y = u(x).v(x)$

  $\frac{dy}{dx} = v.\frac{du}{dx} + u.\frac{dv}{dx}$ or $y' = v.u' + u.v'$
- Use the quotient rule, when differentiating a function divided by another function. "Bottom multiplied by the derivative of the top, *minus* top multiplied by the derivative of the bottom, all divided by the bottom squared."

  $y = \frac{u(x)}{v(x)}$

  $\frac{dy}{dx} = \frac{v.\frac{du}{dx} - u.\frac{dv}{dx}}{v^2}$ or $y' = \frac{v.u' - u.v'}{v^2}$

- Use the chain rule, when differentiating composite functions. "Derivative of the outside, multiplied by derivative of the inside."

  $y = f(u(x))$

  $y' = f'(u(x)).u'(x)$ or $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$
- Calculate the trigonometric ratios of all six trigonometric functions, where θ is the angle, *r* the hypotenuse, *x* the adjacent side, and *y* the opposite side of a right-angled triangle.
  - Sine: $\sin\theta = \frac{y}{r}$
  - Cosine: $\cos\theta = \frac{x}{r}$
  - Tangent: $\tan\theta = \frac{y}{x} = \frac{\sin\theta}{\cos\theta}$
  - Cosecant: $\text{cosec}\,\theta = \frac{1}{\sin\theta}$
  - Secant: $\sec\theta = \frac{1}{\cos\theta}$
  - Cotangent: $\cot\theta = \frac{1}{\tan\theta}$
- Convert between exponents and logarithms, and use the logarithmic laws:
  - $a = b^c \Leftrightarrow c = \log_b a$ where *a* is the power, *b* is the base (or radix) and *c* is the exponent (or index).
  - Products: $\ln ab = \ln a + \ln b$
  - Quotients: $\ln \frac{a}{b} = \ln a - \ln b$
  - Exponents: $\ln a^n = n \ln a$
- Differentiate logarithmic functions:
  - $\frac{d}{dx}\ln(ax) = \frac{1}{ax}\frac{d}{dx}ax = \frac{1}{x}$
  - $\frac{d}{dx}\ln f(x) = \frac{1}{f(x)}\frac{d}{dx}f(x) = \frac{f'(x)}{f(x)}$
- Sketch the graphs of the trigonometric functions and calculate the trigonometric ratios.
- An inverse function is the reverse of a function.
  - $f(x) = y \Leftrightarrow f^{-1}(y) = x$
  - The graph of an inverse function is a reflection of the function about the line $y = x$.

Activity 2.1 SB page 30

1. $$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{4(x+h)^3 - 4x^3}{h}$$

$$= 4 \lim_{h\to 0} \frac{(x^3 + 3x^2h + 3xh^2 + h^3 - x^3)}{h}$$

$$= 4(3x^2)$$

$$= 12x^2$$

2. $$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{3}{8}(x+h)^4 - \frac{3}{8}x^4}{h}$$

$$= \frac{3}{8}\lim_{h\to 0} \frac{(x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4) - x^4}{h}$$

$$= \frac{3}{8}\lim_{h\to 0} (4x^3 + 6x^2h + 4xh^2 + h^3)$$

$$= \frac{3}{8}(4x^3) = \frac{3}{2}x^3$$

3. $$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{(-2(x+h)^5) - (2x^5)}{h}$$

$$= -2 \lim_{h\to 0} \frac{(x^5 + 5x^4h + 10x^3h^2 + 10x^2h^3 + 5xh^4 + h^5) - x^5}{h}$$

$$= -2 \lim_{h\to 0} (5x^4 + 10x^3h + 10x^2h^2 + 5xh^3 + h^4)$$

$$= -2(5x^4)$$

$$= -10x^4$$

4. $$v(t) = \frac{t^4}{12}$$

$$v'(t) = \lim_{h\to 0} \frac{v(t+h) - v(t)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{(t+h)^4}{12} - \frac{(t)^4}{12}}{h}$$

$$= \frac{1}{12}\lim_{h\to 0} \frac{(t^4 + 4t^3h + 6t^2h^2 + 4th^3 + h^4) - (t^4)}{h}$$

$$= \frac{1}{12}\lim_{h\to 0} (4t^3 + 6t^2h + 4th^2 + h^3)$$

$$= \frac{1}{12}(4t^3)$$

$$= \frac{t^3}{3}$$

5. $a(t) = \frac{t^3}{3}$

$$a'(t) = \lim_{h \to 0} \frac{a(t+h) - a(t)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{(t+h)^3}{3} - \frac{(t)^3}{3}}{h}$$

$$= \frac{1}{3} \lim_{h \to 0} \frac{(t^3 + 3t^2h + 3th^2 + h^3) - (t^3)}{h}$$

$$= \frac{1}{3} \lim_{h \to 0} (3t^2 + 3th + h^2)$$

$$= \frac{1}{3}(3t^2)$$

$$= t^2$$

6. $y = x^6$

$$y'(x) = \lim_{h \to 0} \frac{y(x+h) - y(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^6 - (x)^6}{h}$$

$$= \lim_{h \to 0} \frac{(x^6 + 6x^5h + 15x^4h^2 + 20x^3h^3 + 15x^2h^4 + 6xh^5 + h^6) - (x^6)}{h}$$

$$= \lim_{h \to 0} (6x^5 + 15x^4h + 20x^3h^2 + 15x^2h^3 + 6xh^4 + h^5)$$

$$= 6x^5$$

Activity 2.2 SB page 33

1. Use the definition of a derivative and substitute the function:

$$\frac{d}{dx} \cos x = \lim_{h \to 0} \frac{\cos(x+h) - \cos(x)}{h}$$

Use the identity:

$\cos x + h \equiv \cos x \cos h - \sin x \sin h$

$$\frac{d}{dx} \cos x = \lim_{h \to 0} \frac{(\cos x \cos h - \sin x \sin h) - \cos(x)}{h}$$

Factorise terms containing $\cos x$ and $\sin x$:

$$\frac{d}{dx} \cos x = \lim_{h \to 0} \frac{\cos x(\cos h - 1) - \sin x(\sin h)}{h}$$

Split into multiple limits. The limit of the difference of two functions is the difference of their limits, and the limit of a product of two functions is the product of their limits:

$$\frac{d}{dx} \cos x = \lim_{h \to 0} \cos x . \lim_{h \to 0} \frac{\cos h - 1}{h} - \lim_{h \to 0} \sin x . \lim_{h \to 0} \frac{\sin h}{h}$$

The limits that contain no factors of h can be evaluated directly:

$$\frac{d}{dx}\cos x = (\cos x).\lim_{h\to 0}\frac{\cos h - 1}{h} - (\sin x).\lim_{h\to 0}\frac{\sin h}{h}$$

Both remaining limits are standard, for which the answers are known:

$\lim_{h\to 0}\frac{\cos h - 1}{h} = 0$ and $\lim_{h\to 0}\frac{\sin h}{h} = 1$.

Therefore:

$$\frac{d}{dx}\cos x = \cos x.0 - \sin x.1$$
$$= -\sin x$$

2. $$\frac{d}{dx}.\frac{1}{2}\sin x = \lim_{h\to 0}\frac{\frac{1}{2}\sin(x+h) - \frac{1}{2}\sin(x)}{h}$$

$$= \frac{1}{2}\lim_{h\to 0}\frac{(\sin x\cos h + \cos x\sin h) - \sin(x)}{h}$$

$$= \frac{1}{2}\lim_{h\to 0}\frac{\sin x(\cos h - 1) + \cos x(\sin h)}{h}$$

$$= \frac{1}{2}\lim_{h\to 0}\sin x.\lim_{h\to 0}\frac{\cos h - 1}{h} + \frac{1}{2}\lim_{h\to 0}\cos x.\lim_{h\to 0}\frac{\sin h}{h}$$

$$= \frac{1}{2}(\sin x).\lim_{h\to 0}\frac{\cos h - 1}{h} + \frac{1}{2}(\cos x).\lim_{h\to 0}\frac{\sin h}{h}$$

Use the standard limits: $\lim_{h\to 0}\frac{\cos h - 1}{h} = 0$ and $\lim_{h\to 0}\frac{\sin h}{h} = 1$

Therefore:

$$\frac{d}{dx}\frac{1}{2}\sin x = \frac{1}{2}\cos x$$

3. $y = 2 + \sin x$

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{y(x+h) - y(x)}{h}$$

$$= \lim_{h\to 0}\frac{(2 + \sin(x+h)) - (2 + \sin x)}{h}$$

$$= \lim_{h\to 0}\frac{(\sin x\cos h + \cos x\sin h) - \sin x}{h}$$

$$= \lim_{h\to 0}\frac{\sin x(\cos h - 1) + \cos x(\sin h)}{h}$$

$$= \lim_{h\to 0}\sin x.\lim_{h\to 0}\frac{\cos h - 1}{h} + \lim_{h\to 0}\cos x.\lim_{h\to 0}\frac{\sin h}{h}$$

$$= \sin x.\lim_{h\to 0}\frac{\cos h - 1}{h} + \cos x.\lim_{h\to 0}\frac{\sin h}{h}$$

$\lim_{h\to 0}\frac{\cos h - 1}{h} = 0$ and $\lim_{h\to 0}\frac{\sin h}{h} = 1$

$$\therefore \frac{dy}{dx} = \cos x$$

4. $I(t) = -3\cos t$

$$I'(t) = \lim_{h\to 0}\frac{I(t+h) - I(t)}{h}$$

$$= \lim_{h\to 0}\frac{(-3\cos(t+h)) - (-3\cos t)}{h}$$

$$= -3\lim_{h\to 0}\frac{(\cos t\cos h - \sin t\sin h) - \cos t}{h}$$

$$= -3\lim_{h\to 0}\frac{\cos t(\cos h - 1) - \sin t(\sin h)}{h}$$

$$= -3\left[\lim_{h\to 0}\cos t.\lim_{h\to 0}\frac{\cos h - 1}{h} - \lim_{h\to 0}\sin t.\lim_{h\to 0}\frac{\sin h}{h}\right]$$

$$= -3\left[\cos t.\lim_{h\to 0}\frac{\cos h - 1}{h} - \sin t.\lim_{h\to 0}\frac{\sin h}{h}\right]$$

$\lim_{h\to 0}\frac{\cos h - 1}{h} = 0$ and $\lim_{h\to 0}\frac{\sin h}{h} = 1$

$\therefore I'(t) = -3[\cos t.0 - \sin t.1] = 3\sin t$

Activity 2.3 SB page 34

1. $y = \frac{4x}{x+1}$

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{\frac{4(x+h)}{(x+h)+1} - \frac{4x}{x+1}}{h}$$

$$= \lim_{h\to 0}\frac{\frac{(4x+4h)(x+1) - (4x)(x+h+1)}{(x+h+1)(x+1)}}{h}$$

$$= \lim_{h\to 0}\frac{\frac{(4x^2+4x+4xh+4h) - (4x^2+4xh+4x)}{(x+h+1)(x+1)}}{h}$$

$$= \lim_{h\to 0}\frac{\frac{4h}{(x+h+1)(x+1)}}{h}$$

$$= \lim_{h\to 0}\frac{4}{(x+h+1)(x+1)}$$

$$= \frac{4}{(x+1)^2}$$

2. $y = \frac{2}{x-2}$

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{\frac{2}{(x+h)-2} - \frac{2}{x-2}}{h}$$

$$= \lim_{h\to 0}\frac{\frac{(2)(x-2) - (2)(x+h-2)}{(x+h-2)(x-2)}}{h}$$

$$= \lim_{h\to 0}\frac{\frac{(2x-4) - (2x+2h-4)}{(x+h-2)(x-2)}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{-2h}{(x+h-2)(x-2)}}{h}$$

$$= \lim_{h \to 0} \frac{-2}{(x+h-2)(x-2)}$$

$$= -\frac{2}{(x-2)^2}$$

3. $y = \frac{1+x}{1-x}$

Use the definition of a derivative and substitute the function in:

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{\frac{1+(x+h)}{1-(x+h)} - \frac{1+x}{1-x}}{h}$$

Combine the fractions in the numerator, multiply out and simplify:

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{\frac{(1+(x+h))(1-x) - (1+x)(1-(x+h))}{(1-(x+h))(1-x)}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{(1-x+x-x^2+h-xh) - (1-x-h+x-x^2-xh)}{(1-x-h)(1-x)}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{2h}{(1-x-h)(1-x)}}{h}$$

$$= \lim_{h \to 0} \frac{2}{(1-x-h)(1-x)}$$

Evaluate the limit:

$$\frac{dy}{dx} = \frac{2}{(1-x)(1-x)}$$

$$= \frac{2}{(1-x)^2}$$

4. $v = \frac{3t+1}{t-5}$

$$\frac{dv}{dt} = \lim_{h \to 0} \frac{v(t+h) - v(t)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{3(t+h)+1}{(t+h)-5} - \frac{3t+1}{t-5}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{(3t+3h+1)(t-5) - (3t+1)(t+h-5)}{(t+h-5)(t-5)}}{h}$$

$$= \lim_{h \to 0} \frac{(3t^2+3th+t-15t-15h-5) - (3t^2+t+3th+h-15t-5)}{(t+h-5)(t-5)h}$$

$$= \lim_{h \to 0} \frac{-16h}{(t+h-5)(t-5)h}$$

$$= \lim_{h \to 0} \frac{-16}{(t+h-5)(t-5)}$$

$$= -\frac{16}{(t-5)^2}$$

5. $z = \frac{6 - x}{3x}$

$$\frac{dz}{dx} = \lim_{h\to 0}\frac{z(x+h) - z(x)}{h}$$

$$= \lim_{h\to 0}\frac{\frac{6-(x+h)}{3(x+h)} - \frac{6-x}{3x}}{h}$$

$$= \lim_{h\to 0}\frac{\frac{(6-x-h)(3x) - (6-x)(3x+3h)}{(3x+3h)(3x)}}{h}$$

$$= \lim_{h\to 0}\frac{\left(18x - 3x^2 - 3xh\right) - \left(18x + 18h - 3x^2 - 3xh\right)}{(3x+3h)(3x)h}$$

$$= \lim_{h\to 0}\frac{-18h}{(3x+3h)(3x)h}$$

$$= \lim_{h\to 0}\frac{-18}{(3x+3h)(3x)}$$

$$= -\frac{18}{(3x)^2}$$

$$= -\frac{2}{x^2}$$

6. $t = \frac{2(\theta + 2)}{3\theta - 1}$

$$\frac{dt}{d\theta} = \lim_{h\to 0}\frac{t(\theta+h) - t(\theta)}{h}$$

$$= \lim_{h\to 0}\frac{\frac{2((\theta+h)+2)}{3(\theta+h)-1} - \frac{2(\theta+2)}{3\theta-1}}{h}$$

$$= \lim_{h\to 0}\frac{\frac{(2\theta+2h+4)(3\theta-1) - (2\theta+4)(3\theta+3h-1)}{(3\theta+3h-1)(3\theta-1)}}{h}$$

$$= \lim_{h\to 0}\frac{\left(6\theta^2 - 2\theta + 6\theta h - 2h + 12\theta - 4\right) - \left(6\theta^2 + 6\theta h - 2\theta + 12\theta + 12h - 4\right)}{(3\theta+3h-1)(3\theta-1)h}$$

$$= \lim_{h\to 0}\frac{-14h}{(3\theta+3h-1)(3\theta-1)h}$$

$$= \lim_{h\to 0}\frac{-14}{(3\theta+3h-1)(3\theta-1)}$$

$$= \frac{-14}{(3\theta-1)^2}$$

Activity 2.4 SB page 37

1. 1.1 $\frac{d}{dx}\cot x = \frac{d}{dx}\frac{\cos x}{\sin x}$

$$= \frac{\sin x.\left(\frac{d}{dx}\cos x\right) - \cos x.\left(\frac{d}{dx}\right)\sin x}{\sin^2 x}$$

$$= \frac{\sin x(-\sin x) - \cos x(\cos x)}{\sin^2 x}$$

$$= -\left(\frac{\sin^2 x + \cos^2 x}{\sin^2 x}\right)$$

Apply the identity $\sin^2 x + \cos^2 x = 1$:

$$\frac{d}{dx}\cot x = \frac{-1}{\sin^2 x}$$

$$= -\text{cosec}^2 x$$

1.2 $\frac{d}{dx}\text{cosec}\, x = \frac{d}{dx}\frac{1}{\sin x}$

$$= \frac{\sin x.\left(\frac{d}{dx}1\right) - 1.\left(\frac{d}{dx}\sin x\right)}{\sin^2 x}$$

$$= \frac{\sin x.(0) - 1.(\cos x)}{\sin^2 x}$$

$$= -\frac{\cos x}{\sin^2 x}$$

$$= -\frac{1}{\sin x}.\frac{\cos x}{\sin x}$$

$$= -\text{cosec}\, x.\cot x$$

2. 2.1 $\frac{d}{dx}(-\tan x) = \frac{d}{dx}\left(-\frac{\sin x}{\cos x}\right)$

$$= -\frac{\cos x.\left(\frac{d}{dx}\sin x\right) - \sin x.\left(\frac{d}{dx}\cos x\right)}{\cos^2 x}$$

$$= -\frac{\cos x(\cos x) - \sin x(-\sin x)}{\cos^2 x}$$

$$= -\frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

Apply the identity $\sin^2 x + \cos^2 x = 1$:

$$\frac{d}{dx}(-\tan x) = -\frac{1}{\cos^2 x}$$

$$= -\sec^2 x$$

2.2 $\frac{d}{dx}2\sec x = \frac{d}{dx}\frac{2}{\cos x}$

$$= \frac{\cos x\left(\frac{d}{dx}2\right) - 2.\left(\frac{d}{dx}\cos x\right)}{\cos^2 x}$$

$$= \frac{\cos x.(0) - 2.(-\sin x)}{\cos^2 x}$$

$$= \frac{2\sin x}{\cos^2 x}$$

$$= 2.\frac{1}{\cos x}.\frac{\sin x}{\cos x}$$

$$= 2\sec x.\tan x$$

2.3 $y = \frac{1}{4}\cot x$

$$= \frac{\cos x}{4\sin x}$$

$$\frac{dy}{dx} = \frac{4\sin x.(-\sin x) - \cos x.(4\cos x)}{(4\sin x)^2}$$

$$= \frac{-(\sin^2 x + \cos^2 x)}{4\sin^2 x}$$

$$= \frac{-1}{4\sin^2 x}$$

$$= -\frac{1}{4}\operatorname{cosec}^2 x$$

2.4 $f(x) = -3\operatorname{cosec} x$

$$= -\frac{3}{\sin x}$$

$$f'(x) = -\frac{\sin x.0 - 3.\cos x}{\sin^2 x}$$

$$= \frac{3\cos x}{\sin^2 x}$$

$$= 3\frac{1}{\sin x}.\frac{\cos x}{\sin x}$$

$$= 3\operatorname{cosec} x\cot x$$

Activity 2.5 SB page 38

1. $(8x - 4)^{100} = 100(8x - 4)^{99}.8 = 800(8x - 4)^{99}$
2. Let $f(u) = e^u$ and $u = x^2 - 1$, then $f'(u) = e^u$ and $u'(x) = 2x$.

   $f'(u(x)) = f'(u).u'(x)$

   $= e^{x^2-1}.2x$

3. This is a double composite function. If $y = \cos u$, $u = 3^v$ and $v = 2x$, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dv}\frac{dv}{dx}.$$

$$\frac{dy}{dx} = -\sin 3^{2x}.3^{2x}.\ln 3.2$$

4. This is a quotient of two composite functions. Use the quotient rule in addition to the chain rule.

$$\frac{d}{dx}\frac{(x^4 + 2x^3)^2}{\sin 2x} = \frac{\sin 2x.(2(x^4 + 2x^3).(4x^3 + 6x^2)) - (x^4 + 2x^3)^2.(\cos(2x).2)}{\sin^2 2x}$$

$$= \frac{2(x^4 + 2x^3)(4x^3 + 6x^2)}{\sin 2x} - \frac{2(x^4 + 2x^3)^2(\cos 2x)}{\sin^2 2x}$$

5. This is a product of two composite functions. Use the product rule in addition to the chain rule.

$$\frac{dy}{dx} = \left(3\tan^2\sqrt{x}.\sec^2\sqrt{x}.\frac{1}{2\sqrt{x}}\right).\sec(3 - x^3) + \tan^3\sqrt{x}.(\sec(3 - x^3)\tan(3 - x^3).(-3x^2))$$

$$= 3\tan^2\sqrt{x}.\sec(3 - x^3)\left(\frac{\sec^2\sqrt{x}}{2\sqrt{x}} - x^2\tan\sqrt{x}.\tan(3 - x^3)\right)$$

6. $f(x) = e^{\sin 5x}$

$f'(x) = e^{\sin 5x}.\cos 5x.5$

Activity 2.6 SB page 45

1. 1.1 $xy - x + 2y = 1$

Find $\frac{dy}{dx}$: $(y + xy') - 1 + 2y' = 0$

$$xy' + 2y' = 1 - y$$

$$y' = \frac{1 - y}{x + 2}$$

1.2 $x^2y^3 = 2x - y$

Find $\frac{dy}{dx}$: $x^2.3y^2.y' + 2xy^3 = 2 - y'$

$$y'(3x^2y^2 + 1) = 2 - 2xy^3$$

$$y' = \frac{2 - 2xy^3}{3x^2y^2 + 1}$$

1.3 $x^2 + xy = y^3$

Find $\frac{dy}{dx}$: $2x + (y + xy') = 3y^2.y'$

$$2x + y = 3y^2y' - xy'$$

$$y' = \frac{2x + y}{3y^2 - x}$$

1.4 $\sin y + x^2 + 4y = \cos x$

$$\text{Find } \frac{dy}{dx}: \frac{d}{dx}\sin y + \frac{d}{dx}x^2 + \frac{d}{dx}4y = \frac{d}{dx}\cos x$$

$$\cos y\frac{dy}{dx} + 2x + 4\frac{dy}{dx} = -\sin x$$

$$(\cos y + 4)\frac{dy}{dx} = -\sin x - 2x$$

$$\frac{dy}{dx} = -\frac{\sin x + 2}{\cos y + 4}$$

1.5 $3xy^2 + \cos y^2 = 2x^3 + 5$

$$\text{Find } \frac{dy}{dx}: \frac{d}{dx}(3xy^2) + \frac{d}{dx}(\cos y^2) = \frac{d}{dx}(2x^3 + 5)$$

$$\left(3y^2 + 3x.2y\frac{dy}{dx}\right) + \left(-\sin(y^2).2y\frac{dy}{dx}\right) = 6x^2$$

$$(6xy - 2y\sin y^2)\frac{dy}{dx} = 6x^2 - 3y^2$$

$$\frac{dy}{dx} = \frac{6x^2 - 3y^2}{6xy - 2y\sin y^2}$$

$$= \frac{3(2x^2 - y^2)}{2y(3x - \sin y^2)}$$

2. 2.1 $5y + xy + 3x\sin y = y^2e^x$

$$5\frac{dy}{dx} + \left(y + x\frac{dy}{dx}\right) + \left(3\sin y + 3x\cos y\frac{dy}{dx}\right) = \left(2y\frac{dy}{dx}e^x + y^2e^x\right)$$

$$(5 + x + 3x\cos y - 2ye^x)\frac{dy}{dx} = y^2e^x - y - 3\sin y$$

$$\frac{dy}{dx} = \frac{y^2e^x - y - 3\sin y}{5 + x + 3x\cos y - 2ye^x}$$

$$= \frac{5^2e^0 - 5 - 3\sin 5}{5 + 0 + 3(0)\cos 5 - 2(5)e^{(0)}}$$

$$= -4{,}575$$

2.2 $e^y = x^2y^3$

$$e^y\frac{dy}{dx} = (2x)y^3 + x^2\left(3y^2\frac{dy}{dx}\right)$$

$$(e^y - 3x^2y^2)\frac{dy}{dx} = 2xy^3$$

$$\frac{dy}{dx} = \frac{2xy^3}{e^y - 3x^2y^2}$$

$$= \frac{2(\sqrt{e})(1)^3}{e^{(1)} - 3(\sqrt{e})^2(1)^2}$$

$$= -\frac{\sqrt{e}}{e} = -\frac{1}{\sqrt{e}}$$

2.3 $\frac{4x}{y} = 5x - y$

$$\frac{4y - 4x\frac{dy}{dx}}{y^2} = 5 - \frac{dy}{dx}$$

$$\left(1 - \frac{4x}{y^2}\right)\frac{dy}{dx} = 5 - \frac{4}{y}$$

$$\frac{dy}{dx} = \frac{5 - \frac{4}{y}}{1 - \frac{4x}{y^2}}$$

$$= \frac{(5y - 4)y}{y^2 - 4x} = \frac{(5(4) - 4)(4)}{(4)^2 - 4(1)}$$

$$= \frac{16}{3}$$

Activity 2.7 SB page 47

1. If $y = x^{\sin x}$, then $\ln y = \sin x.\ln x$

$$\frac{1}{y}\frac{dy}{dx} = \cos x.\ln x + \sin x\frac{1}{x}$$

$$\frac{dy}{dx} = y\left(\cos x.\ln x + \frac{\sin x}{x}\right)$$

$$= x^{\sin x}\left(\cos x.\ln x + \frac{\sin x}{x}\right)$$

2. If $y = (2 - x)^{\sqrt{x}}$, then $\ln y = \sqrt{x}.\ln (2 - x)$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2\sqrt{x}}.\ln(2 - x)\sqrt{x} + \left(-\frac{1}{2 - x}\right)$$

$$\frac{dy}{dx} = y\left(\frac{\ln(2 - x)}{2\sqrt{x}} - \frac{\sqrt{x}}{2 - x}\right)$$

$$= (2 - x)^{\sqrt{x}}\left(\frac{\ln(2 - x)}{2\sqrt{x}} - \frac{\sqrt{x}}{2 - x}\right)$$

3. If $y = x^{\cos 3x}$, then $\ln y = \cos 3x.\ln x$

$$\frac{1}{y}\frac{dy}{dx} = (-3\sin 3x)\ln x + \cos 3x\frac{1}{x}$$

$$\frac{dy}{dx} = y\left(-3\sin 3x.\ln x + \frac{\cos 3x}{x}\right)$$

$$= x^{\cos 3x}\left(-3\sin 3x.\ln x + \frac{\cos 3x}{x}\right)$$

4. $y = \ln\left(e^x(x^2+1)\right)$

$$\frac{dy}{dx} = \frac{1}{(x^2+1)\,e^x}\cdot\frac{d}{dx}\left[(x^2+1)\,e^x\right]$$

$$= \frac{\left(\frac{d}{dx}[x^2+1]e^x + (x^2+1)\frac{d}{dx}e^x\right)e^{-x}}{(x^2+1)}$$

$$= \frac{e^{-x}\left((x^2+1)e^x + 2xe^x\right)}{x^2+1}$$

$$= \frac{x^2+2x+1}{x^2+1}$$

5. $v = (\cos t)^{2t}$

$$\ln v = 2t\ln(\cos t)$$

$$\frac{1}{v}\frac{dv}{dt} = 2\ln(\cos t) + 2t\frac{1}{\cos t}(-\sin t)$$

$$= 2\ln(\cos t) - \frac{2t\sin t}{\cos t}$$

$$\frac{dv}{dt} = 2v(\ln(\cos t) - t\tan t)$$

$$= 2\cos^{2t}t(\ln(\cos t) - t\tan t)$$

6. $e^y = x^{\ln 3x}$

$$y = \ln 3x \ln x$$

$$\frac{dy}{dx} = \left(\frac{3}{3x}\right)\ln x + \ln 3x\left(\frac{1}{x}\right)$$

$$= \frac{1}{x}(\ln x + \ln 3x)$$

$$= \frac{\ln 3x^2}{x}$$

Activity 2.8 SB page 59

1. $y' = x^2 \mathrm{D}(\arcsin x) + \mathrm{D}(x^2)\arcsin x \Leftrightarrow y' = x^2\frac{1}{\sqrt{1-x^2}} + 2x(\arcsin x)$

