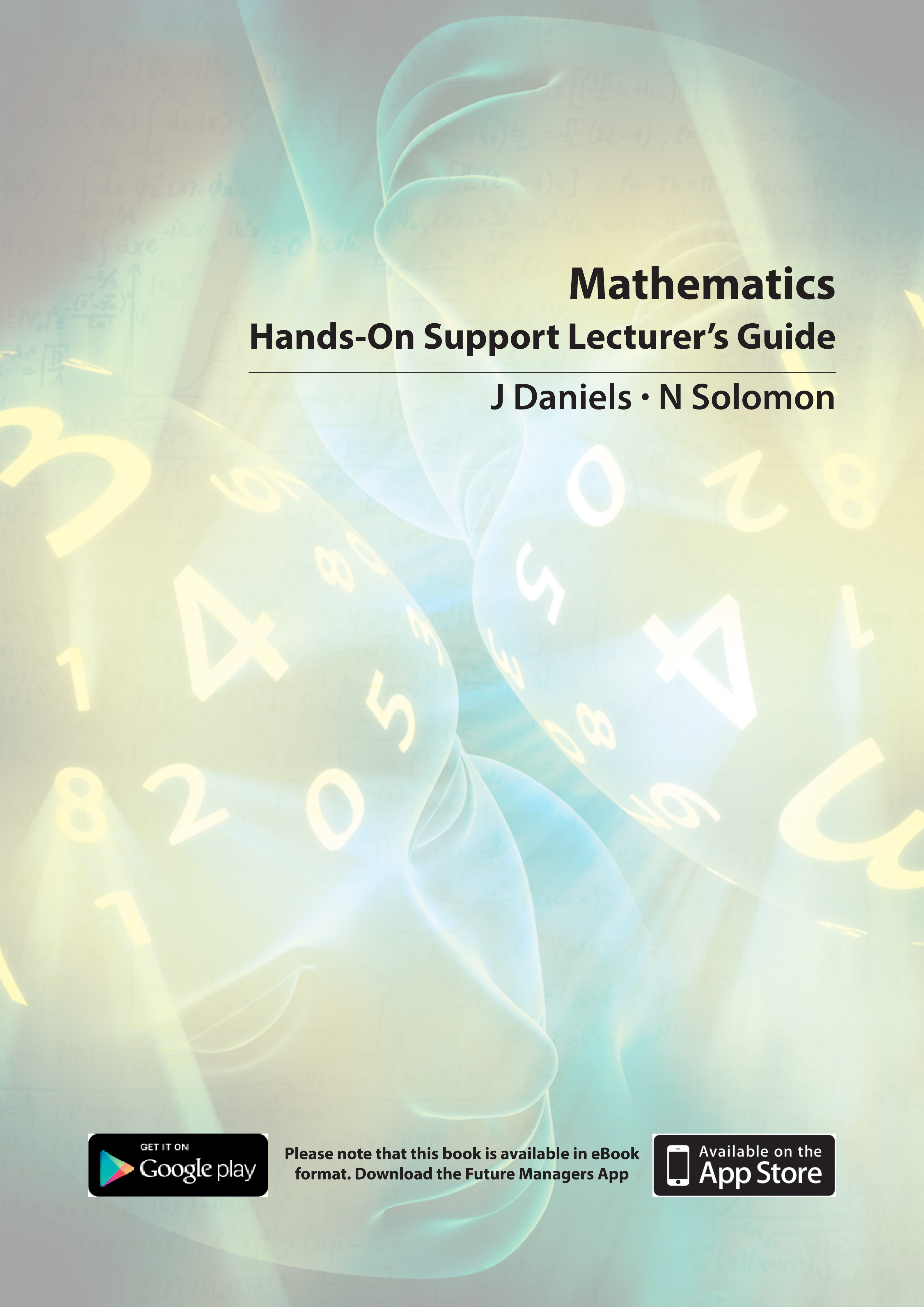

Mathematics

Hands-On Support Lecturer's Guide

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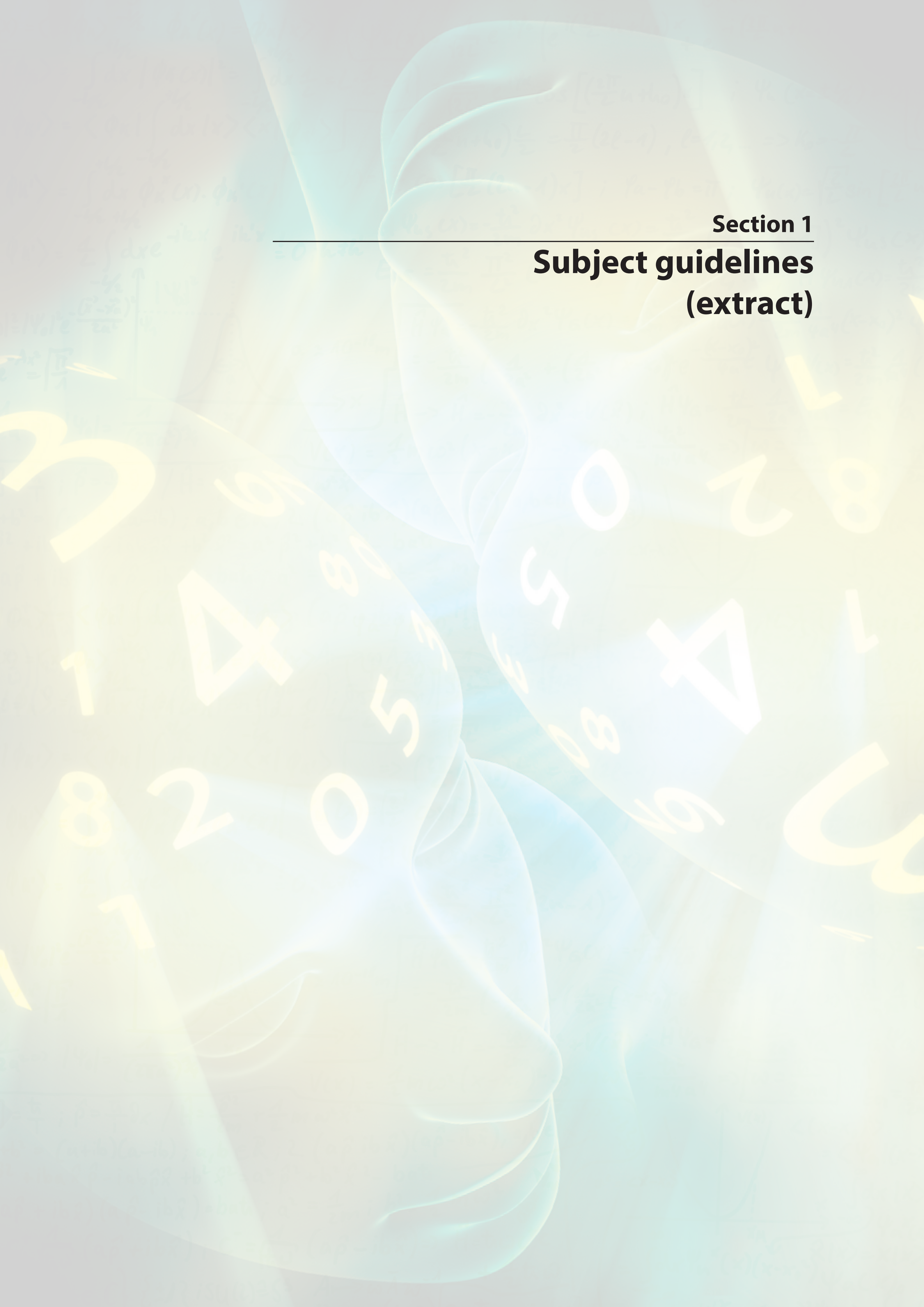


Section 1

Subject guidelines (extract)

1. Introduction

Mathematics enables creative and logical reasoning about problems in the physical and social world and in the context of Mathematics itself. Through mathematical problem solving, students develop an understanding of the world and can use that understanding to great effect in their daily lives.

Knowledge in the mathematical sciences is constructed through the establishment of descriptive, numerical and symbolic relationships. The Subject Outcomes and Assessment Standards for Mathematics are designed to allow all students to become citizens who will be able to confidently deal with Mathematics as and when it impinges on their daily lives, their community and the world in general.

The subject Mathematics (NQF Level 2 – 4) empowers students to:

- Communicate appropriately using numbers, verbal descriptions, graphs, symbols, tables and diagrams.
- Use mathematical process skills to identify, pose and solve problems creatively and critically.
- Organise, interpret and manage mathematical information which demonstrates responsibility and sensitivity to personal and broader societal concerns.
- Work collaboratively in teams and groups to promote understanding in general.
- Collect, analyse and organise quantitative data to evaluate and comment on conclusions.
- Engage responsibly with quantitative arguments relating to local, national and global concerns.

2. Duration of programme

This is a one year instructional programme comprising 200 teaching and learning hours. The subject may be offered on a part-time basis provided all the assessment requirements are adhered to.