2. $y' = \dfrac{\mathrm{D}(1+\arctan x)(2-3\arctan x) - \mathrm{D}(2-3\arctan x)(1+\arctan x)}{(2-3\arctan x)^2}$

$$\Leftrightarrow y' = \frac{2-3\arctan x + 3 + 3\arctan x}{1+x^2}\,\frac{1}{(2-3\arctan x)^2}$$

$$\Leftrightarrow y' = \frac{5}{(1+x^2)(2-3\arctan x)^2}$$

3. $j'(x) = 2 + 10\left(\dfrac{-1}{1+x^2}\right) = \dfrac{2(1+x^2)}{1+x^2} - \dfrac{10}{1+x^2} = \dfrac{2+2x^2-10}{1+x^2} = \dfrac{2(x^2-4)}{1+x^2}$

4. $h'(x) = \frac{d}{dx} a\left(\arctan(x)^2\right)$

$= a\frac{d}{dx} \arctan(x)^2$

$= a.2(\arctan(x)).\frac{d}{dx}[\arctan(x)]$

$= a.2(\arctan\ (x)).\frac{1}{1 + x^2}$

$= \frac{2a \arctan(x)}{1 + x^2}$

5. $y' = -x.\frac{1}{x\sqrt{x^2 - 1}} + \operatorname{cosec}^{-1} x$

6. $y' = x^3.\frac{-1}{\sqrt{1 - x^2}} + 3x^2.\cos^{-1} x$

7. $y' = 4 + \frac{1}{x\sqrt{x^2 - 1}}$

8. $y = \arccos\left(\sin x + e^{5x^2}\right)$

$\frac{dy}{dx} = -\frac{1}{\sqrt{1 - \left(\sin x + e^{5x^2}\right)^2}}\left(\cos x + e^{5x^2}(10x)\right)$

$= -\frac{\cos x + 10xe^{5x^2}}{\sqrt{1 - \left(\sin x + e^{5x^2}\right)^2}}$

9. $y = \left(x^2 + 1\right)^{\arcsin x}$

$\ln y = \arcsin x \ln\left(x^2 + 1\right)$

$\frac{1}{y}\frac{dy}{dx} = \left(\frac{1}{\sqrt{1 - x^2}}\right)\ln\left(x^2 + 1\right) + \arcsin x\left(\frac{1}{x^2 + 1}.2x\right)$

$\frac{dy}{dx} = y\left(\frac{\ln\left(x^2 + 1\right)}{\sqrt{1 - x^2}} + \frac{2x \arcsin x}{x^2 + 1}\right)$

$= \left(x^2 + 1\right)^{\arcsin x}\left(\frac{\left(x^2 + 1\right)\ln\left(x^2 + 1\right) + 2x \arcsin x\sqrt{1 - x^2}}{\left(x^2 + 1\right)\sqrt{1 - x^2}}\right)$

10. $f = \arctan t \Leftrightarrow t = \tan f$ for $-\frac{\pi}{2} < f < \frac{\pi}{2}$

$t = \tan f$

$1 = \sec^2 f\frac{df}{dt}$

$\frac{df}{dt} = \frac{1}{\sec^2 f}$

$= \frac{1}{1 + \tan^2 f}$

$f'(t) = \frac{1}{1 + t^2}$

Summative assessment: Module 2 SB page 60

1. Let $u = x^2$ and $v = \sin\sqrt{x}$ then $y' = 2x\sin(\sqrt{x}) + \frac{1}{2}x^{\frac{3}{2}}.\cos(\sqrt{x})$ (6)

2. Using the chain rule, then $y' = -2.\sin(2(x))$ (3)

3. Using the chain rule, $y' = \cos(Ax + B).A$ (3)

4. Let $u = x^3 + 1$ and $\frac{d}{du}(\ln u) = \frac{1}{u}$then $\frac{\frac{d}{du}(1 + x^3)}{1 + x^3} = \frac{3x^2}{1 + x^3}$ (4)

5. Using the chain rule let $f = \ln(x)$ and $g = \sin x$ then,$f'(g)*g' = \frac{1}{\sin x}.\cos x = \cot x$ (5)

6. Using the log rule, we get $\frac{d}{dx}(\ln x^2) + \frac{d}{dx}(\ln(\sin x)) = 2\frac{d}{dx}(\ln x) + \cot x = \frac{2}{x} + \cot x$ (4)

7. This limit is of indeterminate form $\left[\frac{0}{0}\right]$, so apply L'Hôpital's rule.

$$\lim_{x\to 0}\frac{\arcsin 3x}{\arctan x}$$

$$= \lim_{x\to 0}\frac{\frac{3}{\sqrt{-(3x)^2 + 1}}}{\frac{1}{1 + (x)^2}}$$

$$= \lim_{x\to 0}\left(\frac{3(x^2 + 1)}{\sqrt{1 - 9x^2}}\right)$$

$$= \frac{3(0^2 + 1)}{\sqrt{1 - 9.0^2}}$$

$$= 3$$ (6)

8. Using the chain rule, $y' = \frac{1}{x^2 + 4}.2x$ (4)

9. $y' = x.\frac{d}{dx}\arctan\left(\frac{x}{2}\right) + 1.\arctan\left(\frac{x}{2}\right)$

$$= x.\frac{1}{1 + \left(\frac{x}{2}\right)^2}.\frac{1}{2} + 1.\arctan\left(\frac{x}{2}\right)$$

$$= \frac{x}{2\left(1 + \frac{x^2}{4}\right)} + \arctan\left(\frac{x}{2}\right)$$ (5)

10. $\frac{y}{x} - y\tan x = 7$

$$\left[\frac{\left(\frac{dy}{dx}\right)x - y(1)}{x^2}\right] - \left[\left(\frac{dy}{dx}\right)\tan x + y(\sec^2 x)\right] = [0]$$

$$\left(\frac{1}{x} - \tan x\right)\frac{dy}{dx} = y\sec^2 x + \frac{y}{x^2}$$

$$\frac{1 - x\tan x}{x}\frac{dy}{dx} = \frac{y(x^2\sec^2 x + 1)}{x^2}$$

$$\frac{dy}{dx} = \frac{y(x^2\sec^2 x + 1)}{x(1 - x\tan x)}$$ (6)

TOTAL: [46]

MODULE

3 *Applications of differentiation*

After they have completed this module, students should be able to:

- apply differentiation techniques to Newton's method for the approximation of irrational roots of equations;
- apply differentiation techniques to obtain maximum and minimum values to solve applied optimisation problems;
- apply differentiation techniques to problems concerned with related rates where the rate of change of one quantity is found in terms of the rate of change of a more easily measured quantity; and
- apply differentiation techniques to problems concerned with rates of change.

Introduction

As students know by now, differentiation has many applications such as economics, planetary science, geology and engineering amongst others. In this module they will see some examples of the application of differentiation in science and engineering. They will focus in detail on:

- Newton's method;
- maxima and minima;
- rates of change; and
- related rates.

Students need the following pre-knowledge to successfully complete this module.

Pre-knowledge

Students already know how to:

- Use the first and second derivatives of a function to determine its turning points.
- Draw neat sketch graphs showing the roots and turning points.
- Evaluate the first and second derivatives of a function at a given point.
- Use the chain rule for composite or implicit functions.
- Use implicit differentiation.
- Find or approximate the roots of an equation by:
  - factorising the function;
  - sketching the graph and determining the x-intercepts; or
  - tabulating the function values and identifying the interval where the sign changes, a continuous function must pass through zero in that interval.

- Calculate the rate of change by differentiating.
- Calculate the average displacement, velocity and acceleration of an object as a function of time:
  - Displacement: $s(t)$
  - Velocity: $v = \frac{\Delta s}{\Delta t}$
  - Acceleration: $a = \frac{\Delta v}{\Delta t}$
- Calculate the area and volume of many common shapes.

Table 3.2: Area and volume of common forms

Shape	**Area**	**Volume**
Square	$A = x^2$	
Rectangle	$A = xy$	
Circle	$A = \pi r^2$	
Triangle	$A = \frac{1}{2}xy$, or $A = \frac{1}{2}ab \sin C$	
Cube	$A = 6x^2$	$V = x^3$
Rectangular prism	$A = 2xy + 2xz + 2yz$	$V = xyz$
Sphere	$A = 4\pi r^2$	$V = \frac{4}{3}\pi r^2$
Cylinder	$A = 2\pi r^2 + 2\pi rz$	$V = \pi r^2 z$

Activity 3.1 SB page 70

1. 1.1 Write the equation:

$$x_{n+1} = x_n - \frac{x_n^3 + 4x_n^2 - x_n + 4}{3x_n^2 + 8x_n - 1}, \text{ for } n = 0, 1, 2, 3, \ldots$$

Since the initial value is given, the iterations are as follows:

n	x_n
0	−6,0000
1	−4,9492
2	−4,5143
3	−4,4324
4	−4,4296
5	−4,4296

1.2 Write the equation:

$$x_{n+1} = x_n - \frac{x_n^3 + x_n^2 - x_n}{3x_n^2 + 2x_n - 1}, \text{ for } n = 0, 1, 2, 3, 4$$

Since the initial value is given, the iterations are as follows:

n	x_n
0	2,0000
1	1,7143
2	1,6266
3	1,6181
4	1,6180
5	1,6180

1.3 Write the equation:

$$x_{n+1} = x_n - \frac{2 \sin x_n - x_n}{2 \cos x_n - 1}, \text{ for } n = 0, 1, 2, 3, 4$$

Since the initial value is given, the iterations are as follows:

n	x_n
0	–2,0000
1	–1,9010
2	–1,8955
3	–1,8955
4	–1,8955

1.4 Write the equation:

$$x_{n+1} = x_n - \frac{x_n^2 + x_n - 3}{2x_n + 1,} \text{ for } n = 0, 1, 2, 3, 4$$

Since the initial value is given, the iterations are as follows:

n	x_n
0	2,0000
1	1,4000
2	1,3053
3	1,3028
4	1,3028

2. 2.1 $f(x) = x^3 - 7x - 5$

$f'(x) = 3x^2 - 7 = 0$

$x = \pm\sqrt{\frac{7}{3}}$

$f\left(\sqrt{\frac{7}{3}}\right) = -12{,}128$ and $f\left(-\sqrt{\frac{7}{3}}\right) = 2{,}128$

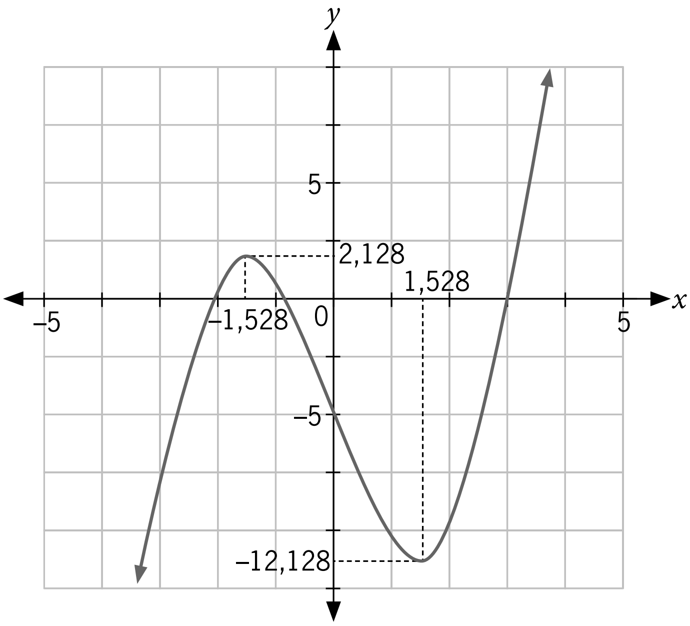
The turning points are (1,528; −12,128) and (−1,528; 2,128).

2.2

x	0	1	2	3	4
***f*(*x*)**	−5	−11	−11	1	31

The function value changes from negative to positive between $x = 2$ and $x = 3$.

2.3



2.4 If the root is estimated as 2,9, use Newton's method to determine a better approximation of this root.

$a = 2{,}9$

$f(a) = -0{,}911$

$f'(a) = 18{,}23$

$e = -\frac{f(a)}{f'(a)} = 0{,}0\ 4997$

$r = a + e$

$= 2{,}9 + 0{,}050$

$= 2{,}950$

Activity 3.2 SB page 81

1. Determine $y'(t)$ of the function $y(t) = 30t^4 - 120t^2$

   $y'(t) = 120t^3 - 240t$

   $= 120t(t^2 - 2) = 0$

   Therefore, there are three extrema, namely where:

   $t = 0; t = \pm\sqrt{2}$

   To determine whether these points are minima or maxima, check the value of the function at a value close to the extrema and compare it to the value of the extrema:

   For $t = 0$ and $t = -1$:

   $y(0) = 30(0)^4 - 120(0)^2 = 0$

   $y(-1) = 30(-1)^4 - 120(-1)^2 = -90$

   Since $y(0)$ is bigger than $y(-1)$, $y(0)$ is a maximum

   For $t = -\sqrt{2}$ and $t = -1{,}5$:

   $y(-\sqrt{2}) = 30(-\sqrt{2})^4 - 120(-\sqrt{2})^2 = -120$

   $y(-1{,}5) = 30(-1{,}5)^4 - 120(-1{,}5)^2 = -118$

   Since $y(-\sqrt{2})$ is less than $y(-1{,}5)$, $y(-\sqrt{2})$ is a minimum

   For $t = +\sqrt{2}$ and $t = 1{,}5$:

   $y(1{,}5) = 30(1{,}5)^4 - 120(1{,}5)^2 = -118$

   $y(\sqrt{2}) = 30(\sqrt{2})^4 - 120(\sqrt{2})^2 = -120$

   Since $y(\sqrt{2})$ is less than $y(1{,}5)$, $y(\sqrt{2})$ is a minimum

2. Let x and y be numbers:

   $x + y = 100$

   $z = x.y^3$

   $z = (100 - y)y^3 = 100y^3 - y^4$

   $\frac{dz}{dy} = 300y^2 - 4y^3 = y^2(300 - 4y) = 0$

   Therefore: $y = 0$ or $y = 75$

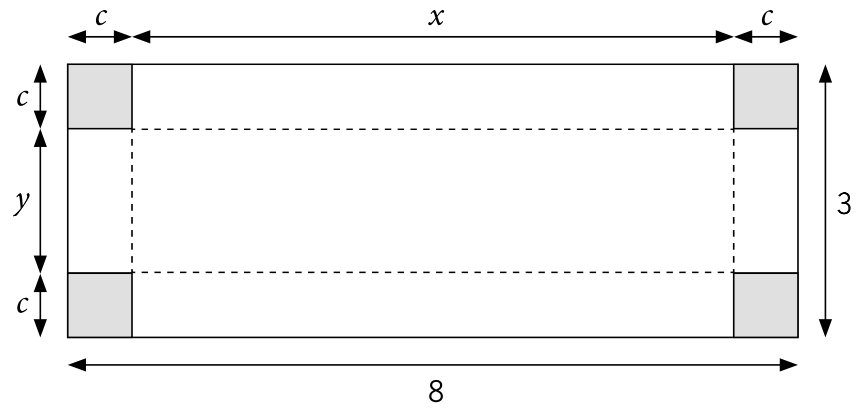
   Since $y = 0$ results in $z = 0$, use $y = 75$ to determine x and z:

   $x = 100 - y = 100 - 75 = 25$

   $z = 25 \times 75^3 = 10\ 546\ 875$

   Thus the numbers are $x = 75$ and $y = 25$ for $z = xy^3$ to be a maximum.

3.



$V = xyc$

$x = 8 - 2c$

$y = 3 - 2c$

$\therefore V = (8 - 2c)(3 - 2c)c$

$= 24c - 22c^2 + 4c^3$

$\frac{dV}{dc} = 24 - 44c + 12c^2$

Maximum is where $\frac{dV}{dc} = 0$:

$\therefore 4(6 - 11c + 3c^2) = 0$

$(3c - 2)(c - 3) = 0$

$\therefore c = 3$ or $c = \frac{2}{3}$

But $c \neq 3$ since $y \neq -3$

Therefore $c = \frac{2}{3}$:

$x = 8 - 2 \times \frac{2}{3} = \frac{20}{3}$

$y = 3 - 2 \times \frac{2}{3} = \frac{5}{3}$

$V = \frac{2}{3}.\frac{5}{3}.\frac{20}{3} = \frac{200}{27}$

4. To find the maximum and the minimum values of a function, we have to find where the derivative = 0.

$h(x) = 3x^2 - 9x + 2$

$h'(x) = 6x - 9 = 0, x = 1\frac{1}{2}$

We consider only the values at the critical points $x = 1{,}5$ and the endpoints $x = 2$ and $x = -2$.

Substitute the critical points in the function and therefore get:

$h(1{,}5) = -4{,}75$

$h(-2) = 32$

$h(2) = -4$

The maximum is 32 and the minimum is –4,75.

5. To find the inflection points, we have to find the second derivative and see where the second derivative function changes sign. To find the interval of concavity we determine where the second derivative is zero.

$f(x) = x^4 - 8x^2 + 16$

$f'(x) = 4x^3 - 16x$

$f''(x) = 12x^2 - 16$

From the equation for f'' we see that $f''(x) = 0$ at points $x = \pm\sqrt{\frac{4}{3}}$. Therefore, the inflection point of f is at the points $x = \sqrt{\frac{4}{3}}$ and $x = -\sqrt{\frac{4}{3}}$.

On the interval $\left(-\infty; -\sqrt{\frac{4}{3}}\right)$, $f''(x) > 0$ meaning that it is concave up.

On the interval $\left(-\sqrt{\frac{4}{3}}; \sqrt{\frac{4}{3}}\right)$, $f''(x) < 0$ meaning that it is concave down.

On the interval $\left(\sqrt{\frac{4}{3}}; \infty\right)$, $f''(x) > 0$ meaning that it is concave up.

6. The least amount of material means that the plate surface area must be a minimum.

$A = 2\pi rh + 2\pi r^2$

$V = \pi r^2 h = 2\ 000 \text{ cm}^3$

$h = \frac{2\ 000}{\pi r^2}$

$A = 2\pi r\left(\frac{2\ 000}{\pi r^2}\right) + 2\pi r^2 = \frac{4\ 000}{r} + 2\pi r^2$

$\frac{dA}{dr} = 0 = -\frac{4\ 000}{r^2} + 4\pi r = \frac{-4\ 000 + 4\pi r^3}{r^2}$

There is a physical constraint that requires $r > 0$. Then $\frac{dA}{dr}$ is defined for all values of r.

$r^3 = \frac{4\ 000}{4\pi}$

$r = \sqrt[3]{\frac{1\ 000}{\pi}} = 6{,}828 \text{ cm}$

$h = \frac{2\ 000}{\pi(6{,}828)^2} = 13{,}656 \text{ cm}$

Activity 3.3 — SB page 84

1. 1.1 Given $x(t) = \frac{1}{4}t^3 - 3t^2 + t + 3$

$v(t) = \frac{dx}{dt}$

$= \frac{3}{4}t^2 - 6t + 1$

$v(5) = \frac{3}{4}(5)^2 - 6(5) + 1$

$= -10{,}25 \text{ m/s} \rightarrow 10{,}25 \text{ m/s}$

1.2 $a(t) = \frac{dv}{dt}$

$= \frac{3}{2}t - 6$

$a(2) = \frac{3}{2}(2) - 6$

$= -3 \text{ m/s}^2 \rightarrow 3 \text{ m/s}$

2. Given $s(t) = \frac{t^2 - 2t}{\sqrt{t}}$

The particle is at rest when $v(t) = 0$

$v(t) = \frac{ds}{dt}$

$= \frac{(2t - 2).\sqrt{t} - \left(\frac{1}{2\sqrt{t}}\right).(t^2 - 2t)}{\sqrt{t^2}}$

$= \frac{\frac{3}{2}t^{\frac{3}{2}} - t^{\frac{1}{2}}}{t}$

$= \frac{3t - 2}{2\sqrt{t}}$

$0 = \frac{3t - 2}{2\sqrt{t}}$

$\therefore t = \frac{2}{3}$

The particle is at rest at time $= \frac{2}{3}$ seconds

3. $f(x) = 18x^5 - 30x^4 - 80x^3 + 300$

$f'(x) = 90x^4 - 120x^3 - 240x^2$

$= 30x^2(3x^2 - 4x - 8)$

> 0

$30x^2 = 0$ or $3x^2 - 4x - 8 = 0$

$x = 0$; $x = \frac{2(1 \pm \sqrt{7})}{3}$

$x = -1{,}097$; $x = 0$; $x = 2{,}431$

Solve the inequality $f'(x) > 0$:

x	***f'*(x)**			**The function is…**
$x < -1{,}097$			> 0	Increasing
$x = -1{,}097$		$= 0$		Not changing (maximum)
$-1{,}097 < x < 0$	< 0			Decreasing
$x = 0$		$= 0$		Not changing (inflection)
$0 < x < 2{,}431$	< 0			Decreasing
$x = 2{,}431$		$= 0$		Not changing (minimum)
$x > 2{,}431$			> 0	Increasing

4. $s(t) = \frac{t}{50} - 2\sin\left(\frac{t}{30}\right) + 2$

$s'(t) = \frac{1}{50} - \frac{1}{15}\cos\left(\frac{t}{30}\right)$

< 0

$\cos\left(\frac{t}{30}\right) < \frac{3}{10}$

Finding all roots on the interval $0 \leq t \leq 300$:

$\frac{t}{30} = 1{,}266 + 2\pi k$ or $\frac{t}{30} = -1{,}266 + 2\pi k$

$t = 37{,}983 + 60\pi k$ or $t = -37{,}983 + 60\pi k$

$t = 37{,}983$; $t = 226{,}479$; $t = 150{,}512$

Solve the inequality $s'(t) < 0$:

t	**s′(t)**			**The function is…**
$0 < t < 37{,}983$	< 0			Decreasing
$t = 37{,}983$		$= 0$		Not changing (minimum)
$37{,}983 < t < 150{,}512$			> 0	Increasing
$t = 150{,}512$		$= 0$		Not changing (maximum)
$150{,}512 < t < 226{,}479$	< 0			Decreasing
$t = 226{,}479$		$= 0$		Not changing (minimum)
$226{,}479 < x < 300$			> 0	Increasing

Therefore the sampling rate is decreasing on the intervals $0 < t < 37{,}983$ and $150{,}512 < t < 226{,}479$.

Activity 3.4 SB page 88

1. 1.1 Find $\frac{dr}{dt}$ when $r = 5$:

$\frac{dA}{dt} = 3$

$A = \pi r^2$

$\therefore \frac{dA}{dr} = 2\pi r$

$\frac{dA}{dt} = \frac{dA}{dr}.\frac{dr}{dt}$

$3 = 2\pi r.\frac{dr}{dt}$

$\therefore \frac{dr}{dt} = \frac{3}{2\pi r}$

$= \frac{3}{10\pi}$ m/min

1.2 Find r when $\frac{dr}{dt} = 0{,}5$:

$$\frac{dA}{dt} = \frac{dA}{dr}.\frac{dr}{dt}$$

$$\therefore 3 = 2\pi r.0{,}5$$

$$r = \frac{3}{\pi}\text{ m}$$

2. 2.1 Find $\frac{dh}{dt}$:

$$\frac{dV}{dt} = 0{,}9\text{ m}^3\text{/min}$$

$$V = Ah$$

$$\therefore \frac{dV}{dh} = A$$

$$\frac{dV}{dt} = \frac{dV}{dh}.\frac{dh}{dt}$$

$$\therefore \frac{dh}{dt} = \frac{\frac{dV}{dt}}{\frac{dV}{dh}}$$

$$= \frac{0{,}9}{100} = 0{,}009\text{ m/min}$$

2.2 Find $\frac{dV}{dt}$.

We know that $\frac{dV}{dh} = A$

$$\frac{dh}{dt} = 0{,}08\text{ m/min}$$

$$\therefore \frac{dV}{dt} = \frac{dV}{dh}.\frac{dh}{dt}$$

$$= A.0{,}08$$

$$= 100 \times 0{,}08$$

$$= 8\text{ m}^3\text{/min}$$

3. $V = \frac{4}{3}\pi r^3$

$$\frac{dV}{dt} = \frac{dV}{dr}.\frac{dr}{dt}$$

$$\frac{dV}{dt} = 4\,000$$

$$\frac{dV}{dr} = 4\pi r^2$$

$$\therefore 4\,000 = 4\pi r^2.\frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{4\,000}{4\pi r^2}$$

When $2r = 200$ cm then $\frac{dr}{dt} = \frac{4\,000}{4\pi(100)^2} = 0{,}032$ cm/s

4. $m = 1\,300$; $v = 30$; $a = 2$

$E = \frac{1}{2}mv^2$

"How fast the kinetic energy changes" is the rate of change of kinetic energy, $\frac{dE}{dt}$. Use the chain rule:

$\frac{dE}{dt} = \frac{dE}{dv}.\frac{dv}{dt}$

$\frac{dE}{dv} = mv$ and $\frac{dv}{dt} = a$

$\therefore \frac{dE}{dt} = (mv).(a)$

$= (1\,300)(30)(2)$

$= 7\,8000$ J/s = 78 kW

Summative assessment: Module 3 SB page 93

1. 1.1 Write down the equation:

$$x_{n+1} = x_n - \frac{x_n^2 + x_n - 2}{2x_n + 1}, \text{ for } n = 0, 1, 2, 3, 4$$

Since the initial value is given, the iterations are as follows:

n	x_n
0	–3,0000
1	–2,2000
2	–2,0118
3	–2,0000
4	–2,0000

(4)

1.2 Write down the equation:

$$x_{n+1} = x_n - \frac{\cos x_n}{-\sin x_n}, \text{ for } n = 0, 1, 2, 3$$

Since the initial value is given, the iterations are as follows:

n	x_n
0	2,0000
1	1,5423
2	1,5708
3	1,5708

(4)

1.3 Write down the equation:

$$x_{n+1} = x_n - \frac{2x_n^3 - 5x_n^2 - 3x_n}{6x_n^2 - 10x_n - 3}, \text{ for } n = 0, 1, 2, 3, 4$$

Since the initial value is given, the iterations are as follows:

n	x_n
0	4,0000
1	3,3208
2	3,0491
3	3,0014
4	3,0000

(4)

2. If $f'(x) > 0$ the function is increasing and if $f'(x) < 0$ the function is decreasing.

2.1 $f(x) = x^2$

$f'(x) = 2x$

$f'(x) < 0$ for $(-\infty; 0)$ therefore, it is decreasing

$f'(x) > 0$ for $(0; \infty)$ therefore, it is increasing (3)

2.2 This function is:

$f(x) = x^3$

$f'(x) = 3x^2$

$f'(x) > 0$ for $(-\infty; +\infty)$ therefore, it is increasing (2)

2.3 This function is:

$f(x) = 3x - 1$

$f'(x) = 3$

$f'(x) > 0$ for $(-\infty; +\infty)$ therefore, it is increasing (2)

2.4 This function is:

$f(x) = x^2 - 1$

$f'(x) = 2x$

$f'(x) < 0$ for $(-\infty; 0)$ therefore, it is decreasing

$f'(x) > 0$ for $(0; \infty)$ therefore, it is increasing (3)

2.5 This function is:

$f(x) = x^4 + 1$

$f'(x) = 4x^3$

$f'(x) < 0$ for $(-\infty; 0)$ therefore, it is decreasing

$f'(x) > 0$ for $(0; \infty)$ therefore, it is increasing (3)

3. Write down the equation for the measurement and volume of the square prism:

   $300 = 4x + y \Rightarrow y = 300 - 4x$

   $V = x^2y \Rightarrow V = x^2(300 - 4x) = 300x^2 - 4x^3$

   Find where the derivative is zero:

   $\frac{dV}{dx} = 600x - 12x^2 = 0$

   $12x(50 - x) = 0 \Rightarrow x = 0$ or $x = 50$

   If $x = 0$ then $V = 0$, if $x = 50$ then $y = 300 - 4(50) = 100$ and $V = 250\,000$ cm^3. (9)

4. To find the inflection points, find the second derivative and see where it changes sign:

   $f(x) = 3x^5 - 10x^3 - 8$

   $f'(x) = 15x^4 - 30x^2$

   $f''(x) = 60x^3 - 60x = 60x(x^2 - 1) = 60x(x - 1)(x + 1)$

   From this equation for f'' we see that $f''(x) = 0$ at points $x = 0$ and $x = \pm 1$.

   We need to test each of these points to determine whether f'' actually goes through the x-axis (changes sign) or whether it just touches the x-axis. To do this we choose one point just before x_0 and one point just after x_0 for each of the $f''(x) = 0$ points.

   For $f(-1)$:

   $f''(-1{,}1) = 60 \times - 1{,}1(-1{,}1 - 1)(-1{,}1 + 1) = -14$

   $f''(-0{,}9) = 60 \times - 0{,}9(-0{,}9 - 1)(-0{,}9 + 1) = 10$

   Since f'' changes sign, $f(x)$ has an inflection point at $f(-1)$.

   For $f(0)$:

   $f''(-0{,}1) = 60 \times -0{,}1(-0{,}1 - 1)(-0{,}1 + 1) = 6$

   $f''(0{,}1) = 60 \times 0{,}1(0{,}1 - 1)(0{,}1 + 1) = -6$

   Since f'' changes sign, $f(x)$ has an inflection point at $f(0)$.

   For $f(2)$:

   $f''(0{,}9) = 60 \times 0{,}9(0{,}9 - 1)(0{,}9 + 1) = -10$

   $f''(1{,}1) = 60 \times 1{,}1(1{,}1 - 1)(1{,}1 + 1) = 14$

   Since f'' changes sign, $f(x)$ has an inflection point at $f(1)$. Therefore, $f(x)$ has inflection points $x = -1$; $x = 0$; $x = 1$. (8)

5. Let x and y be the numbers.

   $x + y = a$

   $\therefore x = a - y$

   $z = xy^2$

   $= (a - y)y^2$

   $= ay^2 - y^3$

The minimum of z is where $\frac{dz}{dy} = 0$

$\frac{dz}{dy} = 2ay - 3y^2 = y(2a - 3y)$

$\frac{dz}{dy} = 0$ where $y = 0$ and $y = \frac{2}{3}a$ (disregard $y = 0$)

Then $x = a - \frac{2}{3}a = \frac{1}{3}a$

So, $y = \frac{2}{3}a$ and $x = \frac{1}{3}a$

The numbers are $\frac{1}{3}a$ and $\frac{2}{3}a$. (8)

6. 6.1 To find the velocity, we find the derivative $\frac{ds}{dt}$.

$\therefore$ Velocity $= \frac{ds}{dt}\left(\frac{1}{4}t^4 - 4t\right)$

$= t^3 - 4$ mm.s^{-1} (4)

6.2 To find the acceleration, we find the second derivative $\frac{d^2s}{dt^2}$.

$\therefore$ Acceleration $= \frac{ds}{dt}\left(t^3 - 4\right)$

$= 3t^2$ mm.s^{-2} (4)

7. The piece of wire used for the construction is W = 60 mm.

$W = 4h + 4\pi r + 4r$

$60 = 4h + 4\pi r + 4r$

$4h = 60 - 4\pi r - 4r$

$\therefore h = 15 - \pi r - r$

Substitute $h = 15 - \pi r - r$ into $V = \pi r^2 h$.

$V = \pi r^2 h$

$= \pi r^2 (15 - \pi r - r)$

$= 15\pi r^2 - \pi^2 r^3 - \pi r^3$

For maximum volume $\frac{dV}{dr} = 0$.

$V = 15\pi r^2 - \pi^2 r^3 - \pi r^3$

$\frac{dV}{dr}(15\pi r^2 - \pi^2 r^3 - \pi r^3) = 30\pi r - 3\pi^2 r^2 - 3\pi r^2$

$\therefore \pi r(30 - 3\pi r - 3r) = 0$

$r \neq 0$ and $r = \frac{30}{3\pi + 3}$

$= 2{,}415$ m (4)

8. 8.1 $V = \frac{4}{3}\pi r^3$

$$\frac{dV}{dt} = 4\pi r^2 . \frac{dr}{dt}$$

$$\therefore \frac{dr}{dt} = \frac{\frac{dV}{dt}}{4\pi r^2}$$

$$= \frac{200}{4\pi(25 + 15)^2}$$

$$= 9{,}947 \times 10^{-3} \text{ mm.s}^{-1}$$ (4)

8.2 $A = 4\pi r^2$

$$\frac{dA}{dt} = 8\pi r . \frac{dr}{dt}$$

$$= 8\pi(25 + 15)\left(9{,}947 \times 10^{-3}\right)$$

$$= 10 \text{ mm}^2.\text{s}^{-1}$$ (4)

TOTAL: [70]

MODULE

4 *Integration techniques*

After they have completed this module, students should be able to:

- identify and apply basic integration techniques;
- integrate by inspection functions that contain their respective derivatives under the integrand;
- integrate by means of algebraic substitution using the substitution rule to replace a relatively complicated integral by a simpler integral;
- integrate trigonometric functions, particularly:
  - using the identity $\sin^2 x + \cos^2 x = 1$ to convert between even powers of sine and cosine
  - writing integrands involving powers of sine and cosine in terms of only one sine or only one cosine factor;
- integrate algebraic fractions by first performing long division when the highest degree of the numerator is greater than or equal to the highest degree of the denominator;
- use partial fractions to integrate proper algebraic fractions when the highest degree of the numerator is less than the highest degree of the denominator; and
- integrate by parts when the integrand consists of the product of two functions, where neither is a derivative of the other.

Introduction

Integration is the opposite of differentiation and as such allows, among many other uses, calculating areas and averaging continuous functions. Students will first summarise the basic integration technique that was covered previously in N4 to refresh their memory. We will then expand on this to discuss other integration techniques, which are the tools that help us to simplify and integrate functions.

Students need the following pre-knowledge to successfully complete this module.

Pre-knowledge

Basic terms used during integration:

- Indefinite and definite integrals: Definite integrals are integrals where the limits have been defined; indefinite integrals do not have defined limits and contain an arbitrary constant, C.
- The S-shaped symbol (∫) is used to denote "find the integral of".
- C is a constant known as the constant of integration.
- *dx* stands for "with respect to *x*" and is written at the end of the function to be integrated.
- The integrand, $f(x)$, is the function being integrated.

Students should know the following:

- Laws of exponents
- Logarithmic laws
- Trigonometric identities
- Standard derivatives:

Table 4.1: Standard derivatives

$f(x))$	$f'(x)$
k kx^n	0 $n.kx^{n-1}$
ka^x ka^{nx}	$ka^x \ln a$ $n.ka^{nx} \ln a$
ke^x ke^{nx}	ke^x $n.ke^{nx}$
$k \ln x$ $k \ln(nx))$	$\frac{k}{x}$ $\frac{k}{x}$
$k \log_a x$ $k \log_a(nx))$	$\frac{k}{x \ln a}$ $\frac{k}{x \ln a}$
$k \sin(nx)$ $k \cos(nx)$ $k \tan(nx)$ $k \cot(nx)$ $k \sec(nx)$ $k \operatorname{cosec}(nx))$	$nk \cos(nx)$ $-nk \sin(nx)$ $nk \sec^2(nx)$ $-nk \operatorname{cosec}^2(nx))$ $nk \sec(nx) \tan(nx)$ $-nk \operatorname{cosec}(nx) \cot(nx))$

Activity 4.1 — SB page 106

1. $\int e^{8x}\,dx = \frac{1}{8}e^{8x} + C$
2. $\int \frac{1}{5x-9}\,dx = \frac{1}{5}\int \frac{5}{5x-9}\,dx = \frac{1}{5}\ln|5x-9| + C$
3. $\int 3x^5\,dx = 3\frac{x^6}{6} + C = \frac{x^6}{2} + C$
4. $\int \sin 5x\,dx = -\frac{1}{5}\cos 5x + C$
5. $\int x\,dx = \frac{x^2}{2} + C$
6. The approximate form is tan x, because $\frac{d}{dx}\tan x = \sec^2 x$.

   Adjust the approximate form:

   $\frac{d}{dx}\tan 3x = \sec^2 3x.3$

   This is the original integrand multiplied by a factor of $\frac{1}{2}$. Include a coefficient of 2 in the antiderivative: 2 tan $3x$.

   Add a constant of integration.

   $\frac{d}{dx}(2\tan 3x + C) = 2\sec^2 3x.3 + 0$

   $= 6\sec^2 3x$

   $\therefore \int 6\sec^2 3x\,dx = 2\tan 3x + C$

Activity 4.2 — SB page 109

1. $\int x^2\sqrt{1+x^3}\,dx$

   Let $u = 1 + x^3$ then $du = 3x^2\,dx$

   $\frac{1}{3}\int \sqrt{u}\,du$

   $= \frac{1}{3}\left(\frac{2}{3}\right)u^{\frac{3}{2}} + C$

   $= \frac{2}{9}u^{\frac{3}{2}} + C$

   $= \frac{2}{9}\sqrt{(1+x^3)^3} + C$

2. $\int \frac{15}{3-2x}\,dx$

   Let $u = 3 - 2x$ then

   $du = -2dx$

   $-\frac{15}{2}\int \frac{1}{u}\,du$

   $= -\frac{15}{2}\ln|u| + C$

   $= -\frac{15}{2}\ln|3-2x| + C$

3. $\int \sin(2x+4)\,dx$

   Let $u = 2x + 4$ then $du = 2dx$

   $\frac{1}{2}\int \sin u\,du$

   $= -\frac{1}{2}\cos u + C$

   $= -\frac{1}{2}\cos(2x+4) + C$

4. $\int e^x\sqrt{1+e^x}\,dx$

   Let $u = 1 + e^x$ then $du = e^x\,dx$

   $\int \sqrt{u}\,du$

   $= \frac{2}{3}u^{\frac{3}{2}} + C$

   $= \frac{2}{3}(1+e^x)^{\frac{3}{2}} + C$

   $= \frac{2}{3}\sqrt{(1+e^x)^3} + C$

5. $\int \frac{1}{x^2\left(1+\frac{1}{x}\right)^2}\,dx$

Let $u = 1 + \frac{1}{x}$ then $du = -x^{-2}\,dx$

$\int -\frac{1}{u^2}\,du$

$= u^{-1} + C$

$= \frac{1}{1+\frac{1}{x}} + C$

$= \frac{x}{1+x} + C$

6. $\int xe^{3x^2}dx$

Let $u = 3x^2$ then

$\frac{du}{dx} = 6x$

$\frac{1}{6}du = x\,dx$

$\int xe^{3x^2}dx = \int e^u.\frac{1}{6}du$

$= \frac{1}{6}e^u + C$

$= \frac{1}{6}e^{3x^2} + C$

Activity 4.3 — SB page 119

1. $\int \sin^2 x\,dx = \frac{1}{2}\int(1 - \cos 2x)\,dx$

$\frac{1}{2}\int(1 - \cos 2x)\,dx = \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right) + C$

2. $\int \tan(2x)\,dx = \frac{1}{2}\ln|\sec 2x| + C$

Let $u = 2x$, $du = 2\,dx$

$\int \tan(2x)\,dx = \frac{1}{2}\int \tan(u)\,du$

$= \frac{1}{2}\ln \sec u + C$

$= \frac{1}{2}\ln \sec 2x + C$

3. $\int \cos^2(ax)\,dx$

Let $u = ax$ then $du = a.dx$

$\int \cos^2(ax)\,dx = \int \cos^2 u.\frac{du}{a}$

$= \frac{1}{a}\int 1 + \frac{\cos 2u}{2}\,du$

$= \frac{1}{2a}\left(u + \frac{\sin 2u}{2}\right) + C$

$= \frac{1}{2a}\left(ax + \frac{\sin 2ax}{2}\right) + C$

$= \frac{x}{2} + \frac{\sin 2ax}{4a} + C$

4. $\int \sin 3x.\cos 4x\,dx$

$\int \sin(3x)\cos(4x)\,dx = \frac{1}{2}\int(\sin(3x - 4x) + \sin(3x + 4x)dx$

$= \frac{1}{2}\int(-\sin x + \sin 7x)\,dx$

$= \frac{1}{2}(\cos x - \frac{1}{7}\cos 7x) + C$

5. $\int \frac{1}{\sqrt{9 - x^2}}\, dx$

$\int \frac{dx}{\sqrt{a^2 - b^2x^2}} = \frac{1}{b} \sin^{-1}\left(\frac{bx}{a}\right) + C$

$a = 3$ and $b = 1$

$\frac{1}{b} \sin^{-1}\left(\frac{bx}{a}\right) + C = \sin^{-1}\left(\frac{x}{3}\right) + C$

$\int \frac{1}{\sqrt{9 - x^2}}\, dx = \sin^{-1}\left(\frac{x}{3}\right) + C$

6. $\int \frac{1}{a^2 + b^2x^2}\, dx = \frac{1}{ab} \tan^{-1}\left(\frac{b}{a}x\right) + C$

$a^2 = 4 \Rightarrow a = 2$ and $b^2 = 16 \Rightarrow b = 4$

$\int \frac{3}{4 + 16x^2}\, dx = 3.\frac{1}{(2)(4)} \tan^{-1}\left(\frac{(4)}{(2)}x\right) + C$

$= \frac{3}{8} \tan^{-1}(2x) + C$

Activity 4.4 SB page 122

1. 1.1 $\frac{x^3 + 2x^2 - 5}{2x^2 + 4} = \frac{1}{2}x + 1 - \frac{2x + 9}{2x^2 + 4}$

The quotient is $\frac{1}{2}x + 1$ and the remainder is $-2x - 9$.

1.2 $\frac{5x^2 - 3x + 4}{x - 5} = 5x + 22 + \frac{114}{x - 5}$

The quotient is $5x + 22$ and the remainder is 114.

1.3 $\frac{3x^2 + 1}{x^2 - 3} = 3 + \frac{10}{x^2 - 3}$

The quotient is 3 and the remainder is 10.

1.4 $\frac{x^2 - x + 12}{x^2 - x} = 1 + \frac{12}{x^2 - x}$

The quotient is 1 and the remainder is 12.

2. 2.1 This is an improper fraction. Use long division:

$$\begin{array}{r|l} & -x \\ 1 - x^2 & x^3 + 1 \\ & \underline{x^3 - x} \\ & \quad x + 1 \end{array}$$

Rewriting and simplifying:

$$\frac{x^3+1}{1-x^2} = -x + \frac{x+1}{1-x^2}$$

$$= -x + \frac{x+1}{(1-x)(1+x)}$$

$$= -x + \frac{1}{1-x}$$

Integrating:

$$\int \frac{x^3+1}{1-x^2}\,dx = -\int x\,dx + \int \frac{1}{1-x}\,dx$$

$$= -\frac{1}{2}x^2 + \ln(1-x) + C$$

2.2 The degree of the numerator is the same as that of the denominator. Use long division:

$$\begin{array}{r} 7 \\ x-1\overline{)7x-6} \\ \underline{7x-7} \\ 1 \end{array}$$

Rewriting:

$$\frac{7x-6}{x-1} = 7 + \frac{1}{x-1}$$

Integrating:

$$\int \frac{7x-6}{x-1}\,dx = \int 7\,dx + \int \frac{1}{x-1}\,dx$$

$$= 7x + \ln(x-1) + C$$

2.3 The degree of the numerator is the same as that of the denominator.

Expand the brackets:

$$\frac{6(x-2)}{3-x} = \frac{6x-12}{-x+3}$$

Use long division:

$$\begin{array}{r} -6 \\ -x+3\overline{)6x-12} \\ \underline{6x-18} \\ 6 \end{array}$$

Rewrite:

$$\frac{6(x-2)}{3-x} = -6 + \frac{6}{3-x}$$

Integrate:

$$\int\frac{6(x-2)}{(3-x)}dx = -\int 6\,dx + \int\frac{6}{3-x}dx$$

$$= -6x + 6\ln(3-x) + C$$

2.4 $\frac{x^3+9x+1}{x^2+9} = x + \frac{1}{x^2+9}$

$$\therefore \int\frac{x^3+9x+1}{x^2+9}dx = \int x\,dx + \int\frac{1}{x^2+9}dx$$

$$= \frac{1}{2}x^2 + \frac{1}{3}\tan^{-1}\left(\frac{1}{3}x\right) + C$$

Activity 4.5 SB page 130

1. $\int\frac{1}{(x+2)(x+1)}dx = \int\frac{A}{x+2}\,dx + \int\frac{B}{x+1}dx$

$$1 = A(x+1) + B(x+2)$$

Equate like terms on both sides of the equal sign and solve simultaneously to get B = 1 and A = −1

Then

$$\int\frac{1}{(x+2)(x+1)}dx = \int\frac{-1}{x+2}\,dx + \int\frac{1}{x+1}dx$$

$$= -\ln(x+2) + \ln(x+1) + C$$

2. $\int\frac{3x+2}{(x-1)(x+7)} = \int\frac{A}{x-1} + \int\frac{B}{x+7}$

$$3x + 2 = A(x+7) + B(x-1)$$

Equate like terms on both sides of the equal sign and solve simultaneously to get B = $\frac{19}{8}$ and A = $\frac{5}{8}$

Then

$$\int\frac{3x+2}{(x-1)(x+7)} = \frac{5}{8}\int\frac{1}{x-1}dx + \frac{19}{8}\int\frac{B}{x+7}dx$$

$$= \frac{5}{8}\ln(x-1) + \frac{19}{8}\ln(x+7) + C$$

3. $\int\frac{1}{(x+3)^2(x-1)}dx = \int\frac{A}{x+3}dx + \int\frac{B}{(x+3)^2}dx + \int\frac{C}{(x-1)}dx$

$$1 = A(x+3)(x-1) + B(x-1) + C(x+3)^2$$

Equate like terms on both sides of the equal sign and solve simultaneously to get B = $-\frac{1}{4}$ and A = $\frac{-1}{16}$ and C = $\frac{1}{16}$

Then

$$\int \frac{1}{(x+3)^2(x-1)}dx = -\frac{1}{16}\int \frac{1}{x+3}dx + \frac{-1}{4}\int \frac{1}{(x+3)^2}dx + \frac{1}{16}\int \frac{1}{(x-1)}dx$$

$$= -\frac{1}{16}\ln(x+3) - \frac{1}{4(x+3)} + \frac{1}{16}\ln(x-1) + C$$

$$= \frac{1}{16}\ln\frac{x-1}{x+3} - \frac{1}{4(x+3)} + C$$

4. $\int \frac{x+1}{x(x-7)^2}dx = \int \frac{A}{x}dx + \int \frac{B}{x-7}\, dx + \int \frac{C}{(x-7)^2}dx$

$$x + 1 = A(x-7)^2 + Bx(x-7) + Cx$$

Equate like terms on both sides of the equal sign and solve simultaneously to get $B = -\frac{1}{49}$ and $A = \frac{1}{49}$ and $C = \frac{8}{7}$

Then

$$\int \frac{x+1}{x(x-7)^2}dx = \frac{1}{49}\int \frac{1}{x}dx + \frac{-1}{49}\int \frac{1}{x-7}\, dx + \frac{8}{7}\int \frac{1}{(x-7)^2}dx$$

$$= \frac{1}{49}\ln x - \frac{1}{49}\ln(x-7) - \frac{8}{7(x-7)} + C$$

$$= \frac{1}{49}\ln\frac{x}{x-7} - \frac{8}{7(x-7)} + C$$

5. This is an improper fraction, since the degree of the numerator (4) is greater than the degree of the denominator (2). Expand the brackets:

$$\frac{24x^3(x-2)}{(x-1)(2x+1)} = \frac{24x^4 - 48x^3}{2x^2 - x - 1}$$

Use long division:

$$\begin{array}{r} 12x^2 - 18x - 3 \\ 2x^2 - x - 1 \overline{)\,24x^4 - 48x^3} \\ \underline{24x^4 - 12x^3 - 12x^2} \\ -36x^3 + 12x^2 \\ \underline{-36x^3 + 18x^2 + 18x} \\ -6x^2 - 18x \\ \underline{-6x^2 + 3x + 3} \\ -21x - 3 \end{array}$$

Rewrite:

$$\frac{24x^3(x-2)}{(x-1)(2x+1)} = 12x^2 - 18\text{x} - 3 + \frac{-21x-3}{(x-1)(2x+1)}$$

The remaining fraction is a proper fraction because the degree of the numerator (1) is less than the degree of the denominator (2). Decompose it into partial fractions:

$$\frac{-21x-3}{(x-1)(2x+1)} = \frac{A}{x-1} + \frac{B}{2x+1} - 21x - 3$$

$$= A(2x+1) + B(x-1)$$

$$= (2A + B)x + (A - B)$$

Equate like terms on both sides of the equal sign and solve simultaneously to get B = –5 and A = –8, then

$$\frac{-21x-3}{(x-1)(2x+1)} = \frac{-8}{x-1} + \frac{-5}{2x+1}$$

Hence,

$$\frac{24x^3(x-2)}{(x-1)(2x+1)} = 12x^2 - 18x - 3 - \frac{8}{x-1} - \frac{5}{2x+1}$$

Integrate:

$$\int\frac{24x^3(x-2)}{(x-1)(2x+1)}dx = \int 12x^2\,dx - \int 18x\,dx - \int 3\,dx - \int\frac{8}{x-1}dx - \int\frac{5}{2x+1}dx$$

$$= 4x^3 - 9x^2 - 3x - 8\ln(x-1) - \tfrac{5}{2}\ln(2x+1) + C$$

6. Using long division:

$$\frac{x^3+2x^2+2x-1}{2x^2+8} = \tfrac{1}{2}x + 1 - \frac{2x+9}{2(x^2+4)}$$

$$\therefore \int\frac{x^3+2x^2+2x-1}{2x^2+8}dx = \int\tfrac{1}{2}x\,dx + \int dx - \int\frac{x}{x^2+4}dx - \tfrac{9}{2}\int\frac{1}{x^2+4}dx$$

$$= \tfrac{1}{4}x^2 + x - \tfrac{1}{2}\ln(x^2+4) - \tfrac{9}{4}\tan^{-1}\left(\tfrac{1}{2}x\right) + C$$

Activity 4.6 SB page 133

1. $\int 3xe^x\,dx$

Let $u = 3x$ then $\frac{du}{dx} = 3$ and let $\frac{dv}{dx} = e^x$ then $v = e^x$

$\int 3xe^x\,dx = 3xe^x - \int e^x 3\,dx$

$\int 3xe^x\,dx = 3xe^x - 3e^x + C$

2. $\int x^3(\ln x)^2 dx$

Let $u = (\ln x)^2$, then $\frac{du}{dx} = \frac{2\ln x}{x}$

Let $\frac{dv}{dx} = x^3$, then $v = \frac{1}{4}x^4$

$$\int x^3(\ln x)^2\,dx = (\ln x)^2\tfrac{1}{4}x^4 - \int\tfrac{1}{4}x^4\frac{2\ln x}{x}\,dx$$

$$= (\ln x)^2\tfrac{1}{4}x^4 - \tfrac{1}{2}\int x^3\ln x\,dx$$

Use integration by parts to simplify further.

$$\int x^3(\ln x)^2\,dx = \tfrac{1}{4}(\ln x)^2x^4 - \tfrac{1}{2}\left[\tfrac{1}{4}x^4\ln x - \tfrac{1}{16}x^4\right] + C$$

3. $\int x^3 \cos \pi x \, dx$

Let $f = x^2$ and $g' = \cos \pi x$. Then $f' = 2x$ and $g = \frac{1}{\pi}\sin \pi x$.