Students with special education needs (LSEN) must be catered for in a way that eliminates barriers to learning.

3. Subject level outcomes

Students will be able to:

- Perform advanced operations on complex numbers and solve problems using complex numbers.
- Investigate and represent a wide range of algebraic expressions and functions and solve related problems.
- Use the Cartesian co-ordinate system to derive and apply equations.
- Explore, interpret and justify geometric relationships.
- Solve problems by constructing and interpreting trigonometric models.
- Analyse and interpret data to establish statistical models to solve related problems.

- Use experiments, simulation, and probability distribution to set and explore probability models.
- Use mathematics to plan and control financial instruments.

4. Assessment

4.1 Internal assessment (25 percent)

Detailed information regarding internal assessment and moderation is outlined in the current ICASS Guideline document provided by the DHET.

Distribution of internal assessment components.

Three formal written tests and one internal examination	70% of ICASS

Two assignments and one practical assessment	30% of ICASS

Possible spread of internal assessment during the year

Term 1	Term 2	Term 3	Term 4	Total
2	2 – 3	*2 – 3	0 – 1	7

*One of these must be an internal examination

4.2 External assessment (75 percent)

A national examination is conducted annually in October/November by means of a paper/papers set and moderated externally.

5. Weighted values of topics

Topics	Weighted value	*Teaching hours
1. Complex numbers	10	10
2. Functions and algebra	40	40
3. Space, shape and measurement	25	35
4. Data handling and probability models	15	18
5. Finance	10	7
Total	**100**	**110**

*Teaching hours refer to the minimum hours required for face to face instruction and teaching. This number excludes time spent on revision, test series and internal and external examination/assessment. The number of the allocated teaching hours is influenced by the topic weighting, complexity of the subject content and the duration of the academic year.

6. Calculation of final mark

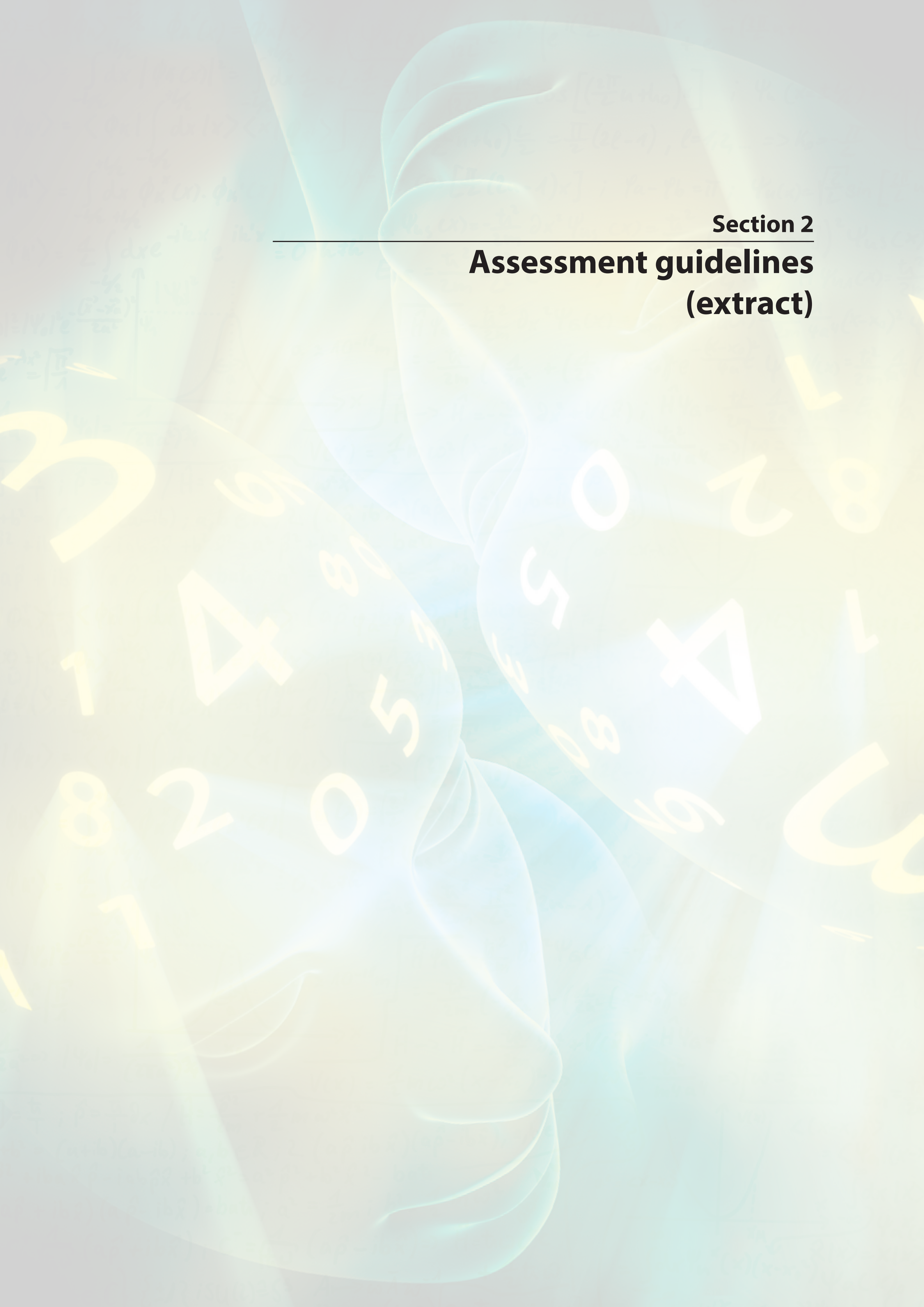
Continuous assessment:	Student’s mark/100 × 25/1 = a mark out of 25 (a)
Examination mark:	Student’s mark/100 × 75/1 = a mark out of 75 (b)
Final mark:	**(a) + (b) = a mark out of 100**

7. Pass requirements

The student must obtain a minimum of 30 percent in the subject. A pass will be condoned at 25 percent if it is the only subject stopping the student from obtaining a level 4 certificate.

Section 2

Assessment guidelines (extract)



1. Introduction

This section will outline the most relevant parts of assessment process for the NCV programme. The full Assessment Guidelines are available on the DHET website.

2. Instruments and tools for collecting evidence

All evidence collected for assessment purposes is kept or recorded in the student's Portfolio of Evidence (PoE).

The following table summarises a variety of methods and instruments for collecting evidence. A method and instrument is chosen to give students ample opportunity to demonstrate the Subject Outcome has been attained. This will only be possible if the chosen methods and instruments are appropriate for the target group and the Specific Outcome being assessed.

	Methods for collecting evidence		
	Observation-based (less structured)	**Task-based** (structured)	**Test-based** (more structured)
Assessment instruments	• Observation • Class questions • Lecturer, student, parent discussions	• Assignments or tasks • Projects • Investigations or research • Case studies • Practical exercises • Demonstrations • Role-play • Interviews	• Examinations • Class tests • Practical examinations • Oral tests • Open-book tests
Assessment tools	• Observation sheets • Lecturer's notes • Comments	• Checklists • Rating scales • Rubrics	• Marks (e.g. %) • Rating scales (1–7)
Evidence	• Focus on individual students • Subjective evidence based on lecturer observations and impressions	**Open middle:** Students produce the same evidence but in different ways. **Open end:** Students use same process to achieve different results.	Students answer the same questions in the same way, within the same time.

3. Internal assessment

ICASS Tasks for Mathematics

(ICASS guideline: implementation January 2013)

Tasks	Time-frame	Type of assessment activity (the time and proposed mark allocation can be increased but not reduced)	Scope of assessment Do not confuse the weightings of topics in the Subject Guidelines with the % contribution to the year mark.	Contribution to the year mark
1	Term 1	Test 1 hour (50 marks)	Topics completed in term 1	10%
2	Term 1	**Assignment	Assignment on one or more topics completed to date	10%
3	Term 2	Test 1 hour (50 marks)	Topics completed in term 2	10%
4	Term 2	**Assignment	Topics completed in term 2	10%
5	Term 2	*Test 2 hours (70 marks)	Topics completed in term 1 and 2	20%
6	Term 3	Practical assessment/ **assignment	Topics completed, any related Subject Outcomes, for example: 1. Work with practical problems involving the construction of scatter plots, lines of best fit by regression analysis and predictions based on those results. 2. Work with practical problems involving tax tables. 3. Construct at least three circle geometrical riders and prove (by using a protractor) some of the major circle theorems. 4. Sketch the graph of a function e.g. $y = x$ for a specified domain, example $[-3 ; 3]$ and calculate the area by first using area of the triangles and then by using integration. Compare results of at least three different equations. 5. Work with practical activities involving probability models. Use dice, coins and cards.	10%
7	Term 3	*Internal Examination External examination papers serve as guidelines for content duration and mark allocation Paper 1 Paper 2	All topics completed to date Paper 1 = 15% Paper 2 = 15%	30%
			Total	**100%**

* The internal examination (Term 3) and the test (Term 2) can be swapped around to allow the examination to be written either during the second term or the third term. If the examination is written at the end of the

second term, at least 60% of the curriculum must have been covered. If the examination is written in the third term at least 80%–90% of the curriculum must have been covered.