$\int x^2 \cos \pi x \, dx = x^2 . \frac{1}{\pi}\sin \pi x - \frac{2}{\pi}\int x \ \sin \pi x \, dx$

For $\int x \ \sin \pi x \, dx$, let $f = x$ and $g' = \sin \pi x$. Then $f' = 1$ and $g = -\frac{1}{\pi}\cos \pi x$

$$\int x^2 \cos \pi x \, dx = \frac{1}{\pi}x^2 \sin \pi x - \frac{2}{\pi}\left[x\left(-\frac{1}{\pi}\cos \pi x\right) - \int 1.\left(-\frac{1}{\pi}\cos \pi x\right)dx\right]$$

$$= \frac{1}{\pi}x^2 \sin \pi x + \frac{2}{\pi^2}x \cos \pi x - \frac{1}{\pi}\int \cos \pi x \, dx$$

$$= \frac{1}{\pi}x^2 \sin \pi x + \frac{2}{\pi^2}x \cos \pi x - \frac{1}{\pi^2}\sin \pi x + C$$

4. $\int (\theta - \pi)\sin\left(\frac{\theta}{2}\right)d\theta$

Let $f = \theta - \pi$ and $g' = \sin\left(\frac{\theta}{2}\right)$. Then $f' = 1$ and $g = -2 \cos\left(\frac{\theta}{2}\right)$.

$$\int (\theta - \pi)\sin\left(\frac{\theta}{2}\right)d\theta = (\theta - \pi)\left(-2 \cos\left(\frac{\theta}{2}\right)\right) - \int 1.\left(-2 \cos\left(\frac{\theta}{2}\right)\right)d\theta$$

$$= 2(\pi - \theta)\cos\left(\frac{\theta}{2}\right) + 4 \sin\left(\frac{\theta}{2}\right) + C$$

5. $\int 2xe^{3x}dx$

Let $f = 2x$ and $g' = e^{3x}$. Then $f' = 2$ and $g = \frac{1}{3}e^{3x}$

$$\int 2xe^{3x}dx = (2x)\left(\frac{1}{3}e^{3x}\right) - \frac{2}{3}\int e^{3x}dx$$

$$= \frac{2}{3}xe^{3x} - \frac{2}{9}e^{3x} + C$$

$$= \frac{2}{3}e^{3x}\left(x - \frac{1}{3}\right) + C$$

6. $\int t\sqrt{t-6}\ dt$

Let $f = t$ and $g' = \sqrt{t-6} = (t-6)^{\frac{1}{2}}$. Then $f' = 1$ and $g = \frac{2}{3}(t-6)^{\frac{3}{2}}$.

$$\int t\sqrt{t-6}\ dt = t.\frac{2}{3}(t-6)^{\frac{3}{2}} - \frac{2}{3}\int (t-6)^{\frac{3}{2}}\ dt$$

$$= \frac{2}{3}t(t-6)^{\frac{3}{2}} - \frac{2}{3}.\frac{2}{5}(t-6)^{\frac{5}{2}} + C$$

$$= \frac{2}{3}t(t-6)^{\frac{3}{2}} - \frac{4}{15}(t-6)^{\frac{5}{2}} + C$$

Summative assessment: Module 4 — SB page 138

1. 1.1 $\int \frac{1}{\sqrt{5x-9}}\ dx$

$= \int (5x-9)^{-\frac{1}{2}}\ dx$

$= -\frac{2}{15}\ (5x-9)^{-\frac{3}{2}} + C$

$= -\frac{2}{15\sqrt{(5x-9)^3}} + C$ (2)

1.2 $\int \cos 7x \, dx$

$= \frac{1}{7} \sin 7x + C$ (2)

1.3 $\int e^{8x} \, dx = \frac{1}{8} e^{8x} + C$ (2)

2. 2.1 $\int (x + 5)^5 dx$

Let $u = x + 5$, then $du = dx$

$\int (x + 5)^5 \, dx = \int u^5 \, du$

$= \frac{1}{6} u^6 + C$

$= \frac{(x + 5)^6}{6} + C$ (4)

2.2 $\int \cos (5x + 3) \, dx$

Let $u = 5x + 3$, then $du = 5 \, dx$

$\int \cos (5x + 3) \, dx = \frac{1}{5} \int \cos u \, du$

$= \frac{1}{5} \sin u + C$

$= \frac{1}{5} \sin (5x + 3) + C$ (4)

2.3 $\int \frac{1}{1 - 3x} \, dx$

Let $u = 1 - 3x$, then $du = -3dx$

$\int \frac{1}{1 - 3x} \, dx = \frac{-1}{3} \int \frac{1}{u} \, du$

$= -\frac{1}{3} \ln | u | + C$

$= -\frac{1}{3} \ln | 1 - 3x | + C$ (4)

3. 3.1 $\int \sqrt{9 - 2x^2} \, dx$

Substitute $x = \frac{3 \sin(u)}{\sqrt{2}}$ and $dx = \frac{3 \cos(u)}{\sqrt{2}} \, du$ and $u = \sin^{-1}\left(\frac{\sqrt{2}x}{3}\right)$

Then $\int \sqrt{9 - 2x^2} \, dx = \frac{3}{\sqrt{2}} \int 3 \cos^2(u) \, du$

$= \frac{x}{2}\sqrt{9 - 2x^2} + \frac{9}{2\sqrt{2}} \sin^{-1}\left(\frac{\sqrt{2}}{3} x\right) + C$ (4)

3.2 $\int \frac{1}{16x^2 + 1} \, dx$

Substitute $u = 4x$ and $du = 4 \, dx$

Then $\frac{1}{4} \int \frac{1}{u^2 + 1} \, du = \frac{1}{4} \tan^{-1}(u) + C$

$= \frac{1}{4} \tan^{-1}(4x) + C$ (4)

4. 4.1 $\int \frac{3}{x(x+1)}\,dx = \int \frac{A}{x}\,dx + \int \frac{B}{x+1}\,dx$

$$3 = A(x+1) + Bx$$

We equate like terms on both sides of the equal sign and solve simultaneously to get B = –3 and A = 3,

Then $\int \frac{3}{x(x+1)}\,dx = \int \frac{3}{x}\,dx - \int \frac{3}{x+1}\,dx$

$$= 3\ln|x| - 3\ln|x+1| + C \qquad (4)$$

4.2 $\int \frac{x}{(2x+3)(x-4)}\,dx = \int \frac{A}{2x+3}\,dx + \int \frac{B}{x-4}\,dx$

$$x = A(x-4) + B(2x+3)$$

We equate like terms on both sides of the equal sign and solve simultaneously to get $B = \frac{4}{11}$ and $A = \frac{3}{11}$,

Then $\int \frac{x}{(2x+3)(x-4)}\,dx = \frac{3}{11}\int \frac{1}{2x+3}\,dx + \frac{4}{11}\int \frac{1}{x-4}\,dx$

$$= \frac{3}{22}\ln|2x+3| + \frac{4}{11}\ln|x-4| + C \qquad (4)$$

4.3 $\int \frac{2x+1}{x^2+x+1}\,dx = \int \frac{2x+1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\,dx$

$$= \ln\left|\left(x+\frac{1}{2}\right)^2+\frac{3}{4}\right| + C$$

$$= \ln(x^2+x+1) + C \qquad (4)$$

4.4 $\int \frac{x}{x^2+x+1}\,dx = \int \frac{2x+1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\,dx$

$$= \int \frac{x+\frac{1}{2}-\frac{1}{2}}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\,dx$$

$$= \int \frac{x+\frac{1}{2}}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\,dx - \int \frac{\frac{1}{2}}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\,dx$$

$$= \frac{1}{2}\int \frac{2x+1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\,dx - \frac{1}{2}\int \frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\,dx$$

$$= \frac{1}{2}\ln\left|\left(x+\frac{1}{2}\right)^2+\frac{3}{4}\right| - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2\left(x+\frac{1}{2}\right)}{\sqrt{3}}\right) + C$$

$$= \frac{1}{2}\ln(x^2+x+1) - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{1}{\sqrt{3}}(2x+1)\right) + C \qquad (4)$$

4.5 $\int \frac{x^3}{x^2 - 4}\, dx$

$$x^2 - 4 \overline{\big)\, x^3} \quad \text{quotient } x$$

$$\underline{(x^3 - 4x)}$$

$$4x$$

$$\int \frac{x^3}{x^2-4}\, dx = \int \left(x + \frac{4x}{x^2-4} \right) - dx$$

$$= \int x\, dx + \int \frac{2}{x-2}\, dx + \int \frac{2}{x+2}\, dx$$

$$= \frac{1}{2}x^2 + 2 \ln | x - 2 | + 2 \ln | x + 2 | + C \qquad (4)$$

4.6 $\int \frac{x+1}{x^2+x+3}\, dx$

$$= \int \frac{x + \frac{1}{2}}{\left(x + \frac{1}{2}\right)^2 + \frac{11}{4}}\, dx + \frac{1}{2} \int \frac{1}{\left(x + \frac{1}{2}\right)^2 + \frac{11}{4}}\, dx$$

$$= \frac{1}{2} \ln \left| \left(x + \frac{1}{2}\right)^2 + \frac{11}{4} \right| + \frac{1}{2} \left(\frac{1}{\sqrt{11}} \tan^{-1}\left(\frac{1}{\sqrt{11}}\left(x + \frac{1}{2}\right)\right)\right) + C \qquad (4)$$

5. 5.1 $\int t^3 e^t dt$

Let $u = t^3$ then $u' = 3t^2$ and $v' = e^t$ then $v = e^t$

$\int t^3 e^t dt = t^3 e^t - 3 \int t^2 e^t dt$

$\int t^2 e^t dt$

Let $u = t^2$ then $u' = 2t$ and $v' = e^t$ then $v = e^t$

$\int t^3 e^t dt = t^3 e^t - 3[t^2 e^t - 2 \int t e^t dt]$

$= t^3 e^t - 3t^2 e^t + 6[t e^t - \int e^t dt]$

$= t^3 e^t - 3t^2 e^t + 6t e^t - 6e^t + C$ (5)

5.2 $\int x e^{-x}\, dx$

Let $u = x$ then $u' = 1$ and $v' = e^{-x}$ then $v = -e^{-x}$

$\int x e^{-x}\, dx = -x e^{-x} + \int e^{-x}\, dx$

$= -x e^{-x} - e^{-x} + C$ (5)

TOTAL: [60]

MODULE

5 The definite integral

After they have completed this module, students should be able to:

- calculate basic definite integrals;
- calculate a definite integral using a change of limits when the variable is changed;
- determine definite integrals with infinity as a limit; and
- apply the Laplace transform to functions.

Introduction

Integration, particularly the definite integral, can be thought of as a method used to accurately add a large number of very small pieces together. Integration is also known as finding the area under a curve. In Module 4, when solving an indefinite integral, the result was another function. This function is the formula for calculating the sum of the small pieces, or the sum of the areas of all the small strips at each point under the curve. A definite integral is used to calculate the sum of the small pieces between two points and results in a numerical value.

Students need the following pre-knowledge to successfully complete this module.

Pre-knowledge

- evaluate functions;
- solve indefinite integrals; and
- use substitution to solve indefinite integrals.

Activity 5.1 SB page 144

1. $\int_{-1}^{2}(x^2 - 3x + 4)\,dx = \left[\frac{x^3}{3} - \frac{3x^2}{2} + 4x\right]_{-1}^{2} = 10{,}5$

2. $\int_{0}^{2\pi} \sin x\,dx = -\cos x\Big|_{0}^{2\pi} = 0$

3. $\int_{1}^{2} \frac{1}{x}\,dx = \ln 2 - \ln 1 = \ln 2$

4. $\int_0^{\pi} \cos^2(2x)\, dx = \frac{1}{2}\left[x + \frac{1}{4}\sin 4x\right]_0^{\pi} = \frac{\pi}{2}$

5. $\int_0^4 \frac{1}{\sqrt{x}}\, dx = \left.\frac{x^{\frac{1}{2}}}{\frac{1}{2}}\right|_0^4 = 2\sqrt{x}\Big|_0^4 = 2\sqrt{4} - 2\sqrt{0} = 4$

6. $\int_0^{\pi} \frac{\cos 2x}{\cos x + \sin x}\, dx$

$= \int_0^{\pi} \frac{\cos^2 x - \sin^2 x}{\sin x + \cos x}\, dx$

$= \int_0^{\pi} \frac{(\cos x + \sin x)(\cos x - \sin x)}{\sin x + \cos x}\, dx$

$= \int_0^{\pi} (\cos x - \sin x) dx$

$= \cos \pi - \sin \pi - \cos 0 + \sin 0$

$= -1 - 0 - 1 + 0 = -2$

7. $\int_0^1 (x + e^x) dx$

$= \left|\frac{1}{2}x^2 + e^x\right|_0^1$

$= \frac{1}{2} + e - 0 - 1$

$= e - \frac{1}{2} \approx 2{,}22$

8. $\int_0^{\pi} x \sin^2 x\, dx$

$= \frac{1}{2}\int_0^{\pi} x\,(1 - \cos 2x) dx$

$= \frac{1}{2}\int_0^{\pi} (x - x \cos 2x) dx$

$= \frac{1}{2}\left(\int_0^{\pi} x\, dx - \int_0^{\pi} x \cos 2x\, dx\right)$

$= \left|\frac{1}{2}\left(\frac{1}{2}x^2 - \frac{1}{2}x \sin 2x - \frac{1}{4}\cos 2x\right)\right|_0^{\pi}$

$= \left(\frac{1}{4}\pi^2 - \frac{\pi}{2} \times 0 - \frac{1}{4} \times 1\right) - \left(\frac{1}{2} \times 0 - \frac{1}{2} \times 0 - \frac{1}{4} \times 1\right)$

$= \frac{\pi^2}{4} \approx 2{,}47$

Activity 5.2 SB page 147

1. $\int_{-1}^{1} (3x^2 - 2x + 2)\, dx = x^3 - x^2 + 2x\Big|_{-1}^{1}$

$= (1 - 1 + 2) - (-1 - 1 - 2) = 6$

2. $\int_1^2 \frac{x}{(x+1)^2}\,dx,$

Let $u = x + 1$, $x = u - 1$, $du = dx$. Limits: $x = 1 \Rightarrow u = 2$, $x = 2 \Rightarrow u = 3$

$$\therefore \int_1^2 \frac{x}{(x+1)^2}\,dx = \int_2^3 \frac{u-1}{u^2}\,du$$

$$= \int_2^3 \left(\frac{u}{u^2} - \frac{1}{u^2}\right) du = \left(\ln u + \frac{1}{u}\right)\Big|_2^3$$

$$= \ln 3 - \ln 2 + \frac{1}{3} - \frac{1}{2} = 0{,}239$$

3. $\int_1^2 \frac{3x^2}{4x^3+1}\,dx.$

Let $u = 4x^3 + 1$, $du = 12x^2dx \therefore 3x^2\,dx = \frac{du}{4}$. If $x = 1 \Rightarrow u = 5$, $x = 2 \Rightarrow u = 33$.

$$\therefore \int_1^2 \frac{3x^2}{4x^3+1}\,dx = \int_5^{33} \frac{1}{4} \times \frac{1}{u}\,du$$

$$= \frac{1}{4}\ln u\Big|_5^{33}$$

$$= \frac{1}{4}\ln\frac{33}{5} = 0{,}472$$

4. $\int_0^2 2x\left(x^2+4\right)^2 dx.$

Let $u = x^2 + 4$, $du = 2x\,dx$. If $x = 0 \Rightarrow u = 4$, $x = 2 \Rightarrow u = 8$.

$$\therefore \int_0^2 2x\left(x^2+4\right)^2 dx = \int_4^8 u^2 du$$

$$= \frac{1}{3}u^3\Big|_4^8$$

$$= \frac{1}{3}(8^3 - 4^3) = 149{,}333$$

5. $\int_0^{\sqrt{\pi}} x \sin x^2 dx.$

Let $u = x^2$, then $du = 2x.dx$. If $x = 0$ then $u = 0$, $x = \sqrt{\pi}$ then $u = \pi$.

$$\int_0^{\sqrt{\pi}} x \sin x^2\,dx = \int_0^{\pi} \frac{1}{2}\sin u\,du$$

$$= -\frac{1}{2}\cos u = -\frac{1}{2}\cos x^2\Big|_0^{\sqrt{\pi}}$$

$$= -\frac{1}{2}\cos\pi - \frac{1}{2}\cos 0$$

$$= -\frac{1}{2}(-1) + \frac{1}{2}(1) = 1$$

6. $\int_0^1 \frac{4}{(x+3)^2}\,dx = \frac{-4}{(x+3)}\Big|_0^1 = -\frac{4}{4} - -\frac{4}{3} = \frac{1}{3}.$

Alternatively: Let $u = x + 3$, $du = dx.x = 0 \Rightarrow u = 3$, $x = 1 \Rightarrow u = 4$.

$$\therefore \int_0^1 \frac{4}{(x+3)^2}\,dx = \int_3^4 \frac{4}{u^2}\,du = -\frac{4}{u}\Big|_3^4 = -\frac{4}{4} - -\frac{4}{3} = \frac{1}{3}$$

7. $\int_0^{\frac{\pi}{2}} \sin x \times \cos^3 x \, dx$

$= -\frac{1}{4}\cos^4 x\Big|_0^{\frac{\pi}{2}}$

$= -\frac{1}{4}(0^4 - 1^4)$

$= \frac{1}{4}$

8. $\int_1^e \frac{\ln x}{x}\, dx = \frac{1}{2}\ln^2 x\Big|_1^e$

$= \frac{1}{2} - \frac{1}{2}(0) = \frac{1}{2}$

Activity 5.3 SB page 152

1. $\int_1^\infty \frac{1}{x}\, dx = \ln|x|_1^\infty = \ln\infty - \ln 1 = \infty \Rightarrow$ divergent function

2. $\int_1^\infty \frac{1}{x^2}\, dx = -\frac{1}{x}\Big|_1^\infty = -\frac{1}{\infty} - -\frac{1}{1} = 1$

3. $\int_{-\infty}^{+\infty} x^2 e^{-x^3}\, dx = \frac{1}{3}e^{-x^3}\Big|_{-\infty}^{+\infty} = \lim_{m\to+\infty}\frac{1}{3}e^{-m^3} - \lim_{n\to-\infty}\frac{1}{3}e^{-n^3} = 0 - \infty \Rightarrow$ diverges

4. $\int_{-\infty}^{+\infty} xe^{-x^2}dx = \int_{-\infty}^{0} xe^{-x^2}dx + \int_0^{+\infty} xe^{-x^2}dx$

$= \left|-\frac{1}{2}e^{-x^2}\right|_{-\infty}^{0} - \left|\frac{1}{2}e^{-x^2}\right|_0^{+\infty}$

$= -\frac{1}{2}e^{-0} - \frac{1}{2}e^{-(-\infty)^2} + \frac{1}{2}e^{-(+\infty)^2} + \frac{1}{2}e^0$

$= -\frac{1}{2} - 0 + 0 + \frac{1}{2}$

$= 0$

5. $\int_1^\infty \frac{1}{2x-1}\, dx$

$= \left|\frac{1}{2}\ln(2x-1)\right|_1^\infty$

$= \frac{1}{2}\ln(2\times\infty - 1) - \frac{1}{2}\ln(2.1 - 1)$

$= \infty - 0 = \infty$

6. $\int_0^\infty \frac{1}{1+x^2}\, dx$

$= \left|\tan^{-1}x\right|_0^\infty$

$= \tan^{-1}\infty - \tan^{-1}0$

$= \frac{\pi}{2} - 0 = \frac{\pi}{2}$

Activity 5.4 **SB page 155**

1. $f(t) = 1$, from $t = b$ to ∞.

$$F(s) = \int_b^\infty e^{-st} \times 1\, dt$$
$$= -\frac{e^{-st}}{s}\Big|_b^\infty$$
$$= -\frac{1}{se^\infty} - -\frac{e^{-sb}}{s}$$
$$= \frac{e^{-bs}}{s}$$

2. $\mathcal{L}\{e^t\}$, from $t = 0$ to ∞.

$$F(s) = \int_0^\infty e^{-st} \times e^t\, dt$$
$$= \int_0^\infty e^{(1-s)t}\, dt$$
$$= \frac{e^{(1-s)t}}{1-s}\Big|_0^\infty$$

Case 1 where $s > 1$: $F(s) = \frac{1}{1-s}\left(e^{(1-s)t} - e^0\right) = \frac{1}{1-s}(0 - 1) = \frac{1}{s-1}$

Case 2 where $s \leq 1$: $F(s) = \frac{1}{1-s}\left(e^{(1-s)t} - e^0\right) = \frac{1}{1-s}(\infty - 1) \Rightarrow$ diverging function

3. $\mathcal{L}\{3t\} = \int_0^\infty e^{-st} 3t\, dt$

Let $u = 3t$, then $u' = 3$, and let $v' = e^{-st}$, then $v = \frac{e^{-st}}{-s}$.

Substitute in $\int uv' = uv - \int u'v$

$$F(s) = 3t\frac{e^{-st}}{-s} - \int_0^\infty 3 \times \frac{e^{-st}}{-s} dt$$
$$= -3\left[\frac{t}{s}e^{-st}\right]_0^\infty + \frac{3}{s}\int_0^\infty e^{-st}\, dt$$

But $\int_0^\infty e^{-st}\, dt = \mathcal{L}\{1\} = \frac{1}{s}$

$$\therefore F(s) = -3(0) - -3\frac{0}{s}e^{-\infty} + \frac{3}{s} \times \frac{1}{s}$$
$$= \frac{3}{s^2}$$

4. 4.1 Given : $F(s) = \frac{5}{s}$

If $F(s) = \frac{c}{s}$ then $f(t) = c$

$\therefore F(s) = \frac{5}{s} \Rightarrow f(t) = 5$

4.2 Given: $F(s) = \frac{1}{s-4}$

If $F(s) = \frac{1}{s-a}$ then $f(t) = e^{at}$

$\therefore F(s) = \frac{1}{s-4} \Rightarrow f(t) = e^{4t}$

4.3 Given: $F(s) = \frac{1}{(2s)^2}$

If $F(s) = \frac{1}{s^2}$ then $f(t) = t$

$\therefore F(s) = \frac{1}{(2s)^2} = \frac{1}{4s^2} \Rightarrow f(t) = \frac{t}{4}$

4.4 Given: $F(s) = \frac{4s-2}{s^2}$

Rewrite: $\frac{4s-2}{s^2} = \frac{4}{s} - \frac{2}{s^2}$

If $F(s) = \frac{4}{s}$ then $f(t) = 4$

If $F(s) = -\frac{2}{s^2}$ then $f(t) = -2t$

$\therefore f(t) = 4 - 2t$

Summative assessment: Module 5 SB page 156

1. 1.1 $\int_0^1 6 \times x^2\, dx = 6\frac{x^3}{3}\Big|_0^1 = 2$ (2)

1.2 $\int_{-1}^{2} (3x^2 - 5x + 2)\, dx = 3\frac{x^3}{3} - 5\frac{x^2}{2} + 2x\Big|_{-1}^{2}$

$= \left((2)^3 - 5\frac{(2)^2}{2} + 2(2)\right) - \left((-1)^3 - 5\frac{(-1)^2}{2} + 2(-1)\right)$

$= 8 - 10 + 4 + 1 + \frac{5}{2} + 2$

$= \frac{15}{2} = 7{,}5$ (3)

1.3 $\int_4^5 \frac{1}{x}\, dx = \ln x\,\big|_4^5 = 0{,}22$ (2)

1.4 $\int_0^4 4x\sqrt{16 - x^2}\, dx$

Let $u = 16 - x^2$, $du = -2x\, dx \therefore dx = \frac{du}{-2x}$.

Limits: $16 - 4^2 = 0$, $16 - 0^2 = 16$.

$\int_0^4 4x\sqrt{16 - x^2}\, dx = \int_0^4 4x\sqrt{u}\, \frac{du}{-2x}$

$= -2\int_{16}^{0} \sqrt{u}\, du$

$= -2\left(\frac{2}{3}u^{\frac{3}{2}}\right)\Big|_{16}^{0}$

$= -\frac{4}{3}\left(0^{\frac{3}{2}} - 16^{\frac{3}{2}}\right) = \frac{256}{3}$ (4)

1.5 $\int_0^{\frac{\pi}{6}} \frac{\sin x}{\cos^2 x}\, dx$

Let $u = \cos x$, $du = -\sin x\, dx \therefore dx = \frac{du}{-\sin x}$.

Limits: $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$, $\cos 0 = 1$.

$$\int_0^{\frac{\pi}{6}} \frac{\sin x}{\cos^2 x}\, dx = \int_0^{\frac{\pi}{6}} \frac{\sin x}{(-\sin x)\,u^2}\, du$$

$$= -\int_1^{\frac{\sqrt{3}}{2}} \frac{du}{u^2} = \frac{1}{u}\Big|_1^{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}} - 1 \qquad (4)$$

1.6 $\int_0^{\infty} \frac{1}{(x+1)^{\frac{3}{2}}}\, dx$

Let $u = x + 1$, $du = dx$. Limits: $0 + 1 = 1$, $\infty + 1 = \infty$

$$\int_0^{\infty} \frac{1}{(x+1)^{\frac{3}{2}}}\, dx = \int_1^{\infty} \frac{1}{u^{\frac{3}{2}}}\, du$$

$$= \frac{u^{-\frac{1}{2}}}{-\frac{1}{2}}\Bigg|_1^{\infty}$$

$$= -\frac{2}{\sqrt{u}}\Bigg|_1^{\infty}$$

$$= -\frac{2}{\sqrt{\infty}} - -\frac{2}{\sqrt{1}} = 2 \qquad (4)$$

1.7 $\int_{-\infty}^{+\infty} \frac{16}{16 + x^2}\, dx = \frac{16}{4}\tan^{-1}\frac{x}{4}\Big|_{-\infty}^{+\infty}$

$$= 4\left(\tan^{-1}\frac{+\infty}{4} - \tan^{-1}\frac{-\infty}{4}\right)$$

$$= 4\left(\frac{\pi}{2} - -\frac{\pi}{2}\right) = 4\pi \qquad (3)$$

2. $F(s) = \mathcal{L}\{e^{at}\} = \int_0^{\infty} e^{-st} e^{at}\, dt$

$$= \int_0^{\infty} e^{t(a-s)}\, dt$$

$$= \left[\frac{e^{t(a-s)}}{a-s}\right]_0^{\infty} \qquad (6)$$

There are two cases here: $s > a$ and $s \leq a$

For $s > a$, $\therefore a - s < 0$: $F(s) = \frac{e^{-\infty}}{a-s} - \frac{e^0}{a-s} = 0 - \frac{1}{a-s} = \frac{1}{s-a}$	For $s \leq a$, $\therefore a - s \geq 0$: $F(s) = \frac{e^{\infty}}{a-s} - \frac{e^0}{a-s} = \infty - \frac{1}{a-s} = \infty$ This function diverges.

3. If $\mathcal{L}\{t\} = \frac{1}{s^2}$, what is $\mathcal{L}\{4t\}$?

$F(s) = \mathcal{L}\{4t\} = \int_0^\infty e^{-st} 4t \, dt$

Let $u = 4t$, then $u' = 4$, and let $v' = e^{-st}$, then $v = \frac{e^{-st}}{-s}$.

$\int uv' = uv - \int u'v$

$\therefore F(s) = 4t \times \frac{e^{-st}}{-s} - \int_0^\infty 4 \times \frac{e^{-st}}{-s} dt$

$= -\left[\frac{4t}{s} e^{-st}\right]_0^\infty + \frac{4}{s}\int_0^\infty e^{-st} dt$

$= -0 - -\frac{0}{s} e^{-\infty} + \frac{4}{s} \times \frac{1}{s}$

$= \frac{4}{s^2}$ (4)

4. $F(s) = \mathcal{L}\{5\} = \int_b^\infty e^{-st} 5 \, dt$

$= 5\left[\frac{e^{-st}}{-s}\right]_b^\infty$

$= \frac{5}{s} \times \left[\frac{-1}{e^\infty} - \frac{-1}{e^{sb}}\right]$

$= \frac{5}{s}\left(0 + \frac{1}{e^{sb}}\right) = \frac{5}{s e^{sb}}$ (3)

TOTAL: [35]

MODULE

6 Areas and volumes

After they have completed this module, students should be able to:

- sketch a function on a given interval;
- calculate and sketch the points of intersection of two functions;
- use a definite integral to calculate the area:
  - of a given function i.e. the area between a curve and a reference axis;
  - of two given functions i.e. the area between two curves; and
- use a definite integral to calculate the volume:
  - of a solid of revolution;
  - between two functions.

Introduction

In Module 5 different integration strategies were used to solve definite integrals. In this module we will expand on using definite integrals to solve complex problems. The focus is on determining the areas and volumes represented by algebraic and trigonometric functions.

Students need the following pre-knowledge to successfully complete this module.

Pre-knowledge

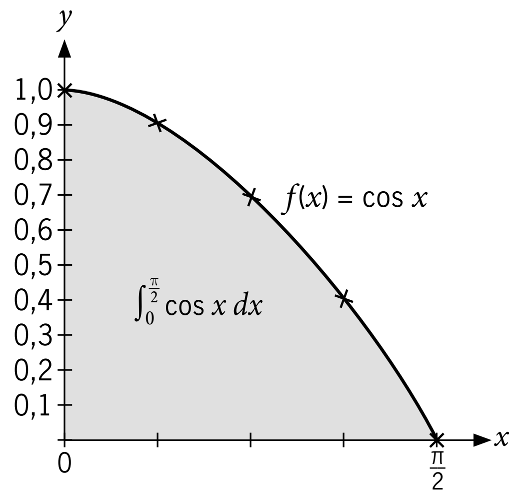
- use different strategies to solve integrals;
- solve definite integrals; and
- find the roots and intersection points of functions.