** The assignment must be completed within 5 days. A clear instruction sheet outlining the task and the resources required to complete the task must be given to students.

4. Recording and reporting

Mathematics is assessed according to seven levels of competence. The level descriptions are explained in the following table.

Scale of achievement for the fundamental component

Rating code	Rating	Marks (%)
7	Outstanding	80 – 100
6	Meritorious	70 – 79
5	Substantial	60 – 69
4	Adequate	50 – 59
3	Moderate	40 – 49
2	Elementary	30 – 39
1	Not achieved	0 – 29

The planned/scheduled assessments should be recorded in the **Lecturer's Portfolio of Assessment (PoA)** for each subject. The minimum requirements for the Lecturer's Portfolio of Assessment should be as follows:

- Lecturer information
- A contents page
- Subject and assessment guidelines
- Year plans/Work schemes/Pace setters
- A subject assessment plan
- Instrument(s) (tests, assignments, practical) and tools (memorandum, rubric, checklist) for each assessment task
- A mark/result sheet for assessment tasks

The college must standardise these documents.

The minimum requirements for the **Student's Portfolio of Evidence (PoE)** should be as follows:

- Student information/identification
- A contents page/list of content (for accessibility)
- A subject assessment schedule
- A record/summary of results showing all the marks achieved per assessment for the subject
- The evidence of marked assessment tasks and feedback according to the assessment schedule
- Where tasks cannot be contained as evidence in the **Portfolio of Evidence** (PoE), its exact location must be recorded and it must be readily for moderation purposes.

5. Specifications for external assessment in Mathematics – Level 4

A national examination is conducted in October/November each year by means of a paper(s) set and moderated externally. The examination will be structured as follows:

Level 4	Knowledge	Comprehension and application	Analysis
	30%	50%	20%

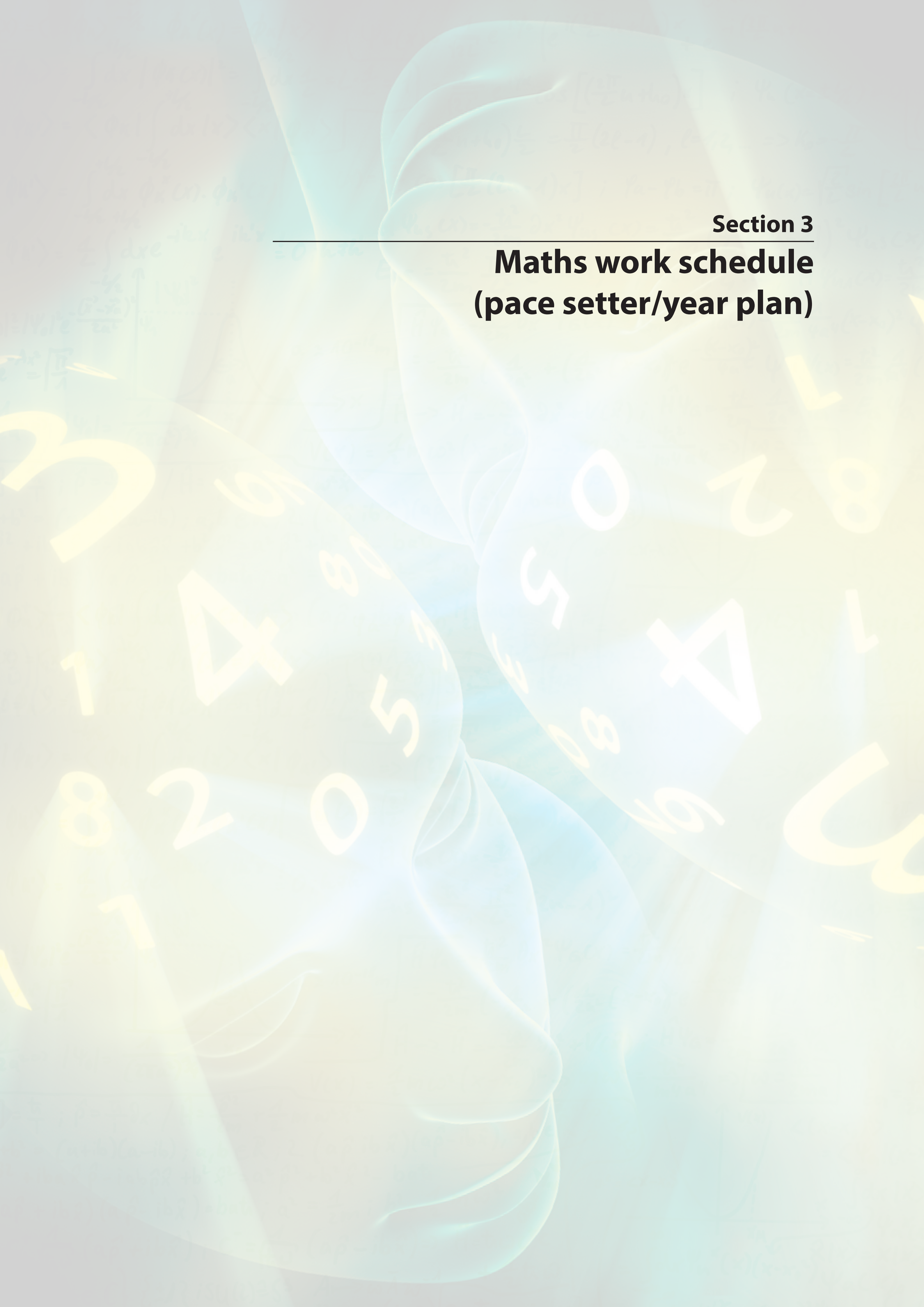
Proposed mark distribution between paper 1 and paper 2 is proposed for setting national examination papers:

Paper 1 (3 hours)

Topics	Weighted value
1. Complex numbers	20
2. Functions and algebra	
2.1 Functions and algebra	25
2.2 Linear programming	15
2.3 Differentiation	25
2.4 Integration	15
Total	**100**

Paper 2 (3 hours)

Topics/themes	Weighted value
3. Space, shape and measurement	
3.1 Geometry	25
3.2 Trigonometry	25
4. Statistics and probability models	
4.1 Statistics	15
4.2 Probability	15
5. Financial Mathematics	20
Total	**100**



Section 3

Maths work schedule (pace setter/year plan)

Topic 1 Complex numbers

(approximately 10 hours)

Outcomes	Duration
1.1 Work with complex numbers	
1.1.1 Perform addition, subtraction, multiplication and division on complex numbers in standard form (including *i*-notation).	1
1.1.2 Perform multiplication and division on complex numbers in polar form.	2
1.1.3 Use De Moivre's theorem to raise complex numbers to powers (excluding fractional powers)	1
1.1.4 Convert the form of complex numbers where needed to enable performance of advanced operations on complex numbers (a combination of standard and polar form may be assessed in one expression)	2
1.2 Solve problems using complex numbers	
1.2.1 Solve identical complex numbers in rectangular/standard form using the concept of simultaneous equations.	2
1.2.2 Use complex numbers to solve equations that cannot be solved using the real number system by applying: • Factorisation • Quadratic formula	2
Total number of hours	**10**

Topic 2 Functions and algebra

(approximately 40 hours)

Outcomes	Duration
2.1 Work with algebraic expressions using the remainder and the factor theorems.	
2.1.1 Use and apply the remainder and the factor theorem. • Find the remainder • Prove that an expression is a factor • Find an unknown variable in order to make an expression, a factor or to leave a remainder.	3
2.1.2 Factorise third degree polynomials including examples that require the factor theorem. (Long division or any other method may be used.)	2
2.2 Use a variety of techniques to sketch and interpret information for the inverse graphs of functions.	
2.2.1 Determine the equations of the inverses of the functions: $y = ax + q$ $y = ax^2$ $y = a^x; a > 0$ ($y = a^x$ may be left with x as the subject of the formula. **Note:** No logarithms required.)	2
2.2.2 Sketch the inverse graphs of the functions: $y = ax + q$ $y = ax^2$ $y = a^x; a > 0$ **Note**: Sketching the graphs using point by point plotting is an option.	2
2.2.3 Obtain the equation of any of the following inverse graphs given as a sketch. $y = ax + q$ $y = ax^2$ $y = a^x; a > 0$	1
2.2.4 Identify characteristics as listed below in respect of the following functions. $y = ax + q$ $y = ax^2$ $y = a^x; a > 0$ • Domain and range • Intercepts with axes • Turning points, minima and maxima • Asymptotes • Shape and symmetry • Functions or non-functions • Continuous or discontinuous • Intervals at which a function increases or decreases	2

2.3	**Use mathematical models to investigate linear programming problems.**	
2.3.1	Find and formulate the linear constraints from a given problem.	
2.3.2	Solve linear programming problems by optimising a function in two variables, subject to one or more linear constraints, using the search line method.	3
2.4	**Investigate and use instantaneous rate of change of a variable when interpreting models in both mathematical and real-life situations.**	
2.4.1	Establish the derivatives of the following functions from first principles: $f(x) = b$ $f(x) = ax + b$ $f(x) = ax^2 + b$ $f(x) = x^3$ $f(x) = ax^3$ $f(x) = \frac{1}{x}$ $f(x) = \frac{a}{x}$ **Note:** The binomial theorem does not form part of the curriculum.	3
2.4.2	Find the derivatives of the function in the form: $f(x) = ax^n$ $f(x) = a \ln kx$ $f(x) = ae^{kx}$ $f(x) = a \sin kx$ $f(x) = a \cos kx$ $f(x) = a \tan kx$ where $f(x) = ax^n$ $\quad$ $f'(x) = nax^{n-i}$ $f(x) = \ln kx$ $\quad$ $f'(x) = \frac{k}{x}$ $f(x) = e^{kx}$ $\quad$ $f'(x) = ke^{kx}$ $f(x) = a \sin kx$ $\quad$ $f'(x) = ka \cos kx$ $f(x) = a \cos kx$ $\quad$ $f'(x) = -ka \sin kx$ Examples to include are $3x^2$; $\frac{3}{x^{-3}}$; $\frac{-2}{\sqrt[3]{x^2}}$; $2 \ln 3x$; $\frac{1}{2} e^{-2x}$; $2 \sin 3x$; $\frac{1}{3} \cos \frac{x}{2}$; $-4 \tan x$ etc.	3
2.4.3	Use the constant, sum and/or difference, product, quotient and chain rules for differentiation. **Note:** Combination of rules in the same problem are excluded.	3
2.4.4	Find the equation of the tangent to a graph at a specific point.	1

2.4.5	Solve practical problems involving rates of change. **Note:** velocity and acceleration may be included.	2
2.4.6	Draw graphs of cubic functions by determining: • y-intercept • roots (x-intercepts) • turning points using derivatives.	3
2.4.7	Determine/prove maximum and minimum turning points by using second order derivatives (only quadratic and cubic functions).	1
2.4.8	Determine the point of inflection of cubic graphs by using second order derivatives.	1
2.5	**Analyse and represent mathematical and contextual situations using integrals and find areas under curves by using integration rules.**	
2.5.1	Find the integrals of the following: $\int ax^n\,dx$ $\quad$ $\int \frac{a}{x}\,dx$ $\int ae^{kx}\,dx$ $\quad$ $\int a\sin kx\,dx$ $\int a\cos kx\,dx$ $\quad$ $\int a\sec^2 kx\,dx$ where: $\int ax^n\,dx = \frac{ax^{n+1}}{n+1} + c$ $\int \frac{a}{x}\,dx = a\ln x + c$ $\int ae^{kx}\,dx = \frac{ae^{kx}}{k} + c$ $\int a\sin kx\,dx = \frac{-a\cos kx}{k} + c$ $\int a\cos kx\,dx = \frac{a\sin kx}{k} + c$	3
Note:	• Simplifications may be required where necessary. • Integrals of polynomials may be assessed. • Integration by parts is excluded.	
2.5.2	Use the upper and lower limits to calculate definite integrals.	1
2.5.3	Determine the area under a curve by: • working from a given graph or sketching a graph • working with an area bounded by a curve, the x-axis, an upper and a lower limit • splitting the area into two intervals when the graph crosses the x-axis.	4
Note:	• Integrals in respect of the x-axis only. • Areas between two curves are excluded. • The y-axis ($x = 0$) can be used as an upper or lower limit.	
	Total number of hours	**40**

Topic 3 Space, shape and measurement (approximately 35 hours)

Outcomes	Duration
3.1 Use the Cartesian coordinate system to derive and apply equations.	
3.1.1 Use the Cartesian coordinate system to derive and apply the equation of a circle (any centre).	2
3.1.2 Use the Cartesian coordinate system to derive and apply the equation of a tangent to a circle given a point on the circle. **Note:** • Straight lines to be written in the following forms only: $y = mx + c$; $y - y_1 = m(x - x_1)$ and/or $ax + by + c = 0$ (general form) • Learners are expected to know and be able to use as an axiom "the tangent to a circle is perpendicular to the radius drawn to the point of contact".	4
3.2 Explore, interpret and justify geometric relationships.	
3.2.1 Use geometry of straight lines and triangles to solve problems and to justify relationships in geometric figures. Concepts to include are: • angles of a triangle • exterior angles • straight lines • vertically opposite angles • corresponding angles • co-interior angles • alternate angles	4
3.2.2 State and apply the major theorems of circles. • If a line is drawn from the centre of a circle to the midpoint of a chord, then that line is perpendicular to the chord. • If a line is drawn from the centre of the circle perpendicular to the chord, then it bisects the chord. • If an arc subtends an angle at the centre of the circle and at any point on the circumference, then the angle at the centre is twice the measure of the angle at the circumference. • If the diameter of a circle subtends an angle at the circumference, then the angle subtended is a right angle triangle.	10

- If an angle subtended by a chord at a point on the circumference is a right angle, then the chord is a diameter.
- Angles in the same segment of a circle are equal.
- The opposite angles of a cyclic quadrilateral are supplementary.
- An exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.
- If the exterior angle of a quadrilateral is equal to the interior opposite angle the quadrilateral will be a cyclic quadrilateral.
- The four vertices of a quadrilateral in which the opposite angles are supplementary will be a cyclic quadrilateral.
- If a tangent to a circle is drawn, then it is perpendicular to the radius at the point of contact.
- If a line is drawn perpendicular to a radius at the point where the radius meets the circle, then it is a tangent to the circle.
- If two tangents are drawn from the same point outside a circle then they are equal in length.
- The angle between a tangent to a circle and a chord drawn from the point of contact is equal to an angle in the alternate segment. (tan-chord theorem)

Note: Proofs of the above theorems are excluded.

3.3 Solve problems by constructing and interpreting trigonometric models.

3.3.1 Use the compound angle identities, 3
$\sin(\alpha \pm \beta) = \sin \alpha.\cos \beta \pm \sin \beta.\cos \alpha$ and
$\cos(\alpha \pm \beta) = \cos \alpha.\cos \beta \mp \sin \alpha.\sin \beta$ to derive and apply the following double angle identities,
$\sin 2\alpha = 2 \sin \alpha.\cos \alpha$
$\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha$
$\cos 2\alpha = 2 \cos^2 \alpha - 1$
$\cos 2\alpha = 1 - 2 \sin^2 \alpha$

3.3.2 Determine the specific solutions of trigonometric expressions using compound and double angle identities without a calculator. (e.g. sin 120°, cos 75° etc.) 3

3.3.3 Use compound angle identities to simplify trigonometric expressions and to prove trigonometric equations. 2

3.3.4	Determine the specific solutions of trigonometric equations by using knowledge of compound angles and identities. **Note:** • Solutions: [0; 360°] • Identities limited to: $\tan \theta = \frac{\sin \theta}{\cos \theta}$ $\sin^2 \theta + \cos^2 \theta = 1$ • Double and compound angle identities are included. **Note:** Radians are excluded.	3
3.3.5	Solve problems from a given diagram in two and three dimensions by applying the sine and cosine rule. **Note:** Area formula and compound angle identities are excluded.	3
	Total number of hours	**35**

Topic 4 Data handling and probability models

(approximately 18 hours)

Outcomes	Duration
4.1 Represent, analyse and interpret data using various techniques	
4.1.1 Identify situations or issues that can be dealt with through statistical methods. **Range:** Data given should include problems relating to health, social, economic, cultural, political and environmental issues. **Note:** Not for examination purposes but for class activities only.	0,5
4.1.2 Discuss the use of appropriate and efficient methods to record, organise and interpret given data by making use of: • Manageable data sample sizes (less than or equal to 10) and which are representative of the population. • Graphical representations and numerical summaries which are consistent with the data, and clear and appropriate to the situation and target audience. **Note:** Discussion only, not expected to draw again. • Compare different representations of given data.	1
4.1.3 Justify and apply statistics to answer questions about problems.	0,5
4.1.4 Discuss new questions that arise from the modelling of data.	0,5
4.1.5 Take a position on an issue by comparing different representations of given data.	0,5
4.2 Use variance and regression analysis to interpolate and extrapolate bivariate data	
4.2.1 Calculate variance and standard deviation manually for small sets of data only.	2
4.2.2 Interpret the meaning of variance and standard deviation for small sets of data only.	0,5
4.2.3 Represent bivariate numerical data as a scatter plot.	1
4.2.4 Identify intuitively whether a linear, quadratic or exponential function would best fit the data.	0,5