Activity 6.1 SB page 165

1. $\int_0^{\frac{\pi}{2}} \cos x \, dx = \sin x \Big|_0^{\frac{\pi}{2}}$

$= 1 - 0$

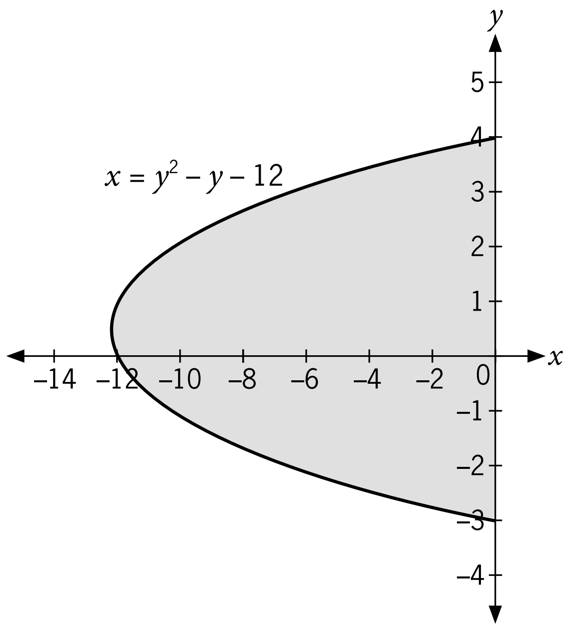
$= 1 \text{ unit}^2$



2. Read the limits from the sketch.

$\int_{-3}^{4} y^2 - y - 12 \, dy = \frac{y^3}{3} - 12y - \frac{y^2}{2}\Big|_{-3}^{4}$

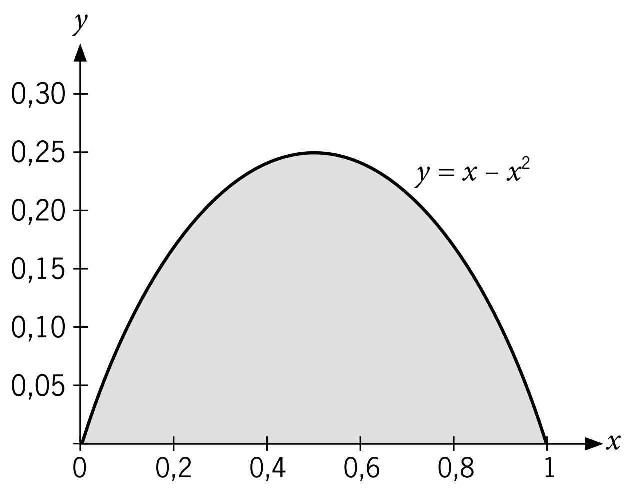
$= -57{,}167 \text{ units}^2$



3. Read the limits from the sketch.

$\int_0^1 (x - x^2) dx = \frac{x^2}{2} - \frac{x^3}{3}\Big|_0^1$

$= \frac{1}{6} \text{ units}^2$



4. $A = \int_0^3 3x^2 - 2x + 3\,dx$

$= x^3 - x^2 + 3x\Big|_0^3$

$= (27 - 9 + 9) - (0 - 0 + 0)$

$= 27 \text{ units}^2$

5. $A = \int_0^{\frac{\pi}{2}} x \sin x\,dx$

$= (\sin x - x\cos x)\Big|_0^{\frac{\pi}{2}}$

$= \left(\sin\frac{\pi}{2} - \frac{\pi}{2}\cos\frac{\pi}{2}\right) - (\sin 0 - 0\cos 0)$

$= 1 - \frac{\pi}{2}.0 - 0 + 0 = 1 \text{ units}^2$

6. $A = \int_0^{\frac{\pi}{2}} x \sin 3x\,dx$

Use integration by parts:

$u = x$, $dv = \sin 3x\,dx$, $v = \int \sin 3x\,dx = -\frac{1}{3}\cos 3x, \frac{du}{dx} = \frac{d}{dx}x = 1, \therefore du = dx$

$\int u\,dv = uv - \int v\,du$

$\therefore \int_0^{\frac{\pi}{2}} x \sin 3x\,dx = x.-\frac{1}{3}\cos 3x + \frac{1}{3}\int_0^{\frac{\pi}{2}} \cos 3x\,dx$

$= \left(-\frac{1}{3}x\cos 3x + \frac{1}{3}.\frac{1}{3}\sin 3x\right)\Big|_0^{\frac{\pi}{2}}$

$= \frac{1}{9}(\sin 3x - 3x\cos 3x)\Big|_0^{\frac{\pi}{2}}$

$= \frac{1}{9}(-1 - 0 - 0 + 0)$

$= -\frac{1}{9} \approx 0{,}11 \text{ units}^2$ (area cannot be a negative value)

Activity 6.2 — SB page 167

1. $x^2 + 2x + 3 = x + 9$

$x^2 + x - 6 = 0$

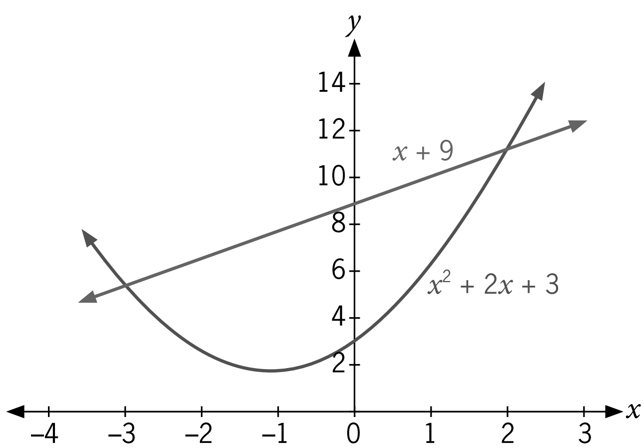
$(x + 3)(x - 2) = 0$

$\therefore x = -3, x = 2$

If $x = -3$, $y = x + 9 = -3 + 9 = 6$

If $x = 2$, $y = x + 9 = 2 + 9 = 11$

$\therefore (x; y) = (-3; 6), (2; 11)$



2. $-4x^2 + 3x + 6 = 3x + 2$

$-4x^2 + 4 = 0$

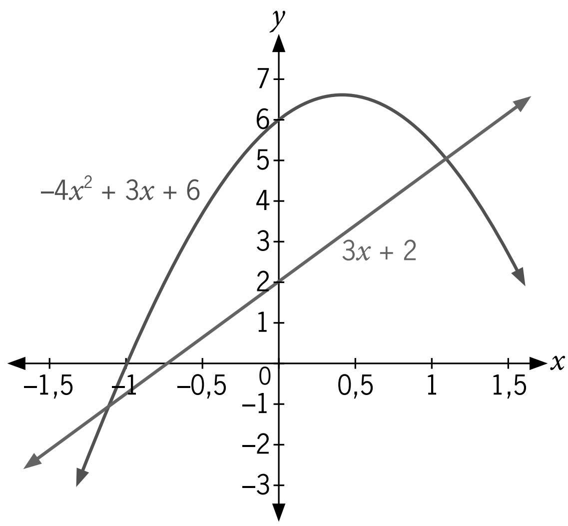
$x^2 = 1$

$\therefore x = \pm 1$

If $x = -1$, $y = 3x + 2 = 3(-1) + 2 = -1$

If $x = 1$, $y = 3x + 2 = 3(1) + 2 = 5$

$\therefore (x; y) = (-1; -1), (1; 5)$



3. $3x^2 - 3x + 2 = -2x + 4$

$3x^2 - x - 2 = 0$

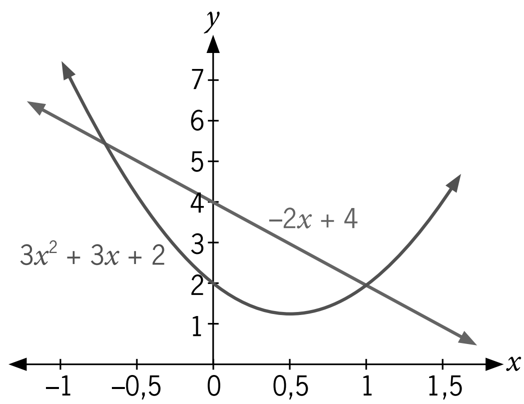
$(3x + 2)(x - 1) = 0$

$\therefore x = -\frac{2}{3}, x = 1$

If $x = -\frac{2}{3}$, $y = -2x + 4 = -2\left(-\frac{2}{3}\right) + 4 = 5\frac{1}{3}$

If $x = 1$, $y = -2x + 4 = -2(1) + 4 = 2$

$\therefore (x; y) = \left(-\frac{2}{3}; 5\frac{1}{3}\right), (1; 2)$



4. $-3x^2 + 6x + 8 = x^2 - 4x + 2$

$-4x^2 + 10x + 6 = 0$

$(4x + 2)(-x + 3) = 0$

$\therefore x = -\frac{1}{2}, x = 3$

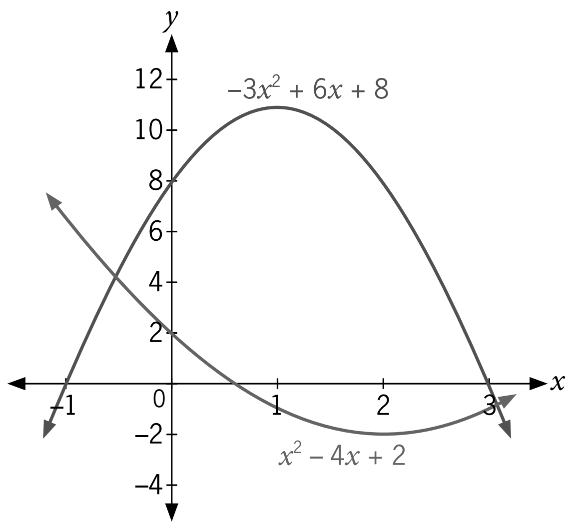
If $x = 3$, $y = x^2 - 4x + 2$

$= 3^2 - 4(3) + 2 = -1$

If $x = -\frac{1}{2}$, $y = \left(-\frac{1}{2}\right)^2 - 4\left(-\frac{1}{2}\right) + 2$

$= \frac{1}{4} + 2 + 2 = 4\frac{1}{4}$

$\therefore (x; y) = \left(-\frac{1}{2}; 4\frac{1}{4}\right), (3; -1)$



5. The points of intersection are where $f(x) = g(x)$.

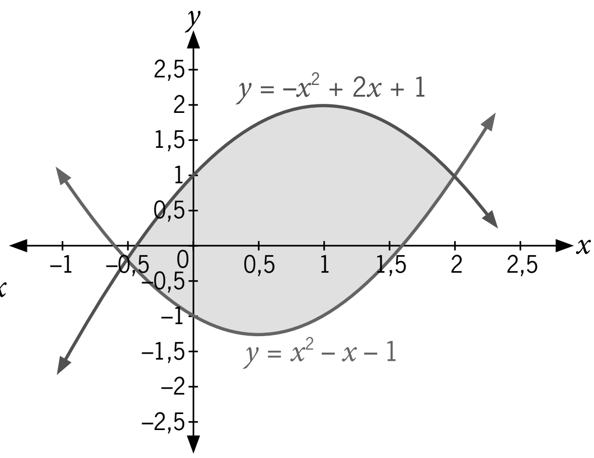
   Therefore set:
   $x + 2 = x^2$
   $\therefore x^2 - x - 2 = 0$
   $(x - 2)(x + 1) = 0$
   $\therefore x = 2$ and $x = -1$

   Determine the y-values:
   $y = x^2$
   $\therefore x = 2 \Rightarrow y = 2^2 = 4$
   $\therefore x = -1 \Rightarrow y = (-1)^2 = 1$

   The coordinates are therefore:
   $(x; y) = (2; 4), (-1; 1)$

Activity 6.3 SB page 171

1. The intersection points are where:
   $x^2 - x - 1 = -x^2 + 2x + 1$
   $2x^2 - 3x - 2 = 0$
   $(2x + 1)(x - 2) = 0$
   $x = -\frac{1}{2}, x = 2$
   $\int_{-\frac{1}{2}}^{2} -x^2 + 2x + 1\,dx - \int_{-\frac{1}{2}}^{2} x^2 - x - 1\,dx$
   $= -\frac{x^3}{3} + x^2 + x - \frac{x^3}{3} + \frac{x^2}{2} + x\Big|_{-\frac{1}{2}}^{2}$
   $= -\frac{2}{3}x^3 + \frac{3}{2}x^2 + 2x\Big|_{-\frac{1}{2}}^{2}$
   $= -\frac{2}{3}(2)^3 + \frac{3}{2}(2)^2 + 2(2) + \frac{2}{3}\left(-\frac{1}{2}\right)^3 - \frac{3}{2}\left(-\frac{1}{2}\right)^2 - 2\left(-\frac{1}{2}\right)$
   $= -\frac{16}{3} + 6 + 4 - \frac{1}{12} - \frac{3}{8} + 1$
   $= \frac{-128 + 11 \times 24 - 2 - 9}{24}$
   $= \frac{125}{24} = 5\frac{5}{24} \sim 5{,}208$ units^2



2. The intersection points are where: $\frac{5}{x^2+1} = 1$

$\therefore x^2 = 4$

$x = \pm 2$

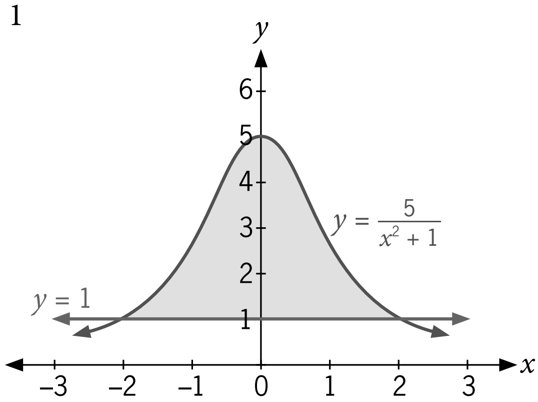
$\int_{-2}^{2} \frac{5}{x^2+1}\,dx - \int_{-2}^{2} 1\,dx$

$= (5\tan^{-1}x - x)\Big|_{-2}^{2}$

$= 5[\tan^{-1}(2) - \tan^{-1}(-2)] - 4$

$= 7{,}071$ units^2

Note: evaluate $\tan^{-1}$ using radians



3. The intersection points are where:

$6 - x = \frac{5}{x}$

$\Rightarrow 6x - x^2 = 5$

$x^2 - 6x + 5 = 0$

$(x-1)(x-5) = 0$

$x = 1, x = 5$

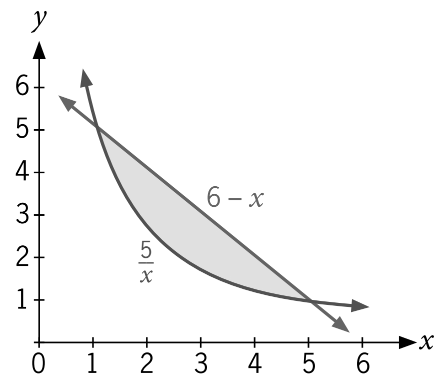
$\int_1^5 6 - x - \frac{5}{x}\,dx = 6x - \frac{x^2}{2} - 5\ln x\Big|_1^5$

$= \left(6(5) - \frac{5^2}{2} - 5\ln 5\right) - \left(6(1) - \frac{1^2}{2} - 5\ln 1\right)$

$= 30 - \frac{25}{2} - 5\ln 5 - 6 + \frac{1}{2} + 5(0)$

$= 12 - 5\ln 5$ units^2

x	1	2	3	4	5
***f*(*x*)**	5	4	3	2	1
***g*(*x*)**	5	2,5	$1\frac{2}{3}$	$1\frac{1}{4}$	1



4. Set:

$x^2 = \sqrt[3]{x}$

also written as: $x^2 = x^{\frac{1}{3}}$

$\therefore x^6 = x$

$x^6 - x = 0$

$x(x^5 - 1) = 0$

$\therefore x = 0$ and $x = 1$

For $0 < x < 1$, $x^6 < x$

To determine the area:

$A = \int_0^1 x^{\frac{1}{3}} - x^2 dx$

$= \left(\frac{3}{4}x^{\frac{4}{3}} - \frac{1}{3}x^3\right)\Big|_0^1$

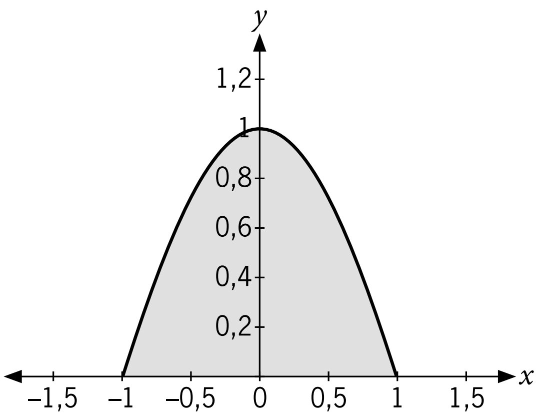
$= \frac{3}{4} - \frac{1}{3} - 0 + 0$

$= \frac{5}{12} \approx 0{,}417$ units^2

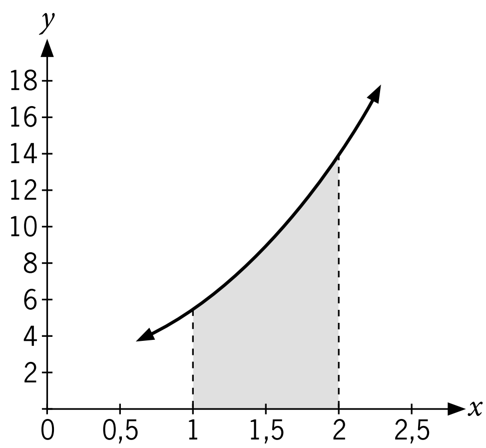
5. Set:
$3 - x^2 = x - 3$
$\therefore x^2 + x - 6 = 0$
$(x + 3)(x - 2) = 0$
$\therefore x = -3$ and $x = 2$
For $-3 \le x \le 2$ the function $3 - x^2 > x - 3$
Determine the area:
$A = \int_{-3}^{2} x^2 + x - 6\, dx$
$= \left(\frac{1}{3}x^3 + \frac{1}{2}x^2 - 6x\right)\Big|_{-3}^{2}$
$= \frac{8}{3} + \frac{4}{2} - 12 + \frac{27}{3} - \frac{9}{2} - 6.3$
$= \frac{16 + 12 - 72 + 54 - 27 - 108}{6}$
$= -\frac{125}{6} \approx 20{,}83$ units^2 (area cannot be a negative value)

Activity 6.4 SB page 175

1. $V_x = \pi\int(f(x))^2\, dx$
Substitute
$f(x) = 1 - x^2$ with limits $x = -1$ to $x = 1$:
$\therefore V = \pi\int_{-1}^{1} (1 - x^2)^2\, dx$
$= \pi\int_{-1}^{1} 1 - 2x^2 + x^4\, dx$
$= \pi\left(x - \frac{2}{3}x^3 + \frac{x^5}{5}\right)\Big|_{-1}^{1}$
$= 2\pi\left(1 - \frac{2}{3} + \frac{1}{5}\right)$
$= \frac{2.(15 - 10 + 3)}{15}\pi = \frac{16}{15}\pi = 3{,}351$ units^3



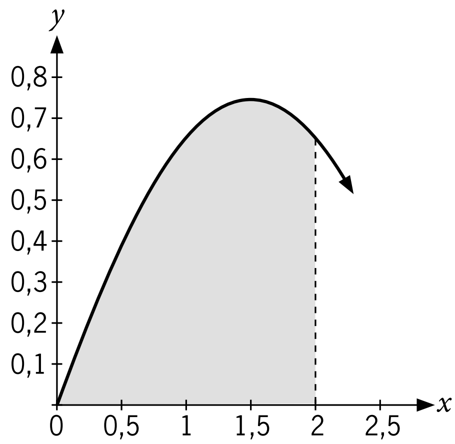
2. $V_x = \pi\int(f(x))^2\, dx$
Substitute
$f(x) = 2e^x$ with limits $x = 1$ to $x = 2$:
$\therefore V = \pi\int_{1}^{2} (2e^x)^2\, dx$
$= 4\pi\int_{1}^{2} e^{2x}\, dx$
$= 4\pi\left(\frac{1}{2}e^{2x}\right)\Big|_{1}^{2}$
$= 2\pi(e^4 - e^2) = 296{,}623$ units^3



3. $V_x = \pi\int(f(x))^2\,dx$

Substitute

$f(x) = x - \frac{x^2}{3}$ with limits $x = 0$ to $x = 2$:



$$\therefore V = \pi\int_0^2 \left(x - \frac{x^2}{3}\right)^2 dx$$

$$= \pi\int_0^2 \left(x^2 - \frac{2}{3}x^3 + \frac{x^4}{9}\right)dx$$

$$= \pi\left(\frac{x^3}{3} - \frac{2x^4}{3.4} + \frac{x^5}{5.9}\right)\Big|_0^2$$

$$= \pi\left(\frac{8}{3} - \frac{16}{6} + \frac{32}{45}\right)$$

$$= \pi\left(\frac{32}{45}\right) = 2{,}234 \text{ units}^3$$

4. $V_x = \pi\int_a^b f(x)^2\,dx$

$$= \pi\int_1^2 (5x^2 + 1)^2\,dx$$

$$= \pi\int_1^2 (25x^4 + 10x^2 + 1)\,dx$$

$$= \pi\left(\frac{25}{5}x^5 + \frac{10}{3}x^3 + x\right)\Big|_1^2$$

$$= \pi\left[\left(5.2^5 + \frac{10}{3}.2^3 + 2\right) - \left(5.1^5 + \frac{10}{3}.1^3 + 1\right)\right]$$

$$= \pi\left(\frac{538}{3}\right) \approx 563{,}4 \text{ units}^3$$

5. $V_x = \pi\int_a^b f(x)^2\,dx$

$$= \pi\int_0^2 (2x^2 - 3x)^2\,dx$$

$$= \pi\int_0^2 (4x^4 - 12x^3 + 9x^2)\,dx$$

$$= \pi\left(\frac{4}{5}x^5 - \frac{12}{4}x^4 + \frac{9}{3}x^3\right)\Big|_0^2$$

$$= \pi\left[\left(\frac{4}{5}.2^5 - 3.2^4 + 3.2^3\right) - \left(\frac{4}{5}.0^5 - 3.0^4 + 3.0^3\right)\right]$$

$$= \pi\left(\frac{8}{5}\right) \approx 5{,}03 \text{ units}^3$$

6. $V_x = \pi\int_a^b f(x)^2\,dx$

$$= \pi\int_1^2 (\sqrt{x \ln x})^2\,dx$$

$$= \pi\int_1^2 x \ln x\,dx$$

Use integration by parts:

$u = \ln x,\ dv = x\,dx,\ v = \frac{1}{2}x^2,\ \frac{du}{dx} = \frac{d}{dx}\ln x = \frac{1}{x},\ \therefore du = \frac{1}{x}\,dx$

$\int u\,dv = uv - \int v\,du$

$$\therefore \pi\int_1^2 x\ln x\,dx = \pi\left(\ln x.\tfrac{1}{2}x^2 - \int\tfrac{1}{2}x^2.\tfrac{1}{x}\,dx\right)\Big|_1^2$$

$$= \pi\left(\tfrac{1}{2}x^2.\ln x - \int\tfrac{1}{2}x\,dx\right)\Big|_1^2$$

$$= \pi\left(\tfrac{1}{2}x^2.\ln x - \tfrac{1}{2}.\tfrac{1}{2}x^2\right)\Big|_1^2$$

$$= \pi\left(\tfrac{4}{2}\ln 2 - \tfrac{4}{4} - \tfrac{1}{2}\ln 1 + \tfrac{1}{4}\right)$$

$$= \pi\left(2\ln 2 - \tfrac{3}{4}\right)$$

$$= 2{,}00 \text{ units}^3$$

Activity 6.5 SB page 178

1. Rewrite the function in terms of y:

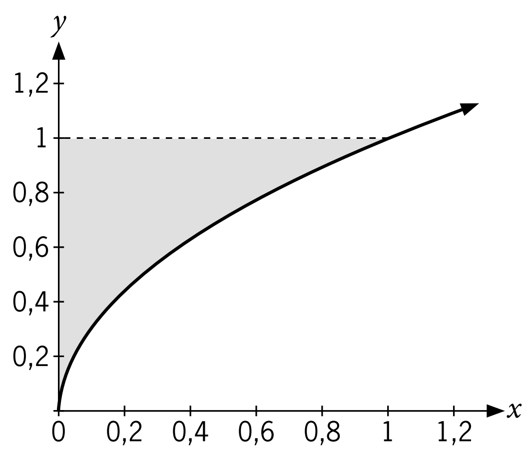
$x = y^2$

$$V_y = \pi\int (f(y))^2\,dy$$

$$= \pi\int_0^1 (y^2)^2\,dy$$

$$= \pi\frac{y^5}{5}\Big|_0^1$$

$$= \frac{\pi}{5} = 0{,}628 \text{ units}^3$$



2. Rewrite the function in terms of y:

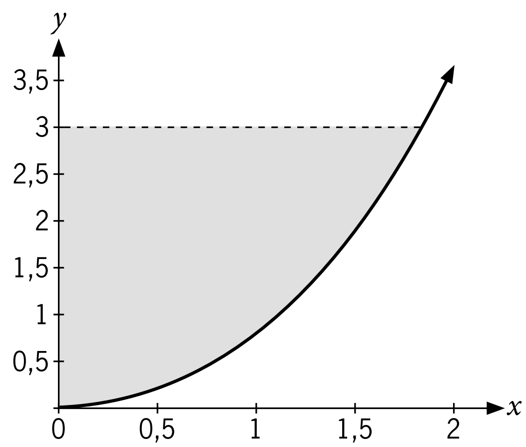
$x = \sqrt{y}$

$$V_y = \pi\int (f(y))^2\,dy$$

$$= \pi\int_0^3 (\sqrt{y})^2\,dy$$

$$= \pi\frac{y^2}{2}\Big|_0^3$$

$$= \frac{9}{2}\pi = 14{,}137 \text{ units}^3$$

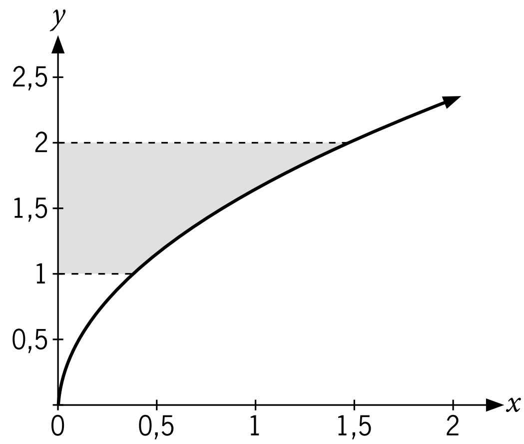


3. $V_y = \pi\int(f(y))^2\,dy$

$= \pi\int_1^2 \left(\frac{y^2}{3}\right)^2 dy$

$= \pi\frac{y^5}{9.5}\Big|_1^2$

$= \pi\left(\frac{32}{45} - \frac{1}{45}\right) = 2{,}164 \text{ units}^3$



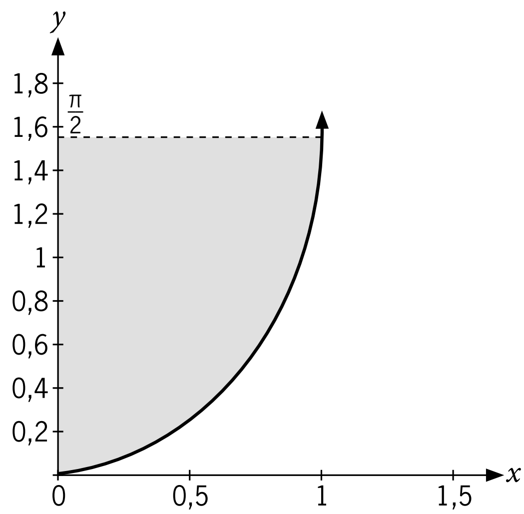
4. $V_y = \pi\int(f(y))^2\,dy$

$= \pi\int_0^{\frac{\pi}{2}} (\sqrt{\sin y})^2\,dy$

$= \pi\int_0^{\frac{\pi}{2}} \sin y\,dy$

$= -\pi.\cos y\Big|_0^{\frac{\pi}{2}}$

$= -\pi(0 - 1) = \pi = 3{,}142 \text{ units}^3$



5. $V_y = \pi\int_a^b f(y)^2\,dy$

$y = x^2 - 4$

$\therefore x = \sqrt{y + 4}$

$V_y = \pi\int_a^b f(y)^2\,dy$

$= \pi\int_2^4 \left(\sqrt{y + 4}\right)^2 dy$

$= \pi\int_2^4 y + 4\,dy$

$= \pi\left(\frac{y^2}{2} + 4y\right)\Big|_2^4$

$= \pi\left(\frac{4^2}{2} + 4.4 - \frac{2^2}{2} - 4.2\right)$

$= 14\pi \approx 43{,}98 \text{ units}^3$

6. $V_x = \pi\int_a^b f(x)^2 dx$

$= \pi\int_1^3 \left[\sqrt{(x+3)^3}\right]^2 dx$

$= \pi\int_1^3 (x+3)^3 dx$

$= \pi\int_1^3 (27 + 27\,x + 9\,x^2 + x^3) dx$

$= \pi\left(27x + \frac{27}{2}x^2 + \frac{9}{3}x^3 + \frac{1}{4}x^4\right)\Big|_1^3$