	4.2.5	Draw the intuitive line of best fit. **Range:** • Data given should include problems related to health, social, economic, cultural, political and environmental issues. • For small sets of data only (limited to 8)	0,5
	4.2.6	Use least squares regression method to determine a function which best fits a given set of bivariate data.	3
	4.2.7	Use the regression line to predict the outcome of a given problem.	0,5
4.3	**Use experiments, simulation and probability to set and explore probability models**		
	4.3.1	Explain and distinguish between the following terminology or events: • Probability • Dependent events • Independent events • Mutually exclusive • Mutually inclusive • Complimentary events	1,5
	4.3.2	Make predictions based on validated experimental or theoretical probabilities taking the following into account: • P(S) = 1 (where S is the sample space) • Disjoint (mutually exclusive) events, and is therefore able to calculate the probability of either of the events occurring by applying the addition rule for disjoint events: P(A or B) = P(A) + P(B) • Complementary events and is therefore able to calculate the probability of an event not occurring • P(A or B) = P(A) + P(B) – P(A and B) (where A and B are events within a sample space) • Correctly identify dependent and independent events (e.g. from two-way contingency tables or Venn diagrams) and therefore appreciate when it is appropriate to calculate the probability of two independent events occurring by applying the product rule for independent events: P(A and B)= P(A).P(B).	2

4.3.3	Draw tree diagrams, Venn diagrams and complete contingency two-way tables to solve probability problems (where events are not necessarily independent) **Range:** • Venn diagrams to be limited to two subsets. • Tree diagrams where the sample space in manageable. (not more than 15 possible outcomes)	3
4.3.4	Interpret and clearly communicate results of the experiments correctly in terms of real context.	0,5
	Total number of hours	**18**

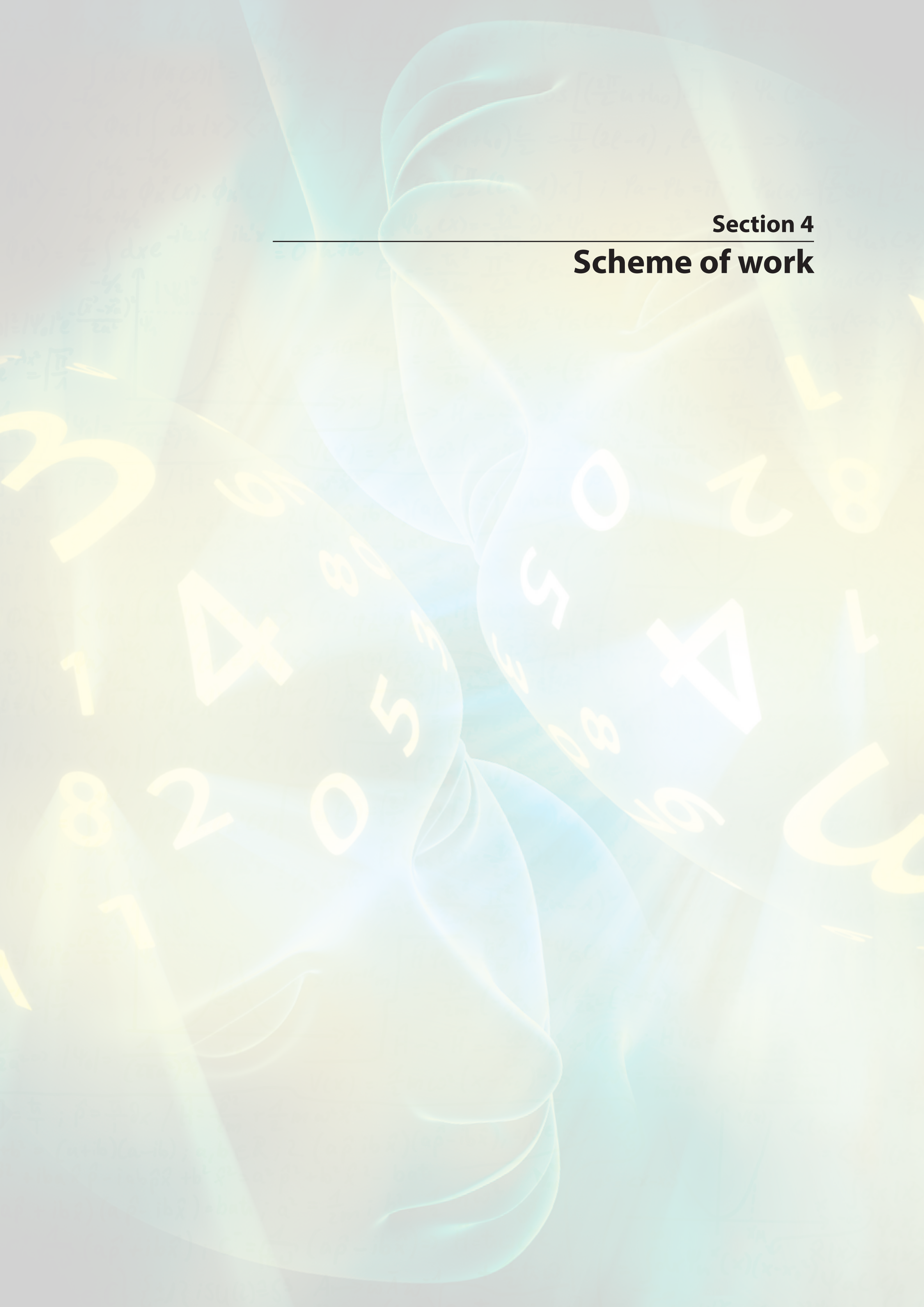
Topic 5 Financial Mathematics

(approximately 7 hours)

Outcomes		Duration
5.1	**Use mathematics to plan and control financial instruments**	
5.1.1	Use simple and compound growth formulae $A = P(l + in)$ and $A = P(1 + i)^n$ and $A = P\left(1 + \frac{r}{100 \times m}\right)^{t \times m}$ to solve problems, including interest, hire-purchase and inflation.	2
5.1.2	Understand, use and interpret tax tables.	2
5.1.3	Use simple and compound decay formulae, $A = P(1 - in)$ and $A = P(1 - i)^n$ to solve problems (straight-line depreciations and depreciation on a reducing balance).	3
	Total number of hours	**7**

Section 4

Scheme of work

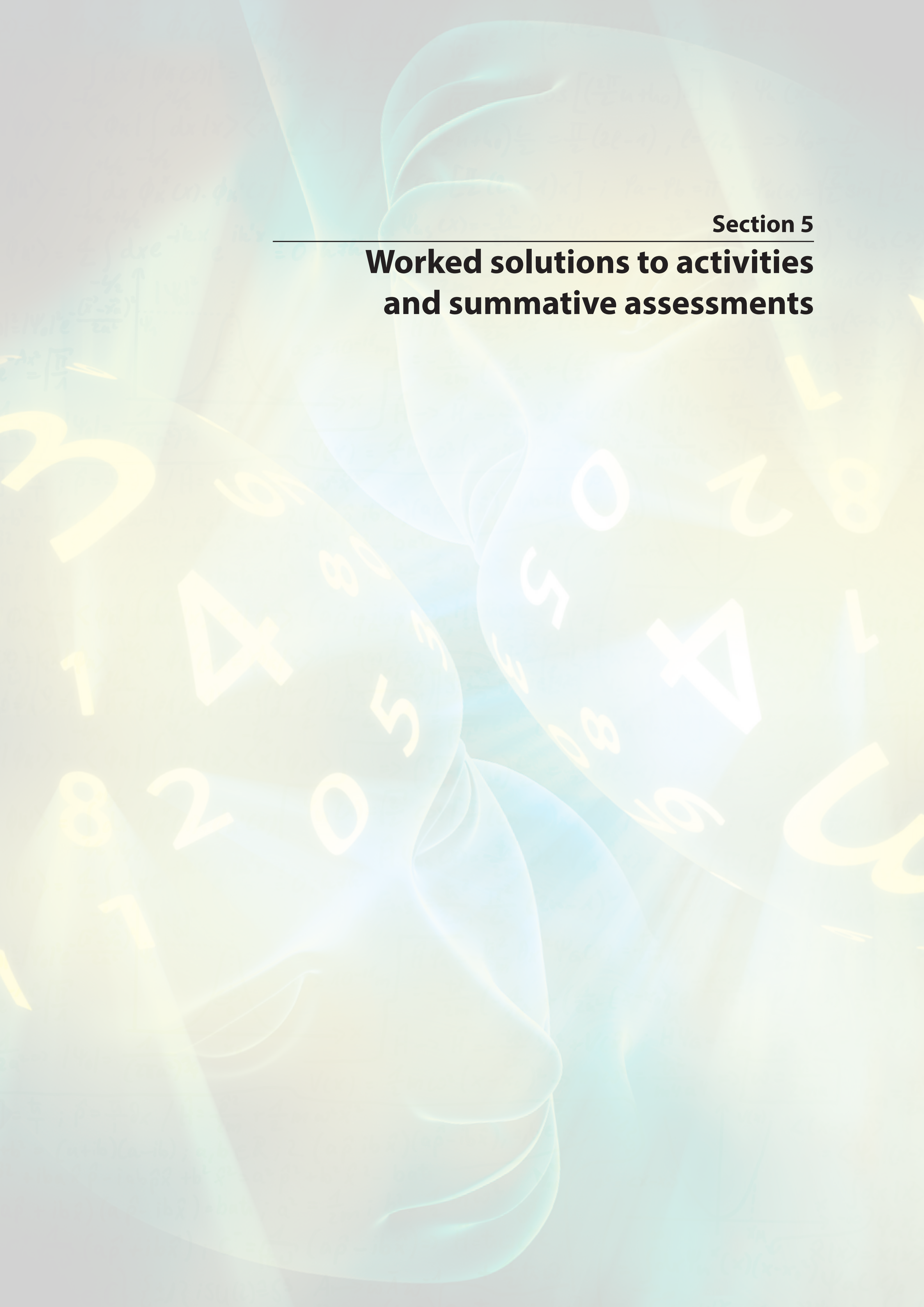


Subject and NC(V) level summarised scheme of work for . . . . . (year)

	Topic	Subject outcome number	Subject outcome	Dates	Days/ hours	Date completed
Term 1						
	Vacation: dates					
Term 2						
	Revision: Dates (one week)					
	Test series/exam: Dates (two weeks)					
	Vacation: Dates					
Term 3						
	Revision: Dates (one week)					
	Internal examinations: Dates (2–3 weeks)					
	Vacation: Dates					
Term 4	Corrections of September exam papers and revision: Dates (10 days)					
	External exams: Dates					

Section 5

Worked solutions to activities and summative assessments



Worked solutions • Chapter 1 Complex numbers

Assessment activity 1.1

1. $i^8 = (i^2)^4$
$= (-1)^4$
$= 1$ or $1 + 0i$

2. $-i^{11}$
$= -(i^2)^5 i$
$= -(-1)^5 i$
$= -(-1)i$
$= i$

3. $5i^9$
$= 5(i^2)^4 i$
$= 5(-1)^4 i$
$= 5(1)i$
$= 5i$

4. $3i^{102}$
$= 3(i^2)^{51}$
$= 3(-1)^{51}$
$= -3$

5. $-i^{40}$
$= -(i^2)^{20}$
$= -(-1)^{20}$
$= -(1)$
$= -1$

6. $\frac{2}{i^6} = \frac{2}{(i^2)^3}$
$= \frac{2}{(-1)^3}$
$= -2$ or $-2 + 0i$

7. $(\sqrt{2}i^6)^2$
$= 2i^{12}$
$= 2(i^2)^6$
$= 2(-1)^6$
$= 2$

8. $i^{10} + i^{12} - i^{23}$
$= (i^2)^5 + (i^2)^6 - (i^2)^{11} i$
$= (-1)^5 + (-1)^6 - (-1)^{11} i$
$= -1 + 1 + i$
$= i$ or $0 + i$

9. $[1 - i]^2$
$= [1 - i][1 - i]$
$= 1 - 2i + i^2$
$= 1 - 2i + (-1)$
$= 0 - 2i$

10. $(i^4 + i^7)^2$
$= [(i^2)^2 + (i^2)^3 i]^2$
$= [(-1)^2 + (-1)^3 i]^2$
$= [1 - i]^2$
$= [1 - i][1 - i]$
$= 1 - 2i + i^2$
$= 1 - 2i + (-1)$
$= -2i$

11. $(\sqrt{3}i)^4 \times i^3 \times i^{-2}$
$= (\sqrt{3})^4 i^4 \times i^3 \times \frac{1}{i^2}$
$= \left(3^{\frac{1}{2}}\right)^4 (i^2)^2 \times (i^2)i \times \frac{1}{-1}$
$= 3^2(-1)^2 \times (-1)i \times (-1)$
$= 9 \times -i \times (-1)$
$= 9i$

12. $(-1 - 6i)(-1 + 6i)$
$= 1 - 36i^2$
$= 1 - 36(-1)$
$= 37$

13. i^{-15}

$= \frac{1}{i^{15}}$

$= \frac{1}{(i^2)^7 i}$

$= \frac{1}{(-1)^7 i} \times \frac{i}{i}$

$= \frac{i}{-i^2}$

$= \frac{i}{-(-1)}$

$= i$

14. $\frac{12i}{6i^7} = \frac{12i}{6(i^2)^3 i}$

$= \frac{2i}{(-1)^3 i}$

$= \frac{2i}{-i} \times \frac{i}{i}$

$= \frac{2i^2}{-i^2}$

$= \frac{2(-1)}{-(-1)}$

$= -2$

15. $(-i)^{38}$

$= i^{38}$

$= (i^2)^{19}$

$= (-1)^{19}$

$= -1$

16. $-i^{38}$

$= -(i^2)^{19}$

$= -(-1)^{19}$

$= 1$

17. $(2i)^5 \times i^9$

$= 32i^5 \times i^9$

$= 32i^{14}$

$= 32(i^2)^7$

$= 32(-1)^7$

$= -32$

18. $(-i)^2 \times 6i^3$

$= i^2 \times 6i^3$

$= (-1) \times 6(i^2)i$

$= (-1) \times 6(-1)i$

$= 6i$

19. $-8i^3 + i^2$

$= -8(i^2)i + (-1)$

$= -8(-1)i + (-1)$

$= -1 + 8i$

20. $30i^8 + 2i - 2^5 i + 5^2 i + 50i^{100}$

$= 30(i^2)^4 + 2i - 32i + 25i + 50(i^2)^{50}$

$= 30(-1)^4 + 2i - 32i + 25i + 50(-1)^{50}$

$= 30 + 2i - 32i + 25i + 50$

$= 80 - 5i$

Assessment activity 1.2

1. **a)** $(2 - 4i) - (-3 + 2i)$

$= 2 - 4i + 3 - 2i$

$= 5 - 6i$

b) $(5 - 2i) - (3 - 4i) - (4 - i)i$

$= 5 - 2i - 3 + 4i - 4i + i^2$

$= 2 - 2i + (-1)$

$= 1 - 2i$

c) $(2 - 3i)(3 + 5i)$

$= 6 + 10i - 9i - 15i^2$

$= 6 + i - 15(-1)$

$= 6 + i + 15$

$= 21 + i$

d) $(-3 + 5i)(-2 + i)^2$

$= (-3 + 5i)(-2 + i)(-2 + i)$

$= (-3 + 5i)(4 - 2i - 2i + i^2)$

$= (-3 + 5i)(4 - 4i - 1)$

$= (-3 + 5i)(3 - 4i)$

$= -9 + 12i + 15i - 20i^2$

$= -9 + 27i - 20(-1)$

$= -9 + 27i + 20$

$= 11 + 27i$

e)

$$\begin{aligned}
&(4 - 5i) - (2i^4 - i^2)\\
&= 4 - 5i - [2(i^2)^2 - (-1)]\\
&= 4 - 5i - [2(-1)^2 + 1]\\
&= 4 - 5i - [2 + 1]\\
&= 4 - 5i - 3\\
&= 1 - 5i
\end{aligned}$$

f)

$$\begin{aligned}
&(2 - i)(-3 + 3i)(4 - 5i)\\
&= (-6 + 6i + 3i - 3i^2)(4 - 5i)\\
&= (-6 + 9i + 3)(4 - 5i)\\
&= (-3 + 9i)(4 - 5i)\\
&= -12 + 15i + 36i - 45i^2\\
&= -12 + 51i - 45(-1)\\
&= -12 + 51i + 45\\
&= 33 + 51i
\end{aligned}$$

2.

$$\begin{aligned}
&z_1 . \bar{z}_2 . z_3\\
&= (2 + 3i)(-3 + i)(-4i)\\
&= (-6 + 2i - 9i + 3i^2)(-4i)\\
&= (-6 - 7i - 3)(-4i)\\
&= (-9 - 7i)(-4i)\\
&= 36i + 28i^2\\
&= 36i + 28(-1)\\
&= 36i - 28\\
&= -28 + 36i
\end{aligned}$$