$= \pi\left(27.3 + \frac{27}{2}3^2 + \frac{9}{3}3^3 + \frac{1}{4}3^4\right) - \pi\left(27.1 + \frac{27}{2}1^2 + \frac{9}{3}1^3 + \frac{1}{4}1^4\right)$

$= 260\pi \approx 816{,}81 \text{ units}^3$

Activity 6.6 SB page 183

1. Determine the intersection points:

$x^2 = 2x$

$\therefore x^2 - 2x = 0$

$x(x-2) = 0$

$x = 0$ and $x = 2$

$V_x = \pi\int_c^d [f(x)^2 - g(x)^2 dx$

$= \pi\int_0^2 (2x)^2 - (x^2)^2 dx$

$= \pi\int_0^2 4x^2 - x^4 dx$

$= \pi\left(\frac{4}{3}x^3 - \frac{1}{5}x^5\right)\Big|_0^2$

$= \pi\left(\frac{32}{3} - \frac{32}{5}\right) = 13{,}404 \text{ units}^3$

2. Determine the intersection points:

$2y^2 = \sqrt{\frac{y}{2}}$

$4y^4 = \frac{y}{2}$

$8y^4 - y = 0$

$y(8y^3 - 1) = 0$

$\therefore y = 0$ and $y^3 = \frac{1}{8} \Rightarrow y = \frac{1}{2}$

$V_y = \pi\int_0^{\frac{1}{2}} \left(\sqrt{\frac{y}{2}}\right)^2 - (2y^2)^2 dy$

$= \pi\int_0^{\frac{1}{2}} \frac{y}{2} - 4y^4 dy$

$= \pi\left(\frac{y^2}{4} - \frac{4y^5}{5}\right)\Big|_0^{\frac{1}{2}}$

$= \pi\left[\frac{1}{16} - \frac{4}{5}\left(\frac{1}{32}\right)\right]$

$= \pi\frac{10-4}{160} = \pi\frac{3}{80} = 0{,}118 \text{ units}^3$

3. Determine the intersection points:

$x^2 - 2x + 2 = -x^2 - 3x + 3$

$\therefore 2x^2 + x - 1 = 0$

$(2x - 1)(x + 1) = 0$

$\therefore x = \frac{1}{2}$ and $x = -1$

$$V_x = \pi \int_c^d [f(x)^2 - g(x)^2]\, dx$$

$$= \pi \int_{-1}^{\frac{1}{2}} (-x^2 - 3x + 3)^2 - (x^2 - 2x + 2)^2\, dx$$

$$= \pi \int_{-1}^{\frac{1}{2}} (x^4 + 3x^3 - 3x^2 + 3x^3 + 9x^2 - 9x - 3x^2 - 9x + 9)$$

$$- (x^4 - 2x^3 + 2x^2 - 2x^3 + 4x^2 - 4x + 2x^2 - 4x + 4)\, dx$$

$$= \pi \int_{-1}^{\frac{1}{2}} 10x^3 - 5x^2 - 10x + 5\, dx$$

$$= \pi \left(10\frac{x^4}{4} - 5\frac{x^3}{3} - 10\frac{x^2}{2} + 5x\right)\Big|_{-1}^{\frac{1}{2}}$$

$$= \pi\left(\frac{10}{64} - \frac{5}{24} - \frac{10}{8} + \frac{5}{2}\right) - \pi\left(\frac{10}{4} + \frac{5}{3} - \frac{10}{2} - 5\right)$$

$$= \pi\left(\frac{15 - 20 - 120 + 240 - 240 - 160 + 480 + 480}{96}\right)$$

$$= \pi\frac{225}{32} = 22{,}089 \text{ units}^3$$

4. For the range, $0 \leq x \leq 1$ it is possible to show that $-2x^2 + 2x \geq -x^2 + x$.

$$V_x = \pi\int_c^d [f(x)^2 - g(x)^2]\, dx$$

$$= \pi\int_0^1 (-2x^2 + 2x)^2 - (-x^2 + x)^2\, dx$$

$$= \pi\int_0^1 (-2x^2 + 2x)^2 - (-x^2 + x)^2\, dx$$

$$= \pi\int_0^1 4x^4 - 8x^3 + 4x^2 - x^4 + 2x^3 - x^2\, dx$$

$$= \pi\int_0^1 3x^4 - 6x^3 + 3x^2\, dx$$

$$= \pi\left(\frac{3}{5}x^5 - \frac{6}{4}x^4 + \frac{3}{3}x^3\right)\Big|_0^1$$

$$= \pi\left(\frac{3}{5} - \frac{3}{2} + 1 - 0 + 0 - 0\right)$$

$$= \pi\frac{6 - 15 + 10}{10}$$

$$= \frac{\pi}{10} \approx 0{,}314 \text{ units}^3$$

5. For the range, $0 \leq y \leq 1$ it is possible to show that $2 \geq y^3 + y$.

$V_y = \pi\int_c^d [f(y)^2 - g(y)^2]dy$

$= \pi\int_0^1 [2^2 - (y^3 + y)^2]dy$

$= \pi\int_0^1 [4 - y^6 - 2y^4 - y^2]dy$

$= \pi\left(4y - \frac{1}{7}y^7 - \frac{2}{5}y^5 - \frac{1}{3}y^3\right)\Big|_0^1$

$= \pi\left(4 - \frac{1}{7} - \frac{2}{5} - \frac{1}{3} - 0 + 0 + 0 + 0\right)$

$= \pi\left(4 - \frac{1}{7} - \frac{2}{5} - \frac{1}{3} - 0 + 0 + 0 + 0\right)$

$= \pi\left(\frac{420 - 15 - 2.21 - 35}{105}\right)$

$= \pi.\frac{328}{105} \approx 9,81 \text{ units}^3$

6. For the range, $0 \leq x \leq \pi$ it is possible to show that $1 \geq \sin x$.

$V_x = \pi\int_c^d [f(x)^2 - g(x)^2]dx$

$= \pi\int_0^\pi (1)^2 - (\sin x)^2 dx$

$= \pi\int_0^\pi 1 - \sin^2 x\, dx$

$= \pi\int_0^\pi \cos^2 x\, dx$

However, $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$, therefore:

$V_x = \frac{\pi}{2}\int_0^\pi (1 + \cos 2x)dx$

$= \frac{\pi}{2}\left(x + \frac{1}{2}\sin 2x\right)\Big|_0^\pi$

$= \frac{\pi}{2}\left(\pi + \frac{1}{2}\sin 2\pi - 0 - \frac{1}{2}\sin 0\right)$

$= \frac{\pi}{2}(\pi + 0 - 0 - 0)$

$= \frac{\pi^2}{2} \approx 4,93 \text{ units}^3$

Summative assessment: Module 6 — SB page 184

1. 1.1 Limits:

$x^2 = x$

$\therefore x(x - 1) = 0$

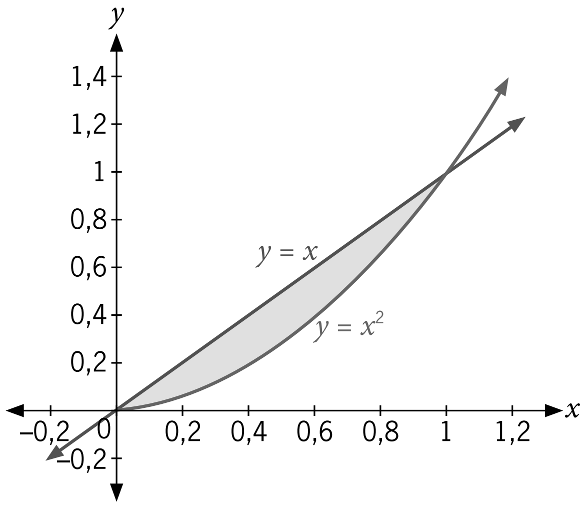
$\Rightarrow x = 0$ and $x = 1$

$A_x = \int_0^1 (x - x^2)\,dx$

$= \frac{x^2}{2} - \frac{x^3}{3}\Big|_0^1$

$= \frac{1}{2} - \frac{1}{3}$

$= \frac{1}{6}$ units^2



(6)

1.2 Limits:

$\frac{x^2}{16} = 1$

$x^2 = 16$

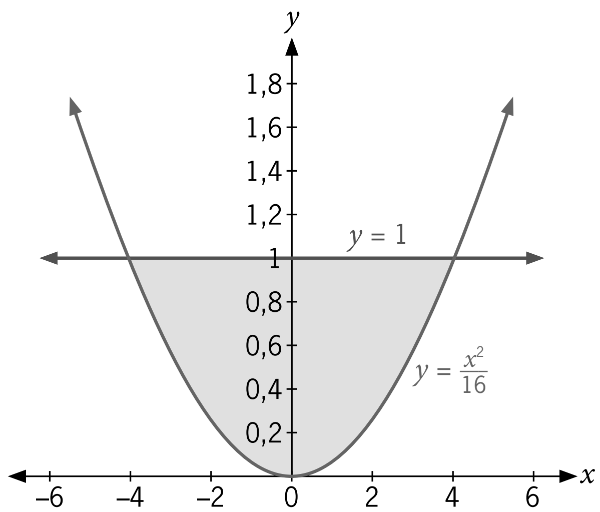
$\therefore x = \pm 4$

$A_x = \int_{-4}^{4} \left(1 - \frac{x^2}{16}\right) dx$

$= \left(x - \frac{x^3}{3{,}16}\right)\Big|_{-4}^{4}$

$= \left(4 - \frac{64}{3{,}16}\right) - \left(-4 - \frac{-64}{3{,}16}\right)$

$= \frac{16}{3}$ units^2



(6)

1.3 Limits:

$9x^2 = x^4$

$\therefore x^4 - 9x^2 = 0$

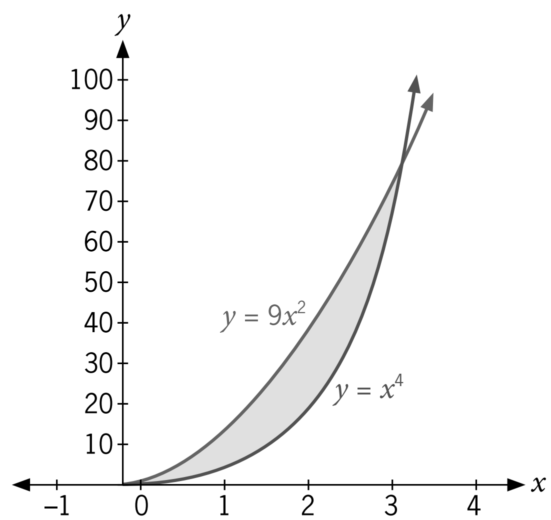
$x^2(x - 3)(x + 3) = 0$

$\Rightarrow x = 0$ and $x = 3$

(discard $x = -3$ as it is not in the first quadrant)

$A_x = \int_0^3 (9x^2 - x^4)\,dx$

$= \frac{9x^3}{3} - \frac{x^5}{5}\Big|_0^3 = 32{,}4$ units^2



(6)

2. $y = h - \frac{h}{r}x$

$\therefore x = r - \frac{r}{h}y$

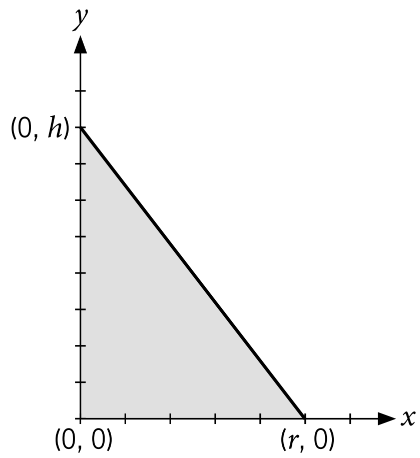
$$V_y = \pi\int_a^b f(y)^2\, dy$$

$$= \pi\int_0^h \left[r\left(1 - \frac{y}{h}\right)\right]^2 dy$$

$$= \pi r^2 \int_0^h \left(1 - \frac{2y}{h} + \frac{y^2}{h^2}\right) dy$$

$$= \pi r^2 \left(y - \frac{2y^2}{2h} + \frac{y^3}{3h^2}\right)\Bigg|_0^h$$

$$= \pi r^2\left(h - h + \frac{h}{3}\right) = \frac{1}{3}\pi r^2 h$$



(5)

3. 3.1 $V_x = \pi\int_a^b f(x)^2\, dx$

$$= \pi\int_0^2 (x + 1)^2\, dx$$

$$= \pi\int_0^2 x^2 + 2x + 1\, dx$$

$$= \pi\left(\frac{x^3}{3} + x^2 + x\right)\Bigg|_0^2$$

$$= \pi\left(\frac{8}{3} + 4 + 2\right) = \frac{26}{3}\pi \text{ units}^3$$ (4)

3.2 $V_y = \pi\int_a^b f(y)^2\, dy$

$$= \pi\int_0^2 \left(\frac{y}{2} + 1\right)^2 dy$$

$$= \pi\int_0^2 \frac{y^2}{4} + y + 1\, dy$$

$$= \pi\left(\frac{y^3}{4.3} + \frac{y^2}{2} + y\right)\Bigg|_0^2$$

$$= \pi\left(\frac{8}{12} + \frac{4}{2} + 2\right) = \frac{14}{3}\pi = 14{,}7 \text{ units}^3$$ (4)

3.3 $V_x = \pi\int_a^b f(x)^2\, dx$

$$= \pi\int_1^3 \left(\frac{x^2}{2} + x\right)^2 dx$$

$$= \pi\int_1^3 \frac{x^4}{4} + x^3 + x^2\, dx$$

$$= \pi\left(\frac{x^5}{4.5} + \frac{x^4}{4} + \frac{x^3}{3}\right)\Bigg|_1^3$$

$$= \pi\left(\frac{243}{20} + \frac{81}{4} + \frac{27}{3} - \frac{1}{20} - \frac{1}{4} - \frac{1}{3}\right)$$

$$= 128{,}072 \text{ units}^3$$ (4)

4. 4.1 Limits: $x^2 = 3x$

$\therefore x^2 - 3x = 0$

$x(x - 3) = 0$

$\Rightarrow x = 0$ and $x = 3$

$$V_x = \pi\int_c^d [f(x)^2 - g(x)^2]\, dx$$

$$= \pi\int_0^3 (3x)^2 - (x^2)^2\, dx$$

$$= \pi\int_0^3 9x^2 - x^4\, dx$$

$$= \pi\left(\frac{9x^3}{3} - \frac{x^5}{5}\right)\Bigg|_0^3 = \pi\left(3(27) - \frac{243}{5}\right)$$

$= 101{,}788$ units^3 (5)

4.2 Limits: $x^2 = 10$

$\therefore x = \pm\sqrt{10}$

$$V_x = \pi\int_c^d [f(x)^2 - g(x)^2]\, dx$$

$$= \pi\int_{-\sqrt{10}}^{\sqrt{10}} 10^2 - (x^2)^2\, dx$$

$$= \pi\int_{-\sqrt{10}}^{\sqrt{10}} 100 - x^4\, dx$$

$$= 2\pi\left(100x - \frac{x^5}{5}\right)\Bigg|_0^{\sqrt{10}}$$

$$= 2\pi\left(100\sqrt{10} - \frac{100\sqrt{10}}{5}\right)$$

$= 160.\pi.\sqrt{10} = 1\ 589{,}534$ units^3 (5)

4.3 Limits: $x = y = 1$ and $x = 2$

$$V_x = \pi\int_c^d [f(x)^2 - g(x)^2]\, dx$$

$$= \pi\int_1^2 x^2 - (1)^2\, dx$$

$$= \pi\left(\frac{x^3}{3} - x\right)\Bigg|_1^2 = \pi\left(\frac{8}{3} - 2 - \frac{1}{3} + 1\right)$$

$= \frac{4}{3}\pi = 4{,}189$ units^3 (5)

TOTAL: [50]

MODULE

7 Second moment of area and moment of inertia (second moment of mass)

After they have completed this module, students should be able to:

- sketch a rectangular or circular lamina on a given interval;
- calculate the second moment of area with respect to a rectangular or circular lamina; and
- calculate the second moment of mass, also known as the moment of inertia, with respect to a rectangular or circular lamina.

Introduction

In this module we continue looking at applications of integration. We will look at ways to evaluate the resistance of an object to change, particularly changes such as bending or torsion.

The second moment of area is an indication of the resistance of a given object against bending or torsion. Different geometric forms with the same surface area resist bending or torsion differently. Students learn how to calculate the second moment of area for different shapes at different distances from the axis of rotation. This moment is also sometimes referred to as the area moment of inertia.

The second moment of mass, or mass moment of inertia of a body is an indication of its resistance to rotation. The calculations are very similar to the second moment of area, except that it relates to the mass distribution of the object to be rotated and not the area of the object. Different shapes with the same mass have different moments of inertia, which students will learn to calculate.

Students need the following pre-knowledge to successfully complete this module.

Pre-knowledge

- use a number of different methods to solve definite integrals;
- sketch a function with respect to an axis; and
- determine the centroid of a shape.

Activity 7.1 SB page 204

1. Using the given formula:

$I_{pa} = Ad^2 + \frac{1}{12}bh^3$

$A = bh$

$\therefore I_{pa} = (bh)\,d^2 + \frac{1}{12}bh^3$

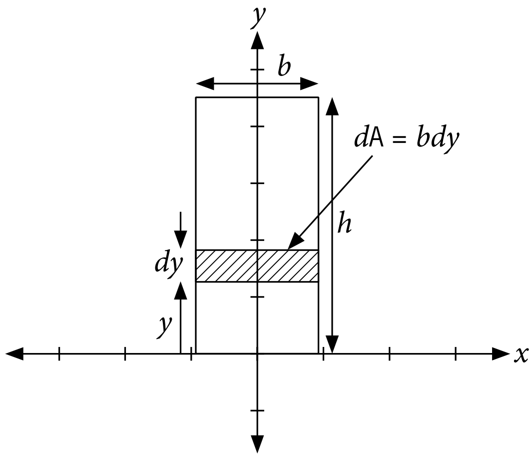
If one edge rests on the x-axis then $d = \frac{h}{2}$

$\therefore I_{pa} = (bh)\left(\frac{h}{2}\right)^2 + \frac{1}{12}bh^3$

$= \frac{1}{4}bh^3 + \frac{1}{12}bh^3$

$= \frac{4}{12}bh^3$

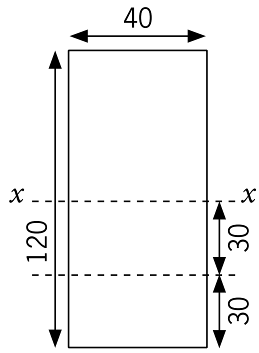
$= \frac{1}{3}bh^3$



2. 2.1 $I_{pa} = Ad^2 + \frac{1}{12}bh^3$

$= 40 \times 120 \times 30^2 + \frac{1}{12}40\,(120)^3$

$= 10\,080\,000 \text{ mm}^4$



2.2 For the web:

$I_{xxw} = \frac{1}{12}bh^3$

$= \frac{1}{12}.6.(100)^3$

$= 500\,000 \text{ mm}^4$

For the flanges:

$I_{xxf} = Ad^2 + \frac{1}{12}bh^3$

$= 80.8.(54)^2 + \frac{1}{12}.80.(8)^3$

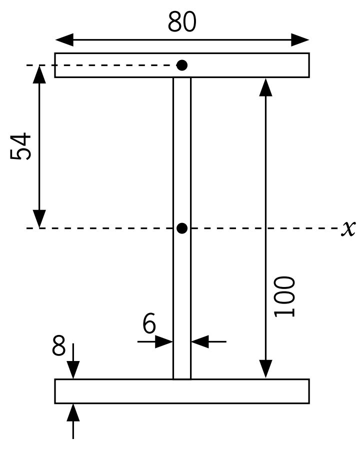
$= 1\,869\,653{,}333 \text{ mm}^4$

The total second moment of area is then:

$I_{xxT} = I_{xxw} + 2 \times I_{xxf}$

$= 500\,000 + 2 \times 1\,869\,653{,}333$

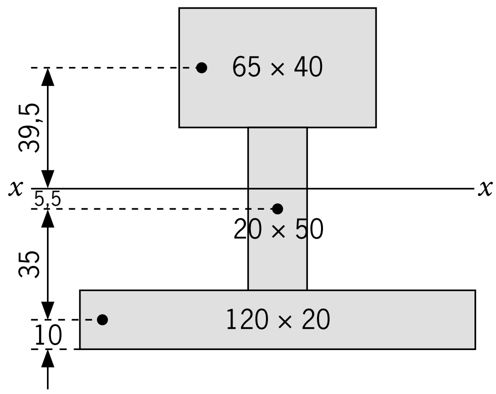
$= 4\,239\,307 \text{ mm}^4$



2.3 $x_t = \dfrac{x_1 A_1 + x_2 A_2 + x_3 A_3}{A_1 + A_2 + A_3}$

$= \dfrac{65.40.90 + 20.50.45 + 120.20.10}{65.40 + 20.50 + 120.20}$

$= 50{,}5 \text{ mm}$



For the running surface:

$I_{xx1} = Ad^2 + \frac{1}{12}bh^3$

$= 65.40.(39{,}5)^2 + \frac{1}{12}.65.(40)^3$

$= 4\ 403\ 317 \text{ mm}^4$

For the web:

$I_{xx2} = Ad^2 + \frac{1}{12}bh^3$

$= 20.50.(5{,}5)^2 + \frac{1}{12}.20.(50)^3$

$= 238\ 583 \text{ mm}^4$

For the base flange:

$I_{xx3} = Ad^2 + \frac{1}{12}bh^3$

$= 120.20.(40{,}5)^2 + \frac{1}{12}.120.(20)^3$

$= 4\ 016\ 600 \text{ mm}^4$

The total second moment of area is then:

$I_{xxT} = I_{xx1} + I_{xx2} + I_{xx3}$

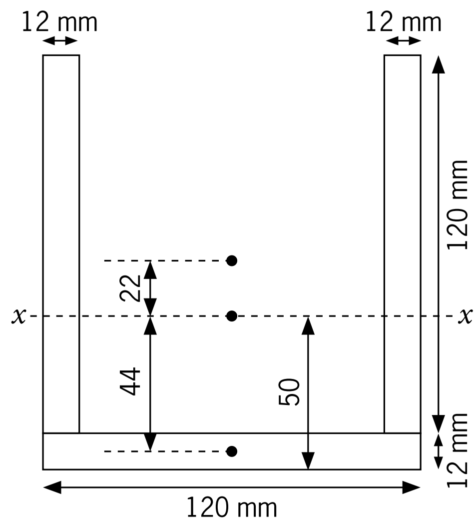
$= 4\ 403\ 317 + 238\ 583 + 4\ 016\ 600$

$= 8\ 658\ 500 \text{ mm}^4$

2.4 $x_t = \dfrac{x_1 A_1 + x_2 A_2 + x_3 A_3}{A_1 + A_2 + A_3}$

$= \dfrac{72.120.12 + 72.120.12 + 6.120.12}{12.120 + 12.120 + 12.120}$

$= 50 \text{ mm}$



For the web (the bottom piece):

$I_{xxw} = Ad^2 + \frac{1}{12}bh^3.$

$= 120.12.(44)^2 + \frac{1}{12}.120.(12)^3$

$= 2\ 805\ 120 \text{ mm}^4$

For the flanges (the vertical bars):

$I_{xxf} = Ad^2 + \frac{1}{12}bh^3$

$= 120.12.(22)^2 + \frac{1}{12}.12.(120)^3$

The total second moment of area is then:

$I_{xxT} = I_{xxw} + 2 \times I_{xxf}$

$= 2\ 805\ 120 + 2 \times 2\ 424\ 960$

$= 7\ 655\ 040\ \text{mm}^4$

2.5 $I_{xx} = \frac{\pi}{4}(R^4 - r^4)$

$= \frac{\pi}{4}(50^4 - 40^4)$

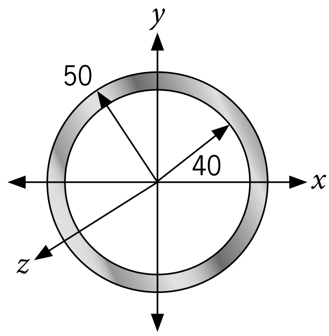
$= 2\ 898\ 119\ \text{mm}^4$

$J_{zz} = \frac{\pi}{2}(R^4 - r^4)$

$= 2 \times I_{xx}$

$= 2 \times 2\ 898\ 119$

$= 5\ 796\ 238\ \text{mm}^4$



Activity 7.2 **SB page 209**

1. Given: 5-kg rectangle of 40 cm high by 15 cm wide
   To find: moment of inertia about the z-axis through its centroid.
   Calculate the area density:

$\sigma = \frac{m}{bh}$

$= \frac{5}{15 \times 40}$

$= 0{,}008\ \text{kg/cm}^2$

Calculate the moment of inertia:

$I_x = \frac{\sigma}{12} bh^3$

$= \frac{0{,}008}{12}.15.40^3$

$= 640\ \text{kg.cm}^2$

$I_y = \frac{\sigma}{12} hb^3$

$= \frac{0{,}008}{12}.40.15^3$

$= 90\ \text{kg.cm}^2$

$J_z = I_x + I_y$

$= 640 + 90$

$= 730\ \text{kg.cm}^2$

2. Given: mass 12 g and diameter 8,4 cm

   To find: moment of inertia

   Calculate the area density:

   $$\sigma = \frac{m}{\pi r^2}$$
   $$= \frac{12}{\pi.4,2^2}$$
   $$= 0,217 \text{ g/cm}^2$$

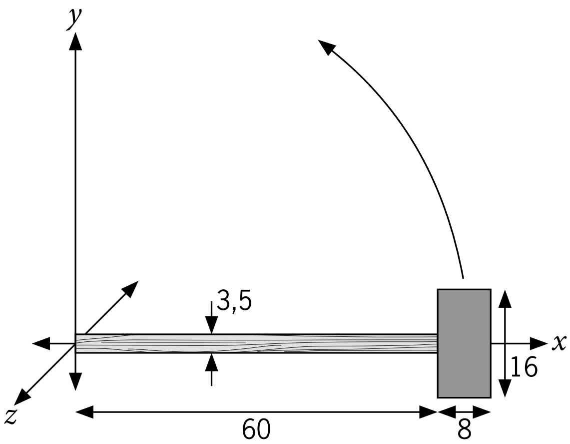
   Calculate the moment of inertia:

   $$J_z = \frac{\pi\sigma}{2} r^4$$
   $$= \frac{\pi.0,217}{2}.4,2^4$$
   $$= 106,066 \text{ g.cm}^2$$

3. **Handle**

   The surface density of the handle is:

   $$\sigma = \frac{m}{A}$$
   $$= \frac{250}{3,5 \times 60}$$
   $$= 1,19 \text{ g/cm}^2$$



   Rotation is around the z-axis.
   For a rectangle shifted to its edge:

   $$I_x = I_y = \frac{1}{3}\sigma h b^3$$
   $$J_{z1} = I_x + I_y$$
   $$= \frac{2}{3}\sigma h b^3$$
   $$= \frac{2}{3}.1,19.3,5.60^3$$
   $$= 600\ 000 \text{ cm}^4$$

   [Note: using the unrounded value for σ (1,190 476) from the previous calculation.]