3. a)

$$\begin{aligned}
&\frac{3 - i}{-i} \times \frac{i}{i}\\
&= \frac{3i - i^2}{-i^2}\\
&= \frac{3i - (-1)}{-(-1)}\\
&= \frac{3i + 1}{1}\\
&= 1 + 3i
\end{aligned}$$

b)

$$\begin{aligned}
&\frac{-i}{3 - 2i} \times \frac{3 + 2i}{3 + 2i}\\
&= \frac{-3i - 2i^2}{9 - 4i^2}\\
&= \frac{-3i - 2(-1)}{9 - 4(-1)}\\
&= \frac{-3i + 2}{13}\\
&= \frac{-3i}{13} + \frac{2}{13}\\
&= \frac{2}{13} - \frac{3}{13}i
\end{aligned}$$

c)

$$\begin{aligned}
&\frac{3 - 2i}{-i + 5} \times \frac{i + 5}{i + 5}\\
&= \frac{3i + 15 - 2i^2 - 10i}{-i^2 + 25}\\
&= \frac{3i + 15 - 2(- 1) - 10i}{- (-1) + 25}\\
&= \frac{-7i + 15 + 2}{1 + 25}\\
&= \frac{-7i + 17}{26}\\
&= \frac{17}{26} - \frac{7}{26}i
\end{aligned}$$

d)

$$\begin{aligned}
&\frac{2 - 4i}{-1 - i} \times \frac{-1 + i}{-1 + i}\\
&= \frac{-2 + 2i + 4i - 4i^2}{1 - i^2}\\
&= \frac{-2 + 6i - 4(-1)}{1 - (-1)}\\
&= \frac{-2 + 6i + 4}{2}\\
&= \frac{2 + 6i}{2}\\
&= \frac{2}{2} + \frac{6}{2i}\\
&= 1 + 3i
\end{aligned}$$

e) $\frac{\sqrt{2} + 3i}{-3i + \sqrt{2}}$

$$= \frac{\sqrt{2} + 3i}{\sqrt{2} - 3i} \times \frac{\sqrt{2} + 3i}{\sqrt{2} + 3i}$$

$$= \frac{2 + 3\sqrt{2}i + 3\sqrt{2}i + 9i^2}{2 - 9i^2}$$

$$= \frac{2 + 6\sqrt{2}i + 9(-1)}{2 - 9(-1)}$$

$$= \frac{2 + 6\sqrt{2}i - 9}{11}$$

$$= \frac{-7 + 6\sqrt{2}i}{11}$$

$$= -\frac{7}{11} + \frac{6\sqrt{2}}{11}i$$

f) $\frac{(3 + i)(-2 + 3i)}{2 - 4i}$

$$= \frac{-6 + 9i - 2i + 3i^2}{2 - 4i}$$

$$= \frac{-6 + 7i + 3(-1)}{2 - 4i}$$

$$= \frac{-6 + 7i - 3}{2 - 4i}$$

$$= \frac{-9 + 7i}{2 - 4i} \times \frac{2 + 4i}{2 + 4i}$$

$$= \frac{-18 - 36i + 14i + 28i^2}{4 - 16i^2}$$

$$= \frac{-18 - 22i + 28(-1)}{4 - 16(-1)}$$

$$= \frac{-18 - 22i - 28}{4 + 16}$$

$$= \frac{-46 - 22i}{20}$$

$$= -\frac{46}{20} - \frac{22}{20}i$$

$$= -\frac{23}{10} - \frac{11}{10}i$$

g) $\frac{2 - i}{3 + i} - \frac{3 - 2i}{4 - i}$

$$= \left[\frac{3 - i}{3 - i}\right] \times \left[\frac{2 - i}{3 + i}\right] - \left[\frac{3 - 2i}{4 - i}\right] \times \left[\frac{4 + i}{4 + i}\right]$$

$$= \left[\frac{6 - 3i - 2i + i^2}{9 - i^2}\right] - \left[\frac{12 + 3i - 8i - 2i^2}{16 - i^2}\right]$$

$$= \left[\frac{6 - 5i + (-1)}{9 - (-1)}\right] - \left[\frac{12 - 5i - 2(-1)}{16 - (-1)}\right]$$

$$= \left[\frac{5 - 5i}{10}\right] - \left[\frac{14 - 5i}{17}\right]$$

$$= \frac{5}{10} - \frac{5}{10}i - \left[\frac{14}{17} - \frac{5}{17}i\right]$$

$$= \frac{1}{2} - \frac{1}{2}i - \frac{14}{17} + \frac{5}{17}i$$

$$= \frac{1}{2} - \frac{14}{17} - \frac{1}{2}i + \frac{15}{17}i$$

$$= -\frac{11}{34} + \frac{13}{34}i$$

h) $\frac{3i}{(1 - i)^2}$

$$= \frac{3i}{(1 - i)(1 - i)}$$

$$= \frac{3i}{1 - 2i + i^2}$$

$$= \frac{3i}{1 - 2i + (-1)}$$

$$= \frac{3i}{-2i} \times \frac{2i}{2i}$$

$$= \frac{6i^2}{-4i^2}$$

$$= \frac{6(-1)}{-4(-1)}$$

$$= -\frac{6}{4}$$

$$= -\frac{3}{2} + 0i$$

i) $(2 - i)^{-2}$

$$= \frac{1}{(2 - i)(2 - i)}$$

$$= \frac{1}{4 - 4i + i^2}$$

$$= \frac{1}{4 - 4i + (-1)}$$

$$= \frac{1}{3 - 4i} \times \frac{3 + 4i}{3 + 4i}$$

$$= \frac{3 + 4i}{9 - 16i^2}$$

$$= \frac{3 + 4i}{9 - 16(-1)}$$

$$= \frac{3}{25} + \frac{4}{25}i$$

j) $\frac{(2 - 2i)(3 + i)}{-i + 2} - \frac{2 + 3i}{1 + i}$

$$= \frac{6 + 2i - 6i - 2i^2}{2 - i} - \frac{2 + 3i}{1 + i}$$

$$= \frac{6 - 4i - 2(-1)}{2 - i} - \frac{2 + 3i}{1 + i}$$

$$= \left(\frac{2 + i}{2 + i} \times \frac{8 - 4i}{2 - i}\right) - \left(\frac{2 + 3i}{1 + i} \times \frac{1 - i}{1 - i}\right)$$

$$= \left(\frac{16 - 8i + 8i - 4i^2}{4 - i^2}\right) - \left(\frac{2 - 2i + 3i - 3i^2}{1 - i^2}\right)$$

$$= \left(\frac{16 + 4}{5}\right) - \left(\frac{2 + \ i + 3}{2}\right)$$

$$= \frac{20}{5} - \frac{5}{2} - \frac{1}{2}i$$

$$= \frac{15}{10} - \frac{1}{2}i$$

$$= 1\frac{1}{2} - \frac{1}{2}i$$

k) $i^{-5} + \frac{1}{i^3} - i^{13}$

$= \frac{1}{i^5} + \frac{1}{i^3} - i^{13}$

$= \frac{1}{(i^2)^2 i} + \frac{1}{(i^2)i} - (i^2)^6 i$

$= \frac{1}{(-1)^2 i} + \frac{1}{(-1)i} - (-1)^6 i$

$= \frac{1}{i} - \frac{1}{i} - i$

$= -i$

$= 0 - i$

l) $-i^6 + \frac{1}{i^9} - i^{-7}$

$= -(i^2)^3 + \frac{1}{(i^2)^4 i} - \frac{1}{i^7}$

$= -(-1)^3 + \frac{1}{(-1)^4 i} - \frac{1}{(i^2)^3 i}$

$= -(-1) + \frac{1}{i} - \frac{1}{(-1)^3 i}$

$= 1 + \frac{1}{i} - \frac{1}{-i}$

$= 1 + \frac{1}{i} + \frac{1}{i}$

$= 1 + \frac{2}{i}$

$= 1 + \frac{2i}{i^2}$

$= 1 + \frac{2i}{-1}$

$= 1 - 2i$

Assessment activity 1.3

1. **a)** $(4{,}5\underline{|60^\circ})(3{,}3\underline{|41^\circ})$

$= 4{,}5 \times 3{,}3\underline{|60^\circ + 41^\circ}$

$= 14{,}85\underline{|101^\circ}$

b) $\dfrac{5{,}4\underline{|87{,}34^\circ}}{\sqrt{2}\underline{|-40{,}65^\circ}}$

$= \dfrac{5{,}4}{\sqrt{2}}\underline{|87{,}34^\circ - (-40{,}65)^\circ}$

$= 3{,}818\underline{|127{,}99^\circ}$

c) $(-6 \text{ cis } 45^\circ)(9\underline{|-135^\circ})$

$= (-6\underline{|45^\circ})(9\underline{|-135^\circ})$

$= -6 \times 9\underline{|45^\circ + (-135^\circ)}$

$= -54\underline{|-90^\circ}$ or $-54\underline{|270^\circ}$

d) $\dfrac{5\underline{|75{,}5^\circ} \times 7\underline{|23{,}4^\circ}}{4\underline{|80^\circ}}$

$= \dfrac{5 \times 7\underline{|75{,}5^\circ + 23{,}4^\circ}}{4\underline{|80^\circ}}$

$= \dfrac{35\underline{|98{,}9^\circ}}{4\underline{|80^\circ}}$

$= \dfrac{35}{4}\underline{|98{,}9^\circ - 80^\circ}$

$= 8{,}75\underline{|18{,}9^\circ}$

e) $\dfrac{(6\underline{|38^\circ})(4\underline{|20^\circ})}{(3\underline{|-80^\circ})(4\underline{|-30^\circ})}$

$= \dfrac{6 \times 4\underline{|38^\circ + 20^\circ}}{3 \times 4\underline{|-80^\circ + (-30^\circ)}}$

$= \dfrac{24\underline{|58^\circ}}{12\underline{|-110^\circ}}$

$= \dfrac{24}{12}\underline{|58^\circ - (-110^\circ)}$

$= 2\underline{|168^\circ}$

2. **a)** $\underline{|45^\circ}\,.2\underline{|30^\circ}\,.3\underline{|-58^\circ}$