   Head

   The surface density is:

   $$\sigma = \frac{m}{A}$$
   $$= \frac{2\ 560}{8 \times 16}$$
   $$= 20 \text{ g/cm}^2$$

   For a rectangle:

   $$I_x = I_y = \frac{1}{12}\sigma h b^3$$
   $$J_z = I_x + I_y = \frac{1}{6}\sigma h b^3$$

For the parallel shift:

$$J_{z2} = A\sigma r^2 + \frac{1}{6}\sigma h b^3$$
$$= 8.16.20.64^2 + \frac{1}{6}.20.16.8^3$$
$$= 10\ 485\ 760 + 27\ 307$$
$$= 10\ 513\ 067 \text{ cm}^4$$

$$J_{zT} = J_{z1} + J_{z2}$$
$$= 600\ 000 + 10\ 513\ 067$$
$$= 11\ 113\ 067 \text{ cm}^4$$

4. Calculate the area density:

$$\sigma = \frac{m}{\pi(R^2 - r^2)}$$
$$= \frac{14\ 000}{\pi.(8^2 - 6^2)}$$
$$= \frac{14\ 000}{\pi.28}$$
$$= \frac{500}{\pi}$$

Subtract the moment of the hollow area from the total area to get the moment for the ring:

$$J_z = \frac{\pi\sigma}{2}R^4 - \frac{\pi\sigma}{2}r^4$$
$$= \frac{\pi.500}{2.\pi}.(8^4 - 6^4)$$
$$= 700\ 000 \text{ g.cm}^2$$

5. $I_x = \frac{1}{12}mh^2 = \frac{1}{12}.640.40^2 = 85\ 333 \text{ g.cm}^2$

$I_y = \frac{1}{12}mb^2 = \frac{1}{12}.640.20^2 = 21\ 333 \text{ g.cm}^2$

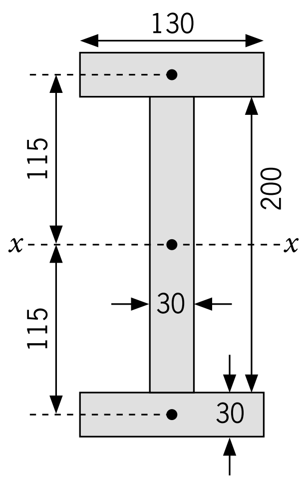
$J_z = I_x + I_y = 85\ 333 + 21\ 333 = 106\ 667 \text{ g.cm}^2$

Summative assessment: Module 7 SB page 210

1. Due to symmetry, the centroid is in the middle of the web section.

For the flanges:

$$I_{xxf} = Ad^2 + \frac{1}{12}bh^3$$
$$= 130.30.(115)^2 + \frac{1}{12}.130.(30)^3$$
$$= 51\ 577\ 500 + 292\ 500$$
$$= 51\ 870\ 000 \text{ mm}^4$$



For the web:

$$I_{xxw} = \frac{1}{12}bh^3$$
$$= \frac{1}{12}.30.(200)^3$$
$$= 20\ 000\ 000 \text{ mm}^4$$

The total second moment of area is then:

$$I_{xxT} = I_{xxf} + I_{xxw}$$
$$= 51\ 870\ 000 \times 2 + 20\ 000\ 000$$
$$= 123\ 740\ 000 \text{ mm}^4 \ (= 12\ 374 \text{ cm}^4)$$ (8)

2. $I_{xx} = \frac{\pi}{4}(R^4 - r^4)$
$$= \frac{\pi}{4}(10^4 - 5^4)$$
$$= 7\ 363 \text{ mm}^4$$ (2)

3. $I_{xx} = Ad^2 + \frac{\pi}{4}r^4$
$$= \pi r^2 d^2 + \frac{\pi}{4}r^4$$
$$= \pi.50^2.100^2 + \frac{\pi}{4}.50^4$$
$$= 83\ 448\ 555 \text{ mm}^4$$
$$= 8\ 345 \text{ cm}^4$$

$$I_{yy} = Ad^2 + \frac{\pi}{4}r^4$$
$$= \pi r^2 d^2 + \frac{\pi}{4}r^4$$
$$= \pi.50^2.150^2 + \frac{\pi}{4}.50^4$$
$$= 181\ 623\ 325 \text{ mm}^4$$
$$= 18\ 162 \text{ cm}^4$$

$$J_{zz} = I_{xx} + I_{yy}$$
$$= 83\ 448\ 555 + 181\ 623\ 325$$
$$= 265\ 071\ 880 \text{ mm}^4$$
$$= 26\ 507 \text{ cm}^4$$ (6)

4.

Round tube	Square tube
$I_{xxR} = \frac{\pi}{4}(R^4 - r^4)$ $= \frac{\pi}{4}(12,5^4 - 10,5^4)$ $= 9\,628 \text{ mm}^4$	For the outer rectangle: $I_{xxo} = \frac{1}{12}bh^3 = \frac{1}{12}.25.(25)^3$ $= 32\,552 \text{ mm}^4$ For the inner rectangle: $I_{xxi} = \frac{1}{12}bh^3$ $= \frac{1}{12}.23.(23)^3$ $= 23\,320 \text{ mm}^4$ The total second moment of area is then: $I_{xxT} = I_{xxo} - I_{xxi}$ $= 32\,552 - 23\,320$ $= 9\,232 \text{ mm}^4$

Therefore, the round tube is more resistant to torsion. (8)

5. Given: 6-kg rectangle of 30 cm high by 20 cm wide
To find: moment of inertia about the z-axis through its centroid.
Calculate the area density:

$$\sigma = \frac{m}{bh}$$
$$= \frac{6}{30 \times 20}$$
$$= 0,01 \text{ kg/cm}^2$$

$$I_x = \frac{\sigma}{12}bh^3$$
$$= \frac{0,01}{12}.20.30^3$$
$$= 450 \text{ kg.cm}^2$$

$$I_y = \frac{\sigma}{12}hb^3$$
$$= \frac{0,01}{12}.30.20^3$$
$$= 200 \text{ kg.cm}^2$$

$$J_z = I_x + I_y$$
$$= 450 + 200$$
$$= 650 \text{ kg.cm}^2$$ (4)

6. Given: mass 9,4 g and radius 2,6 cm
To find: moment of inertia
Calculate the area density:

$$\sigma = \frac{m}{\pi r^2}$$
$$= \frac{9{,}4}{\pi . 2{,}6^2}$$
$$= 0{,}443 \text{ g/cm}^2$$

Calculate the moment of inertia:

$$J_z = \frac{\pi\sigma}{2} r^4$$
$$= \frac{\pi . 0{,}443}{2} . 2{,}6^4$$
$$= 31{,}799 \text{ g.cm}^2$$

(4)

7.

Solid disk flywheel	**Ring flywheel**
Calculate the area density: $\sigma = \frac{m}{\pi r^2}$ $= \frac{50}{\pi . 0{,}3^2}$ $= 176{,}839 \text{ kg/m}^2$	Calculate the area density: $\sigma = \frac{m}{\pi (R^2 - r^2)}$ $= \frac{30}{\pi . (0{,}3^2 - 0{,}26^2)}$ $= 426{,}308 \text{ kg/m}^2$
Calculate the moment of inertia: $J_z = \frac{\pi\sigma}{2} r^4$ $= \frac{\pi . 176{,}8}{2} . 0{,}3^4$ $= 2{,}25 \text{ kg.m}^2$	Subtract the hollow area from the total area to get the moment of inertia: $J_z = \frac{\pi\sigma}{2}(R^4 - r^4)$ $= \frac{\pi . 426{,}3}{2} . (0{,}3^4 - 0{,}26^4)$ $= 2{,}364 \text{ kg.m}^2$

(8)

8. For the polystyrene disk
The area density:

$$\sigma = \frac{m}{\pi r^2}$$
$$= \frac{300}{\pi . 10^2}$$
$$= \frac{3}{\pi}$$

The moment of inertia:

$$J_z = \frac{\pi\sigma}{2} r^4$$
$$= \frac{\pi . 3}{2 . \pi} . 10^4$$
$$= 15\ 000 \text{ g.cm}^2$$

For the copper ring

The area density:

$$\sigma = \frac{m}{\pi(R^2 - r^2)}$$

$$= \frac{300}{\pi.(10^2 - 8^2)}$$

$$= \frac{300}{\pi.36}$$

Subtract the moment of the hollow area from the total area to get the moment for the ring:

$$J_z = \frac{\pi\sigma}{2}R^4 - \frac{\pi\sigma}{2}r^4$$

$$= \frac{\pi.300}{2.\pi.36}.(10^4 - 8^4)$$

$$= 24\ 600 \text{ g.cm}^2$$ (8)

TOTAL: [48]

MODULE

8 Differential equations

After they have completed this module, students should be able to:

- distinguish between first and second order differential equations;
- determine solutions for a first order differential equation:
  - finding the general solution by integrating;
  - finding the general solution by separating variables;
  - finding the particular solution from a general solution with given conditions; and
- determine the particular solution of a second order differential equation from a general solution with given conditions.

Introduction

Mathematics is used to solve many real-life problems. This requires that the problem is described in mathematical terms in order to construct a mathematical model.

Many physical problems involve quantities that change such as population growth and Newton's laws of motion. In previous modules, particularly Module 3, students have seen that a derivative represents a rate of change. When solving problems, it is often useful to develop mathematical models that relate functions to their derivatives. These are called differential equations.

They take the form $\frac{dy}{dx} = f'(x)$ if the function f is independent of the variable y.

We will look at differential equations from two aspects:

- Find the specific differential equation to describe a situation.
- Find the solution of the differential equation for specific values.

Students need the following pre-knowledge to successfully complete this module.

Pre-knowledge

- determine the rate of change of a function by differentiating;
- reverse the process of differentiation by determining the antiderivative of a function; and
- determine the value of a constant by substituting given values into a function.

Activity 8.1 SB page 220

1. 1.1 General solution:

$\frac{dy}{dx} = 10x + 5$

$\int dy = \int 10x + 5\ dx$

$y = 5x^2 + 5x + C$

Particular solution for $y(1) = 5$:

$5 = 5.1^2 + 5.1 + C$

$C = -5$

$\therefore y = 5x^2 + 5x - 5$

1.2 General solution:

$\frac{dy}{dx} = 2x^2 + 1$

$\int dy = \int 2x^2 + 1\ dx$

$y = \frac{2}{3}x^3 + x + C$

Particular solution for $y(1) = 5$:

$5 = \frac{2}{3}.1^3 + 1 + C$

$C = \frac{10}{3}$

$\therefore y = \frac{2}{3}x^3 + x + \frac{10}{3}$

1.3 General solution:

$dy + 5x\ dx = 0$

$\int dy = -5\int x\ dx$

$y = -\frac{5}{2}x^2 + C$

Particular solution for $y(1) = 5$:

$5 = -\frac{5}{2}.1^2 + C$

$C = 7\frac{1}{2}$

$\therefore y = -2\frac{1}{2}x^2 + 7\frac{1}{2}$

1.4 General solution:

$\frac{dy}{dx} = 7$

$\int dy = \int 7dx$

$y = 7x + C$

Particular solution for $y(1) = 5$:

$5 = 7.1 + C$

$C = -2$

$\therefore y = 7x - 2$

2. 2.1 General solution:

$$\frac{dy}{dx} = xy$$

$$\frac{1}{y}dy = x\,dx$$

$$\int\frac{1}{y}dy = \int x\,dx$$

$$\ln y = \frac{x^2}{2} + C_1$$

$$y = e^{\left(\frac{x^2}{2}+C_1\right)}$$

$$= e^{\left(\frac{x^2}{2}\right)}e^{C_1} \quad \text{(Let } e^{C_1} = C_2\text{, replace one arbitrary constant with another)}$$

$$= C_2 e^{\frac{x^2}{2}}$$

Particular solution for $y(1) = 5$:

$$5 = C_2 e^{\frac{1^2}{2}}$$

$$C_2 = \frac{5}{\sqrt{e}}$$

$$\therefore y = \frac{5e^{\frac{x^2}{2}}}{\sqrt{e}}$$

2.2 General solution:

$$\frac{dy}{dx} = \frac{2x}{y}$$

$$\int y\,dy = 2\int x\,dx$$

$$\frac{y^2}{2} = 2\frac{x^2}{2} + C$$

$$y = \sqrt{2}\,x + C$$

Particular solution for $y(1) = 5$:

$$5 = \sqrt{2}.1 + C$$

$$C = 5 - \sqrt{2}$$

$$\therefore y = \sqrt{2}\,x + 5 - \sqrt{2}$$

2.3 General solution:

$$\frac{dy}{dx} = e^{y-x}$$

$$= \frac{e^y}{e^x}$$

$$\int e^{-y}dy = \int e^{-x}dx$$

$$-ye^{-y} = -xe^{-x} + C_1$$

$$\frac{y}{e^y} = \frac{x}{e^x} + C_2$$

Particular solution for $y(1) = 5$:

$$\frac{5}{e^5} = \frac{1}{e^1} + C_2$$

$$C_2 = \frac{5 - e^4}{e^5}$$

$$\therefore \frac{y}{e^y} = \frac{x}{e^x} + \frac{5 - e^4}{e^5}$$

2.4 General solution:

$$\frac{dy}{dx} = -ye^{2x}$$

$$\int \frac{1}{y} dy = -\int e^{2x} dx$$

$$\ln y = -\frac{e^{2x}}{2} + C_1$$

$$y = C_2 e^{-\frac{1}{2}e^{2x}}$$

Particular solution for $y(1) = 5$:

$$5 = C_2 e^{-\frac{1}{2}e^{2\cdot 1}}$$

$$C_2 = 5e^{\frac{1}{2}e^2}$$

$$\therefore y = 5e^{\frac{1}{2}e^2} . e^{-\frac{1}{2}e^{2x}}$$

$$= 5e^{\frac{1}{2}(e^2 - e^{2x})}$$

Activity 8.2 SB page 223

1. 1.1 $\frac{1}{6}\frac{d^2y}{dx^2} = x^2 - x + 1$

$$\therefore \frac{dy}{dx} = \frac{6}{3}x^3 - \frac{6}{2}x^2 + 6x + C_1$$

$$y = \frac{2}{4}x^4 - \frac{3}{3}x^3 + \frac{6}{2}x^2 + C_1x + C_2$$

$$= \frac{1}{2}x^4 - x^3 + 3x^2 + C_1x + C_2$$

1.2 $\frac{1}{x}\frac{d^2y}{dx^2} = 1 - 3x$

$$\frac{d^2y}{dx^2} = x - 3x^2$$

$$\frac{dy}{dx} = \frac{1}{2}x^2 - \frac{3}{3}x^3 + C_1$$

$$y = \frac{1}{6}x^3 - \frac{1}{4}x^4 + C_1x + C_2$$

1.3 $\frac{d^2y}{dx^2} = \frac{1}{x^2} + 2\cos 2x$

$$\frac{dy}{dx} = -\frac{1}{x} + \sin 2x + C_1$$

$$y = -\ln x - \frac{1}{2}\cos 2x + C_1x + C_2$$

2. 2.1 General solution:

$\frac{d^2y}{dx^2} = 4x + 1$

$\frac{dy}{dx} = 2x^2 + x + C_1$ • Integrate and add constant

$y = \frac{2}{3}x^3 + \frac{x^2}{2} + C_1x + C_2$ • Integrate again, add another constant

Particular solution for $\frac{dy}{dx} = 0$, $y = 1$ and $x = 3$:

$0 = 2.3^2 + 3 + C_1$ • Substitute conditions for first integration

$\therefore C_1 = -21$

$1 = \frac{2}{3}3^3 + \frac{3^2}{2} - 21.3 + C_2$ • Substitute conditions for second integration

$\therefore C_2 = 40{,}5$

$y = \frac{2}{3}x^3 + x^2 - 21x + 40{,}5$

2.2 General solution:

$x.\frac{d^2y}{dx^2} = 3x^2 + 4x$

$\frac{d^2y}{dx^2} = 3x + 4$

$\frac{dy}{dx} = \frac{3}{2}x^2 + 4x + C_1$ • Integrate and add constant

$y = \frac{1}{2}x^3 + 2x^2 + C_1x + C_2$ • Integrate again, add another constant

Particular solution for $\frac{dy}{dx} = 1$, $y = 0$ and $x = 2$:

$1 = \frac{3}{2}2^2 + 4.2 + C_1$ • Substitute conditions for first integration

$\therefore C_1 = -14$

$0 = \frac{1}{2}2^3 + 2.2^2 - 14.2 + C_2$ • Substitute conditions for second integration

$C_2 = 16$

$y = \frac{1}{2}x^3 + 2x^2 - 14x + 16$

2.3 General solution:

$8\frac{d^2y}{dx^2} = x + 16$

$\frac{d^2y}{dx^2} = \frac{x}{8} + 2$

$\frac{dy}{dx} = \frac{x^2}{16} + 2x + C_1$ • Integrate and add constant

$y = \frac{x^3}{48} + x^2 + C_1x + C_2$ • Integrate again, add another constant

Particular solution for $\frac{dy}{dx} = 2$, $y = 1$ and $x = 1$:

$2 = \frac{1^2}{16} + 2.1 + C_1$

- Substitute conditions for first integration

$\therefore C_1 = -\frac{1}{16}$

$1 = \frac{1^3}{48} + 1^2 - \frac{1}{16} + C_2$

- Substitute conditions for second integration

$\therefore C_2 = \frac{1}{24}$

$y = \frac{x^3}{48} + x^2 - \frac{x}{16} + \frac{1}{24}$

Summative assessment: Module 8 — SB page 224

1. 1.1 $\int dy = \int (2x + 3)\, dx$

$y = x^2 + 3x + C$

Substituting the initial value $y(1) = 3$, we get:

$3 = 1^2 + 3(1) + C$

$C = -1$

Therefore, the particular solution is $y = x^2 + 3x - 1$ (2)

1.2 $\int dy = \int 3(1 - 4x^2)\, dx$

$y = \int (3 - 12x^2)\, dx$

$y = 3x - 4x^3 + C$

Substituting the initial value $y\left(-\frac{1}{2}\right) = 0$, we get:

$0 = 3\left(-\frac{1}{2}\right) - 4\left(-\frac{1}{2}\right)^3 + C$

$C = 1$

Therefore, the particular solution is $y = 3x - 4x^3 + 1$ (2)

1.3 $\int dy = \int (2x + 1)\, dx$

$y = x^2 + x + C$

Substituting the initial value $y(0) = 3$, we get:

$3 = 0^2 + 0 + C$

$C = 3$

Therefore, the particular solution is $y = x^2 + x + 3$ (2)

1.4 $\int dy = \int \cos(2x)\, dx$

$y = \frac{1}{2}\sin(2x) + C$

Substituting the initial value $y(0) = 1$, we get:

$1 = \frac{1}{2}\sin 0 + C$

$C = 1$

Therefore, the particular solution is $y = \frac{1}{2}\sin(2x) + 1$ (2)

2. 2.1 $\int \frac{dy}{y} = \int e^x\, dx$

$\ln y = e^x + C$

$y = e^{e^x + C}$

$= e^C e^{e^x}$

$= C_2 \cdot e^{e^x}$ (4)

2.2 $\int \frac{1}{3(y-1)^{\frac{2}{3}}} dy = \int 2x\, dx$

$(y-1)^{\frac{1}{3}} = x^2 + C$

$y = 1 + (x^2 + C)^3$ (4)

2.3 $(x^2 - 1)y^2 + x^2 \frac{dy}{dx} = 0$

$\int \frac{1}{y^2}\, dy = \int -\frac{x^2 - 1}{x^2}\, dx$

$\int \frac{1}{y^2}\, dy = \int \left(-1 + \frac{1}{x^2}\right) dx$

$-\frac{1}{y} = -x - \frac{1}{x} + C$ (4)

2.4 $\frac{dy}{dx} = \frac{\text{cosec}^2 y}{\tan^2 x}$

$\int \frac{1}{\text{cosec}^2 y}\, dy = \int \frac{1}{\tan^2 x}\, dx$

$\int \sin^2 y\, dy = \int \cot^2 x\, dx$

$\int \frac{1}{2}(1 - \cos 2y) dy = \int (\text{cosec}^2 x - 1) dx$

$\frac{1}{2}y - \frac{1}{4}\sin 2y = -\cot x - x + C$ (5)

2.5 $e^{2x-5y} = e^{x+2y} . \frac{dy}{dx}$

$\int e^{2x-x}\, dx = \int e^{2y+5y}\, dy$

$\int e^x\, dx = \int e^{7y}\, dy$

$e^x = \frac{1}{7}e^{7y} + C$ (4)

2.6 $\frac{2y}{\sec x \tan x} \frac{dy}{dx} = 1$

$\int 2y\, dy = \int \sec x \tan x\, dx$

$y^2 = \sec x + C$ (3)

3. 3.1 $\frac{d^2y}{dx^2} = 6x^2 - 2x + 1$

$\frac{dy}{dx} = 2; y = 1; x = -1$

$\frac{dy}{dx} = 2x^3 - x^2 + x + C_1$

Substituting the initial value $\frac{dy}{dx} = 2, x = -1$, we get:

$2 = 2(-1)^3 - (-1)^2 - 1 + C_1$

$C_1 = 6$

$\therefore \frac{dy}{dx} = 2x^3 - x^2 + x + 6$

$y = \frac{2}{4}x^4 - \frac{1}{3}x^3 + \frac{1}{2}x^2 + 6x + C_2$

Substituting the initial value $y = 1, x = -1$, we get:

$1 = \frac{1}{2}(-1)^4 - \frac{1}{3}(-1)^3 + \frac{1}{2}(-1)^2 + 6(-1) + C_2$

$C_2 = 1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{2} + 6 = \frac{17}{3}$

$y = \frac{1}{2}x^4 - \frac{1}{3}x^3 + \frac{1}{2}x^2 + 6x + 5\frac{2}{3}$ (6)

3.2 $e^{-2x}\frac{d^2y}{dx^2} = 2$

$\frac{dy}{dx} = 1; y = 10; x = 0$

$\frac{d^2y}{dx^2} = 2e^{2x}$

$\frac{dy}{dx} = e^{2x} + C_1$

Substituting the initial value $\frac{dy}{dx} = 1, x = 0$, we get:

$1 = e^0 + C_1$

$\therefore C_1 = 1 - 1 = 0$

$\therefore \frac{dy}{dx} = e^{2x}$

$y = \frac{1}{2}e^{2x} + C_2$

Substituting the initial value $y = 10, x = 0$, we get:

$10 = \frac{1}{2}e^0 + C_2$

$\therefore C_2 = 10 - \frac{1}{2} = 9\frac{1}{2}$

$y = \frac{1}{2}e^{2x} + 9\frac{1}{2}$ (6)

3.3 $\frac{d^2y}{dx^2} = 4\cos 2x + 6,\ \frac{dy}{dx} = 5,\ y = 2,\ x = 0$

$\therefore \frac{dy}{dx} = 2\sin 2x + 6x + C_1$

Substituting the initial value $\frac{dy}{dx} = 5$, $x = 0$, we get:

$5 = 2(0) + 6(0) + C_1$

$\therefore C_1 = 5$

$\therefore \frac{dy}{dx} = 2\sin 2x + 6x + 5$

$y = \int 6x + 5 + 2\sin 2x\, dx$

$= 3x^2 + 5x - \cos 2x + C_2$

Substituting the initial value $y = 2$, $x = 0$, we get:

$2 = 3(0)^2 + 5(0) - \cos 2.0 + C_2$

$\therefore C_2 = 3$

$y = 3x^2 + 5x + 3 - \cos 2x$ (6)

TOTAL: [50]

Exemplar examination paper

Time: 3 hours **Marks: 100**

INSTRUCTIONS AND INFORMATION

1. Answer all the questions.
2. Read all the questions carefully.
3. Number the answers according to the numbering system used in this question paper.
4. Show all intermediate steps and simplify where possible.
5. All answers must be rounded off to THREE decimals.
6. Questions may be answered in any order, but subsections of questions must be kept together.
7. Sketches must be large, neat and fully labelled.
8. Start each question on a new page.
9. Only use a black or a blue pen.
10. Write neatly and legibly.

QUESTION 1

1.1 Determine the following limits:

1.1.1 $\lim_{x \to 3} \frac{6 - 2x}{\sqrt{x + 22} - \sqrt{10x - 5}}$ (3)

1.1.2 $\lim_{x \to -1} \frac{\frac{1}{5 + 4x} + \frac{1}{x}}{2x + 2}$ (2)

1.2 Given: $\ln y = \lim_{x \to 4} \frac{\sin(x - 4)}{x - 4}$, calculate the numerical value of:

1.2.1 $\ln y$ (2)

1.2.2 y (1)

1.3 Determine the value(s) of x for which $f(x)$ is discontinuous if:

$f(x) = \frac{\tan 3x}{\cos 2x}$ (2)

[10]

QUESTION 2

2.1 Given: $f(x) = -3x^6$

Determine the simplest form of:

2.1.1 $f(x + h)$ (2)

2.1.2 $f(x + h) - f(x)$ (1)

2.1.3 $\frac{f(x + h) - f(x)}{h}$ (1)

2.1.4 $\lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$ (1)

2.2 Determine $\frac{dy}{dx}$ in each of the following cases:

(Simplification not required)

2.2.1 $y = \tan\left[(7 - x^3)(\ln x)^2\right]$ (4)

2.2.2 $y = \sqrt[3]{\cos\left(11 - x^2\right) + \sqrt{\ln x}}$ (4)

2.3 Calculate $\frac{dy}{dx}$ if $y = \frac{\sin(6x + x^2)}{(7 - x^4)^3}$ with the aid of logarithmic differentiation. (4)

2.4 Determine $\frac{dy}{dx}$ of implicit function $\cos\left(x^2 + 3y\right) + xe^{y^2} = 5$ (5)

[22]

QUESTION 3

3.1 Given: $f(x) = 4x^3 - 10x + 3$

3.1.1 One root of the equation $4x^3 - 10x + 3 = 0$ is close to 0,3.

Use Taylor's/Newton's method twice to determine a better approximation of this root (root correct to THREE decimals). (4)

3.1.2 Determine the coordinate of the point of inflection of $f(x)$. (2)

3.1.3 Draw up a table of x and $f(x)$, where x ranges from $x = -2$ to $x = 2$. (2)

3.2 Your customer needs an enclosed rectangular box that will hold 20 m^3. You are not given any dimensions, but the base must be five times longer than it is wide. You determine that the cost of material for the sides is R4/m^2 while the cost of the material for the base is R16/m^2.

Calculate the dimensions of the box that will minimise the cost. (6)

[14]

QUESTION 4

4.1 Determine $\int y\,dx$ in each of the following cases.

4.1.1 $y = \sec^2 3x\left(7 \tan 3x - \tan^2 3x + 5\right)$ (4)

4.1.2 $y = \frac{3x + 4}{\sqrt{1 - 16x^2}}$ (4)

4.1.3 $y = \frac{5x^3 - 23x^2 + 15x + 12}{5x + 2}$ (5)

4.1.4 $y = \cos^3 x \sin x$ (3)

4.1.5 $y = \tan^3 x$ (6)

4.2 Determine $\int y\,dx$ by resolving the integral into partial fractions:

$y = \frac{x + 8}{(x - 3)(x + 1)}$ (5)

[27]

QUESTION 5

5.1 Evaluate the integral:

$\int_2^{10} \left[\frac{(\ln 2x)^2}{x} + \frac{e^x}{e^x + 2} +\right] dx$ **[5]**

QUESTION 6

Given: $y = 4x - 2x^2$ and the x-axis.

6.1 Calculate the coordinates of the points of intersection. (2)

6.2 Make a neat sketch to show the enclosed area, the representative strip and the point of intersection. (2)

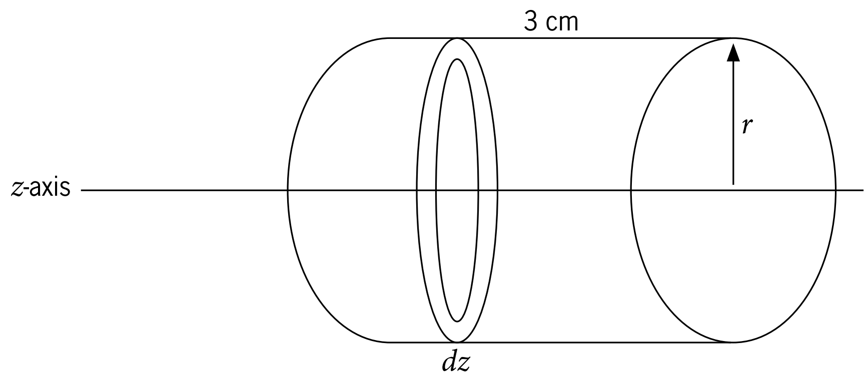
6.3 Calculate the magnitude of the area in QUESTION 6.2. (3)

6.4 Calculate the volume of the solid of revolution formed when the area in QUESTION 6.2 is rotated about the x-axis. (4)

[11]

QUESTION 7

7.1 Calculate the moment of inertia of a flywheel of radius 30 cm and thickness of 3 cm about an axis through its centre and perpendicular to the flywheel. The mass of the flywheel is 15 kg.



[4]

QUESTION 8

8.1 Determine the particular solution of:

$$\frac{dy}{dx} = -\frac{x}{ye^{x^2}} \text{ at } (2; 0)$$ (4)

8.2 Determine the general solution of:

$$\sec x . \frac{d^2y}{dx^2} = 1 + \tan x + \frac{x^2}{\cos x}$$ (3)

[7]

TOTAL: 100

Formula sheet

Any applicable formula may also be used.

Trigonometry

$\sin^2 x + \cos^2 x = 1$ $\cos^2 x = 1 - \sin^2 x$ $\sin^2 x = 1 - \cos^2 x$

$1 + \tan^2 x = \sec^2 x$ $\tan^2 x = \sec^2 x - 1$

$1 + \cot^2 x = \operatorname{cosec}^2 x$ $\cot^2 x = \operatorname{cosec}^2 x - 1$

$\sin 2A = 2 \sin A \cos A$

$\cos 2A = \cos^2 A - \sin^2 A$

$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$

$\sin^2 A = \frac{1}{2}(1 - \cos 2A)$

$\cos^2 A = \frac{1}{2}(1 + \cos 2A)$

$\sin A \pm B = \sin A \cos B \pm \cos A \sin B$

$\cos A \pm B = \cos A \cos B \mp \sin A \sin B$

$\tan A \pm B = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

$\sin A \cos B = \frac{1}{2}[\sin(A + B) + \sin(A - B)]$

$\cos A \sin B = \frac{1}{2}[\sin(A + B) - \sin(A - B)]$

$\cos A \cos B = \frac{1}{2}[\cos(A + B) + \cos(A - B)]$

$\sin A \sin B = \frac{1}{2}[\cos(A - B) - \cos(A + B)]$

$\tan x = \frac{\sin x}{\cos x}$; $\sin x = \frac{1}{\operatorname{cosec} x}$; $\cos x = \frac{1}{\sec x}$

$\cot x = \frac{\cos x}{\sin x}$; $\operatorname{cosec} x = \frac{1}{\sin x}$; $\sec x = \frac{1}{\cos x}$

Binomial theorem

$(x + h)^n = x^n + nx^{n-1}h + \frac{n(n-1)}{2 \times 1}x^{n-2}h^2 + \frac{n(n-1)(n-2)}{3 \times 2 \times 1}x^{n-3}h^3 + \ldots$ for $n \in \mathrm{N}$

Differentiation

$e = -\frac{f(a)}{f'(a)}$

$r = a + e$

Table of differentiation and integration

$f(x)$	$\frac{d}{dx}f(x)$	$\int f(x)\,dx$
a	0	$ax + c$
x^n	nx^{n-1}	$\frac{x^{n+1}}{n+1} + C, \quad (n \neq -1)$
ax^n	$a\frac{d}{dx}x^n = nax^{n-1}$	$a\int x^n\,dx = \frac{ax^{n+1}}{n+1}$
e^x	e^x	$e^x + c$
a^x	$a^x.\ln a$	$\frac{a^x}{\ln a} + C$
$\ln x$	$\frac{1}{x}$	——
$\log_a x$	$\frac{1}{x\ln a}$	——
$\sin x$	$\cos x$	$-\cos x + C$
$\cos x$	$-\sin x$	$\sin x + C$
$\tan x$	$\sec^2 x$	$\ln(\sec x) + C$
$\cot x$	$-\text{cosec}^2 x$	$\ln(\sin x) + C$
$\sec x$	$\sec x \tan x$	$\ln(\sec x + \tan x) + C$
$\text{cosec}\, x$	$-\text{cosec}\, x \cos x$	$\ln(\text{cosec}\, x + \cot x) + C$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$	——
$\cos^{-1} x$	$\frac{-1}{\sqrt{1-x^2}}$	——
$\tan^{-1} x$	$\frac{1}{1+x^2}$	——
$\cot^{-1} x$	$\frac{-1}{1+x^2}$	——
$\sec^{-1} x$	$\frac{1}{\sqrt{x^2-1}}$	——
$\text{cosec}^{-1} x$	$\frac{-1}{x\sqrt{x^2-1}}$	——
$\frac{1}{\sqrt{a^2-x^2}}$	——	$\sin^{-1}\left(\frac{x}{a}\right) + C$
$\frac{1}{a^2+x^2}$	——	$\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$
$\frac{1}{x\sqrt{x^2+a^2}}$	——	$\frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + C$
$\sqrt{a^2-x^2}$	——	$\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + \frac{x}{2}\sqrt{a^2-x^2} + C$
$\frac{1}{x^2-a^2}$	——	$\frac{1}{2a}\ln\left(\frac{x-a}{x+a}\right) + C$
$\frac{1}{a^2-x^2}$	——	$\frac{1}{2a}\ln\left(\frac{a+x}{a-x}\right) + C$

Product rule

$y = u(x).v(x)$

$\frac{dy}{dx} = v.\frac{du}{dx} + u.\frac{dv}{dx}$

$= v.u' + u.v'$

Quotient rule

$y = \frac{u(x)}{v(x)}$

$\frac{dy}{dx} = \frac{v.\frac{du}{dx} - u.\frac{dv}{dx}}{v^2}$

$= \frac{v.u' - u.v'}{v^2}$

Chain rule

$y = f(u(x))$

$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$

Integration

$\int f(x).g'(x)\,dx = f(x).g(x) - \int f'(x).g(x)\,dx$

$\int [f(x)]^n.f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1} + C(n \neq -1)$

$\int \frac{f'(x)}{f(x)}\,dx = \ln f(x) + C$

$\frac{f(x)}{(ax+b)(cx+d)} = \frac{A}{ax+b} + \frac{B}{cx+d}$

$\frac{f(x)}{(x+a)^n} = \frac{A}{x+a} + \frac{B}{(x+a)^2} + \frac{C}{(x+a)^3} + \ldots \frac{Z}{(x+a)^n}$

Applications of integration

Areas

$A_x = \int_a^b y\,dx$; $A_x = \int_a^b (y_T - y_B)\,dx$

$A_y = \int_c^d x\,dy$; $A_y = \int_c^d (x_R - x_L)\,dy$

Volumes

$V_x = \pi\int_a^b y^2\,dx$; $V_x = \pi\int_a^b \left(y_T^2 - y_B^2\right)dx$

$V_y = \pi\int_c^d x^2\,dy$; $V_y = \pi\int_c^d \left(x_R^2 - x_L^2\right)dy$

Second moments of area

$I_x = \int_a^b r^2\,dA$; $I_y = \int_c^d r^2\,dA$

Moments of inertia

Mass = density × volume

$m = \rho V$

Definition: $I = mr^2$

General: $I = \int_a^b r^2\,dm = \rho\int_a^b r^2\,dV$

Exemplar examination paper memorandum

QUESTION 1

1.1 1.1.1 $\lim\limits_{x\to 3}\dfrac{6-2x}{\sqrt{x+22}-\sqrt{10x-5}} \to \left(\dfrac{0}{0}\right)$

$= \lim\limits_{x\to 3}\dfrac{-2}{\dfrac{1}{2\sqrt{x+22}}-\dfrac{10}{2\sqrt{10x-5}}}$ ✓✓

$= \dfrac{20}{9}$ ✓ (3)

1.1.2 $\lim\limits_{x\to -1}\dfrac{\dfrac{1}{5+4x}+\dfrac{1}{x}}{2x+2} \to \left(\dfrac{0}{0}\right)$

$= \lim\limits_{x\to -1}\dfrac{-\dfrac{4}{(5+4x)^2}-\dfrac{1}{x^2}}{2}$ ✓

$= -\dfrac{5}{2}$ ✓ (2)

1.2 1.2.1 $\ln y = \lim\limits_{x\to 4}\dfrac{\sin(x-4)}{x-4} \to \left(\dfrac{0}{0}\right)$

$= \lim\limits_{x\to 4}\dfrac{\cos(x-4)}{1}$ ✓

$= 1$ ✓ (2)

1.2.2 $y = e^1 = e$ ✓ (1)

1.3 $f(x)$ is discontinuous if $\cos 2x = 0$

$\cos 2x = 0$

$2x = 90°$

$\therefore x = 45°$✓ or $x = 135°$✓ (2)

[10]

QUESTION 2

2.1 $f(x) = -3x^6$

2.1.1 $f(x+h) = -3(x+h)^6$

$= -3\left[\dfrac{x^6h^0}{0!}+\dfrac{6x^5h}{1!}+\dfrac{30x^4h^2}{2!}+\dfrac{120x^3h^3}{3!}+\dots\right]$

$= -3x^6 - 18x^5h - 45x^4h^2 - 90x^3h^3 + \dots$ ✓✓ (2)

2.1.2 $f(x+h) - f(x)$

$= -18x^5h - 45x^4h^2 - 90x^3h^3 + \dots$ ✓ (1)

2.1.3 $\dfrac{f(x+h)-f(x)}{h} = -18x^5 - 45x^4h - 90x^3h^2 + \dots$ ✓ (1)

2.1.4 $\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h} = -18x^5$ ✓ (1)

2.2 Determine $\frac{dy}{dx}$ in each of the following cases:

2.2.1 $y = \tan\left[(7 - x^3)(\ln x)^2\right]$

$$\frac{dy}{dx} = \sec^2\left[(7 - x^3)(\ln x)^2\right] \times \left[-3x(\ln x)^2 + (7 - x^3) \times 2\ln x \times \frac{1}{x}\right] \checkmark\checkmark\checkmark$$

$$= \sec^2\left[(7 - x^3)(\ln x)^2\right] \times \left[-3x(\ln x)^2 + \frac{2\ln x(7 - x^3)}{x}\right] \checkmark \qquad (4)$$

2.2.2 $y = \sqrt[3]{\cos(11 - x^2) + \sqrt{\ln x}}$

$$\frac{dy}{dx} = \frac{1}{3}\left[\cos(11 - x^2) + \sqrt{\ln x}\right]^{\frac{-3}{4}} \times \left[-\sin(11 - x^2) \times 2x + \frac{1}{2}(\ln x)^{\frac{-1}{2}} \times \frac{1}{x}\right] \checkmark\checkmark\checkmark$$

$$= \frac{1}{3}\left[\cos(11 - x^2) + \sqrt{\ln x}\right]^{\frac{-3}{4}} \times \left[-2x\sin(11 - x^2) \times 2x + \frac{1}{2x\sqrt{\ln x}}\right] \checkmark \qquad (4)$$

2.3 $y = \frac{\sin(6x + x^2)}{(7 - x^4)^3}$

$$\ln y = \ln\sin(6x + x^2) - 3\ln(7 - x^4)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{\sin(6x + x^2)} \times \cos(6x + x^2) \times (6 + 2x) - \frac{3}{(7 - x^4)} \times (-4x^3) \checkmark\checkmark\checkmark$$

$$\frac{dy}{dx} = y\left[\frac{(6 + 2x)\cos(6x + x^2)}{\sin(6x + x^2)} + \frac{12x^3}{7 - x^4}\right]$$

$$\frac{dy}{dx} = \frac{\sin(6x + x^2)}{(7 - x^4)^3}\left[\frac{(6 + 2x)\cos(6x + x^2)}{\sin(6x + x^2)} + \frac{12x^3}{7 - x^4}\right] \checkmark \qquad (4)$$

2.4 $\cos(x^2 + 3y) + xe^{y^2} = 5$

$$-\sin(x^2 + 3y) \times \left[2x + 3\frac{dy}{dx}\right] + e^{y^2} + 2xye^{y^2}\frac{dy}{dx} = 0 \checkmark\checkmark$$

$$-2x\sin(x^2 + 3y) - 3\sin(x^2 + 3y)\frac{dy}{dx} + e^{y^2} + 2xye^{y^2}\frac{dy}{dx} = 0 \checkmark$$

$$\frac{dy}{dx}\left[2xye^{y^2} - 3\sin(x^2 + 3y)\right] = 2x\sin(x^2 + 3y) - e^{y^2} \checkmark$$

$$\frac{dy}{dx} = \frac{2x\sin(x^2 + 3y) - e^{y^2}}{2xye^{y^2} - 3\sin(x^2 + 3y)} \checkmark \qquad (5)$$

[22]

QUESTION 3

3.1 Given: $f(x) = 4x^3 - 10x + 3$

3.1.1 Let $x_0 = 0{,}3$

$f(0{,}3) = 0{,}108$

$f'(0{,}3) = -8{,}92$ ✓

$x_1 = 0{,}3 - \frac{0{,}108}{-8{,}92}$ ✓

$= 0{,}312$

Now use $x_1 = 0{,}312$ ✓

$f(0{,}312) = 0{,}001$

$f'(0{,}312) = -8{,}831$

$x_2 = 0{,}312 - \frac{0{,}001}{-8{,}831}$

$= 0{,}312$ ✓ (4)

3.1.2 $f(x) = 4x^3 - 10x + 3$

$f'(x) = 12x^2 - 10$

$f''(x) = 24x$

$24x = 0 \rightarrow x = 0$ ✓

$f(0) = 3$

The coordinate of a point of inflection is (0; 3) ✓ (2)

3.1.3

x	−2	−1	0	1	2
$f(x)$	−9	9	3	−3	15

(2)

3.2 Volume = length × width × height

$= 5w \times w \times h = 5w^2 h$

Required volume: $20 = 5w^2 h$

$\therefore h = \frac{20}{5w^2} = \frac{4}{w^2}$ ✓

Cost $= 4(2lh + 2wh) + 16(2lw)$ ✓

$= 4(10wh + 2wh) + 16\left(10w^2\right)$

$= 48wh + 160w^2$

$= \left(48w \times \frac{4}{w^2}\right) + 160w^2$

$\therefore C = \frac{192}{w} + 160w^2$ ✓

$C' = -\frac{192}{w^2} + 320w$ ✓

Cost at a minimum where $C' = 0$.

$-\frac{192}{w^2} + 320w = 0$ ✓

$320w^3 = 192$

$w^3 = \frac{192}{320}$

$\therefore w = 0{,}843;\ l = 4{,}217;\ h = 5{,}629$ ✓ (6)

[14]

QUESTION 4

4.1 Determine $\int y\,dx$ in each of the following cases.

4.1.1 $y = \int \sec^2 3x(7\tan 3x - \tan^2 3x + 5)dx$

Let $u = \tan 3x$; $\frac{du}{dx} = 3\sec^2 3x$; $\frac{du}{3} = \sec^2 3x\,dx$ ✓

$y = \frac{1}{3}\int 7u - u^2 + 5\,du$

$= \frac{1}{3}\left(\frac{7}{2}u^2 - \frac{1}{3}u^3 + 5u\right) + C$

$= \frac{7}{6}\tan^2 3x - \frac{1}{9}\tan^3 3x + \frac{5}{3}\tan 3x + C$ ✓✓✓ (4)

4.1.2 $y = \int \frac{3x + 4}{\sqrt{1 - 16x^2}}\,dx$

$= \int \frac{3x}{\sqrt{1 - 16x^2}}\,dx + \int \frac{4}{\sqrt{1 - (4x)^2}}\,dx$

Let $u = 1 - 16x^2$ and $v = 4x$;

$\frac{du}{dx} = -32x$; $\frac{du}{-32} = x\,dx$ ✓

$\frac{dv}{dx} = 4$; $\frac{dv}{4} = dx$

$y = \frac{-3}{32}\int u^{-\frac{1}{2}}\,du + \frac{4}{4}\int \frac{1}{\sqrt{1 - v^2}}\,dv$ ✓

$= \frac{-3}{16}u^{\frac{1}{2}} + \frac{4}{4}\sin^{-1} v + C$

$= \frac{-3}{16}(1 - 16x^2)^{\frac{1}{2}} + \sin^{-1}4x + C$ ✓✓ (4)

4.1.3 $y = \int \frac{5x^3 - 23x^2 + 15x + 12}{5x + 2}dx$

$= \int x^2 - 5x + 5 + \frac{2}{5x + 2}dx$

$= \frac{x^3}{3} - \frac{5}{2}x^2 + 5x + \frac{2}{5}\ln(5x + 2) + C$ ✓✓✓✓ (5)

4.1.4 $y = \int \cos^3 x \sin x\,dx$

Let $u = \cos x$

$du = -\sin x\,dx$ ✓

$y = -\int u^3 du$ ✓

$= -\frac{1}{4}u^4 + C$

$= -\frac{1}{4}\cos^4 x + C$ ✓ (3)

4.1.5 $y = \int \tan^3 x\,dx$

$\int \tan^3 x\,dx = \int \tan^2 x \tan x\,dx$

$= \int (\sec^2 x - 1)\tan x\,dx$

$= \int \sec^2 x \tan x\,dx - \int \tan x\,dx$ ✓

$= \int \sec^2 x \tan x\,dx - \ln \sec x + C$ ✓

Let $u = \tan x$; $du = \sec^2 x\, dx$ ✓

Then $\int \sec^2 x \tan x\, dx = \int u\, du = \frac{u^2}{2} + C_1$ ✓

$\therefore y = \frac{u^2}{2} - \ln \sec x + C + C_1$

$= \frac{\tan^2 x}{2} - \ln \sec x + K$ ✓✓ (6)

4.2 Determine $\int y\, dx$ by resolving the integral into partial fractions:

$$y = \int \frac{x + 8}{(x - 3)(x + 1)} dx$$

$$\frac{x + 8}{(x - 3)(x + 1)} = \frac{A}{(x - 3)} + \frac{B}{(x + 1)}$$

$x + 8 = A(x + 1) + B(x - 3)$ ✓

$x + 8 = Ax + A + Bx - 3B$

$\therefore A = 2; B = -1$ ✓✓

$y = \int \frac{2}{(x - 3)} - \frac{1}{(x + 1)} dx$

$= 2 \ln(x - 3) - \ln(x + 1) + C$ ✓✓ (5)

[27]

QUESTION 5

5.1 $\int_2^{10} \left[\frac{(\ln 2x)^2}{x} + \frac{e^x}{e^x + 2}\right] dx$

$= \int_2^{10} \frac{(\ln 2x)^2}{x} dx + \int_2^{10} \frac{e^x}{e^x + 2} dx$

Let $u = \ln 2x\, dx$; $du \frac{1}{x} dx$

$z = e^x + 2$; $dz = e^x\, dx$ ✓

$= \int u^2\, du + \int \frac{1}{z} dz$ ✓

$= \left[\frac{(\ln 2x)^3}{3} + \ln(e^x + 2)\right]_2^{10}$ ✓

$= \left[\frac{(\ln 20)^3}{3} + \ln(e^{10} + 2)\right] - \left[\frac{(\ln 4)^3}{3} + \ln(e^2 + 2)\right]$ ✓

$= 15{,}834$ ✓ **[5]**

QUESTION 6

6.1 $4x - 2x^2 = 0$

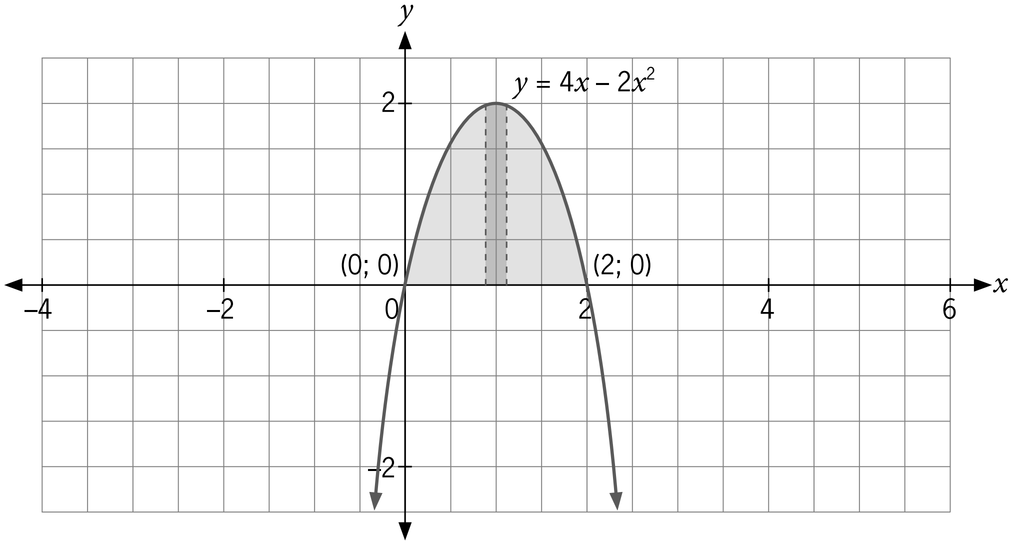
$2x(2 - x) = 0$

$\therefore x = 0$ or $x = 2$

and $y = 0$

The co-ordinates are (0; 0) and (2; 0) ✓✓ (2)

6.2



(2)

6.3 $A = \int_a^b y\,dx$

$A = \int_0^2 4x - 2x^2\,dx$ ✓

$= \left[2x^2 - \frac{2}{3}x^3\right]_0^2$ ✓

$= \left[2(2)^2 - \frac{2}{3}(2)^3\right] - \left[2(0)^2 - \frac{2}{3}(0)^3\right]$

$= 8 - \frac{16}{3}$

$= 2\frac{2}{3} = 2{,}667 \text{ units}^2$ ✓ (3)

6.4 $V = \pi\int_0^2 (4x - 2x^2)^2\,dx$

$= \pi\int_0^2 16x^2 - 16x^3 + 4x^4\,dx$ ✓✓

$= \pi\left[16\frac{x^3}{3} - 4x^4 + 4\frac{x^5}{5}\right]_0^2$ ✓

$= \pi\left[\left[16\frac{2^3}{3} - 4(2)^4 + 4\frac{2^5}{5}\right] - [0]\right]$

$= \frac{64}{15}\pi = 13{,}404 \text{ units}^3$ ✓ (4)

[11]

QUESTION 7

7.1 Moment of inertia is: $\frac{1}{2}r^2\,dm$

$dl_z = \frac{1}{2}r^2\,dm$ ✓

$l_z = \frac{1}{2}r^2\int_0^{0{,}03} dm$ ✓

$= \frac{1}{2}(0{,}03)^2(15)$ ✓

$= 0{,}007 \text{ kg.m}^2$ ✓ **[4]**

QUESTION 8

8.1 $\frac{dy}{dx} = -\frac{x}{ye^{x^2}}$

$y\,dy = -xe^{-x^2}\,dx$

$\frac{y^2}{2} = \frac{1}{2}e^{-x^2} + C$ ✓✓

$\frac{2^2}{2} = \frac{1}{2}e^{-0^2} + C$

$\therefore C = 1\frac{1}{2}$ ✓

$\frac{y^2}{2} = \frac{1}{2}e^{-x^2} + 1\frac{1}{2}$ ✓ (4)

8.2 $\sec x.\frac{d^2y}{dx^2} = 1 + \tan x + \frac{x^2}{\cos x}$

$\frac{d^2y}{dx^2} = \cos x + \sin x + x^2$ ✓

$\frac{dy}{dx} = -\sin x + \cos x + \frac{x^3}{3} + C_1$ ✓

$y = -\cos x - \sin x + \frac{x^4}{12} + C_1 x + C_2$ ✓ (3)

[7]

TOTAL: 100

Glossary

A

Algebraic fraction – a fraction with an algebraic expression in the numerator and denominator

Algorithm – a process or set of rules to be followed in calculations or other problem-solving operations

Antiderivative – reversing the process of differentiation, the indefinite integral

Arbitrarily – not related to a specific value, true for any value close enough to the point

Asymptote – a straight line that continually approaches a given curve but does not meet it

B

Boundary condition – a value that constrains the problem to a particular solution

C

Calculus – the study of how quantities change

Centroid – the centre point of a geometric object of uniform density

Chord – a line segment connecting two points on a curve

Circular lamina – a two-dimensional circular surface in a plane which has both mass and surface density

Coincide – occur at the same place or time; be identical

Concavity – inward-curving or hollow

Continuous – smooth and without interruptions, gaps, or breaks

Converge – move or tend towards, or approach a point

D

Decomposition – breaking the number apart, for example: 254 = 200 + 50 + 4

Definite integral – the area under the graph of a function, with respect to an axis, obtained by calculating the continuous sum of the area between two limits

Derivative – the result of differentiating a function; the rate at which the dependent variable of the function changes with respect to changes in the independent variable

Differentiation – the process of finding the rate of change of a function

Differential equation – an equation that describes the relationship between a function and its derivatives

Discontinuity – a point where the limit of a function is not equal to the function value or the limit does not exist

Discs (**also disk**) – thin round objects

Distinct – clearly different from others

Diverge – to go in different directions

E

Extrema – collective term for extreme values, maximum- or minimum-values

F

First moment of area – a measure of the distribution of the area of a shape in relation to an axis; used to determine the centroid of the shape

First order differential equation – differential equation that contains only first derivatives, $\frac{dy}{dx}$

G

General solution – a function (or set of functions) that satisfies the differential equation

Geometric – relating to lines and shapes

Global – over the whole domain

I

Improper fractions – a fraction where the degree of the numerator is greater than the degree of the denominator

Increment – the amount (often small) by which something is increased

Indefinite integral – the result of integrating a function with no limits, also known as the antiderivative

Indeterminate form – a limit that cannot be determined by substitution

Inertia – the resistance to movement or change in movement

Initial value – another name for the boundary condition, depending on the context

Integrand – the function to integrate

Integrate – to calculate the integral

Integration – the process of finding the area under the curve of a function

L

Lamina – a two-dimensional surface in a plane which has both mass and surface density

Limit – the predicted value of a function based on the values of points close to it

Limits – the points between which a definite integral is evaluated

Linear factor – a first degree polynomial, $ax + b$

Local – on an interval

M

Mathematical model – any description of a system using mathematical concepts and language

Maximum – the largest value of a function in a given interval or on the entire domain

Minimum – the smallest value of a function in a given interval or on the entire domain

Monotone – unchanging

O

Optimal – a solution where the function reaches its maximum (or minimum) value

Optimisation – the process of finding the maxima

Oscillation – repetitive back and forth movement

P

Particular solution – the function that is obtained when particular values are assigned to the arbitrary constants in the general solution of a differential equation

Perimeter – the continuous line forming the boundary of a closed geometrical figure

Point of inflection – a point on a curve at which a change in the direction of the curve occurs

Proper fractions – a fraction where the degree of the numerator is less than the degree of the denominator

Q

Quadratic – a quadratic factor contains terms of the second degree, or square terms

R

Rate of change – how quickly one quantity changes in terms of another quantity

Rectangular lamina – a two-dimensional rectangular surface in a plane which has both mass and surface density

Rigid – resistant to change, does not bend or twist easily

Root –a solution to an equation of the form $f(x) = 0$

S

Second order differential equation – differential equation that includes a second derivative, $\frac{d^2y}{dx^2}$

Solids of revolution – a solid of revolution is a solid form obtained by rotating a plane curve around some straight line that lies on the same plane

Steel profiles – products such as beams, T-sections, U-sections, angles, bars made of steel

T

Tangent – a straight line that touches a curve at a single point and does not cross it

Torsion – twisting of a body that has some resistance, as the result of an applied force

Torus – a solid of revolution generated by revolving a circle about an axis on the same plane

V

Vertices (**singular: vertex**) – point(s) where two or more straight lines meet