$= 1 \times 2 \times 3\underline{|45^\circ + 30^\circ + (-58^\circ)}$

$= 6\underline{|17^\circ}$

$= 6(\cos 17^\circ + i \sin 17^\circ)$

$= 5{,}738 + 1{,}754i$

b) $\dfrac{5\underline{|-105^\circ}}{3\underline{|-50^\circ}}$

$= \dfrac{5}{3}\underline{|-105^\circ - (-50^\circ)}$

$= 1{,}667\underline{|-55^\circ}$

$= 1{,}667(\cos -55^\circ + i \sin -55^\circ)$

$= 0{,}956 - 1{,}366i$

c) $\dfrac{2\underline{|93^\circ} \times 4\underline{|-35^\circ} \times 3\underline{|136^\circ}}{3\underline{|200^\circ}}$

$= \dfrac{2 \times 4 \times 3\underline{|93^\circ + (-35^\circ) + 136^\circ}}{3\underline{|200^\circ}}$

$= \dfrac{24\underline{|194^\circ}}{3\underline{|200^\circ}}$

$= 8\underline{|194^\circ - 200^\circ}$

$= 8\underline{|-6^\circ}$

$= 8[\cos(-6^\circ) + i \sin(-6^\circ)]$

$= 7{,}956 - 0{,}836\, i$

d) $\dfrac{(\sqrt{2}\underline{|-45^\circ})(2\underline{|200^\circ})}{(2\underline{|60^\circ})(\sqrt{2}\,(-135^\circ)}$

$= \dfrac{\sqrt{2}.2\underline{|-45^\circ + 200^\circ}}{2.\sqrt{2}\underline{|60^\circ + (-135^\circ)}}$

$= \dfrac{2\sqrt{2}\underline{|155^\circ}}{2\sqrt{2}\underline{|-75^\circ}}$

$= \dfrac{2\sqrt{2}}{2\sqrt{2}}\underline{|155^\circ - (-75^\circ)}$

$= \underline{|230^\circ}$

$= 1(\cos 230^\circ + i \sin 230^\circ)$

$= -0{,}643 - 0{,}766i$

e) $\left(\dfrac{10\underline{|98^\circ}}{5\underline{|50^\circ}}\right) \times \left(\dfrac{8\underline{|158^\circ}}{4\underline{|-15^\circ}}\right)$

$= \left(\dfrac{10}{5}\underline{|98^\circ - 50^\circ}\right) \times \left(\dfrac{8}{4}\underline{|158^\circ - (-15^\circ)}\right)$

$= (2\underline{|48^\circ})(2\underline{|173^\circ})$

$= 2 \times 2\underline{|48^\circ + 173^\circ}$

$= 4\underline{|221^\circ}$

$= 4(\cos 221^\circ + i \sin 221^\circ)$

$= -3{,}02 - 2{,}624i$

Assessment activity 1.4

1. **a)** $(3\underline{|50^\circ})^4$

$= 3^4\underline{|50^\circ \times 4}$

$= 81\underline{|200^\circ}$

b) $(2{,}5 \text{ cis } 60{,}3^\circ)^5$

$= (2{,}5\underline{|60{,}3^\circ})^5$

$= 2{,}5^5\underline{|60{,}3^\circ \times 5}$

$= 97{,}656\underline{|301{,}5^\circ}$

c) $(\sqrt{4{,}2}\underline{|-80^\circ})^3$

$= (\sqrt{4{,}2})^3\underline{|-80^\circ \times 3}$

$= 8{,}607\underline{|-240^\circ}$

d) $\dfrac{(5\underline{|85^\circ})^3}{(2\underline{|20^\circ})^2}$

$= \dfrac{5^3\underline{|85^\circ \times 3}}{2^2\underline{|20^\circ \times 2}}$

$= \dfrac{125\underline{|255^\circ}}{4\underline{|40^\circ}}$

$= \dfrac{125}{4}\underline{|255^\circ - 40^\circ}$

$= 31{,}25\underline{|215^\circ}$

e) $\dfrac{4 \text{ cis } 45^\circ \times (3 \text{ cis } 60^\circ)^3}{(2 \text{ cis } - 50^\circ)^2}$

$= \dfrac{4 \text{ cis } 45^\circ \times 27 \text{ cis } 180^\circ}{4 \text{ cis } - 100^\circ}$

$= \dfrac{4 \times 27\underline{|45^\circ + 180^\circ}}{4\underline{|-100^\circ}}$

$= \dfrac{108\underline{|225^\circ}}{4\underline{|-100^\circ}}$

$= \dfrac{108}{4}\underline{|225^\circ - (-100^\circ)}$

$= 27\underline{|325^\circ}$

2. $z_1 = 21 \text{ cis } 120^\circ$
$z_2 = 3 \text{ cis } 80^\circ$
$z_3 = \underline{|-108^\circ}$

$\dfrac{(z_1)(z_2)^3}{(z_3)^2} = \dfrac{(21\underline{|120^\circ})(3\underline{|80^\circ})^3}{(\underline{|-108^\circ})^2}$

$= \dfrac{(21\underline{|120^\circ})(27\underline{|240^\circ})}{(\underline{|-216^\circ})}$

$= \dfrac{21 \times 27\underline{|120^\circ + 240^\circ}}{\underline{|-216^\circ}}$

$= \dfrac{567\underline{|360^\circ}}{\underline{|-216^\circ}}$

$= 567\underline{|360^\circ + 216^\circ}$

$= 567\underline{|576^\circ}$

$= 567\underline{|216^\circ}$

3. **a)** $\left(\dfrac{6\underline{|50^\circ}}{3\underline{|25^\circ}}\right)^4 \times \left(\dfrac{2\underline{|60^\circ}}{4\underline{|190^\circ}}\right)^{-3}$

$= \left(\dfrac{6\underline{|50^\circ}}{3\underline{|25^\circ}}\right)^4 \times \left(\dfrac{4\underline{|190^\circ}}{2\underline{|60^\circ}}\right)^3$

$= (2\underline{|25^\circ})^4 \times (2\underline{|130^\circ})^3$

$= (2^4\underline{|25^\circ \times 4})2^3(\underline{|130^\circ \times 3})$

$= (16\underline{|100^\circ})(8\underline{|390^\circ})$

$= 16 \times 8\underline{|100^\circ + 390^\circ}$

$= 128\underline{|490^\circ}$

$= 128\underline{|130^\circ}$

b) $[2(\cos 30^\circ + i \sin 30^\circ)]^3$

$= (2\underline{|30^\circ})^3$

$= 2^3\underline{|30^\circ \times 3}$

$= 8\underline{|90^\circ}$

c) $\left(\dfrac{3 \text{ cis } 78^\circ}{2 \text{ cis } 35^\circ}\right)^3 \times \left(\dfrac{3 \text{ cis } 25^\circ}{4 \text{ cis } 120^\circ}\right)^{-2}$

$= \left(\dfrac{3 \text{ cis } 78^\circ}{2 \text{ cis } 35^\circ}\right)^3 \times \left(\dfrac{4 \text{ cis } 120^\circ}{3 \text{ cis } 25^\circ}\right)^2$

$= \left(\dfrac{3^3\underline{|78^\circ \times 3}}{2^3\underline{|35^\circ \times 3}}\right) \times \left(\dfrac{4^2\underline{|120^\circ \times 2}}{3^2\underline{|25^\circ \times 2}}\right)$

$= \dfrac{27\underline{|234^\circ}}{8\underline{|105^\circ}} \times \dfrac{16\underline{|240^\circ}}{9\underline{|50^\circ}}$

$= \left(\dfrac{27}{8}\ \underline{|234^\circ - 105^\circ}\right) \times \left(\dfrac{16}{9}\ \underline{|240^\circ - 50^\circ}\right)$

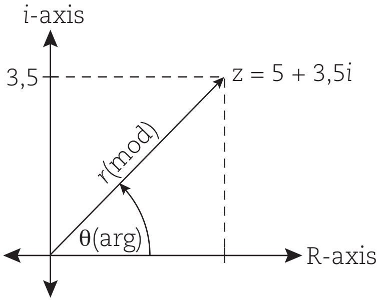
$= \left(\dfrac{27}{8}\ \underline{|129^\circ}\right)\left(\dfrac{16}{9}\ \underline{|190^\circ}\right)$

$= \left(\dfrac{27}{8}\right)\left(\dfrac{16}{9}\right)\underline{|129^\circ + 190^\circ}$

$= 6\,\underline{|319^\circ}$

Assessment activity 1.5

1. **a)** $(5 + 3{,}5i)^5$



$$r(\text{mod}) = \sqrt{(5)^2 + (3{,}5)^2}$$
$$= 6{,}103$$

$$\theta(\text{arg}) = \tan^{-1}\left(\frac{3{,}5}{2}\right)$$
$$= 34{,}992^\circ$$

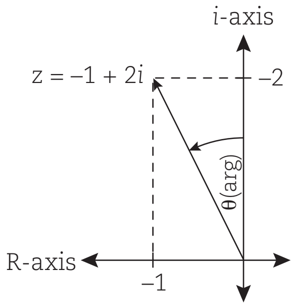
$\therefore\ (5 + 3{,}5i)^5$

$= (6{,}103 \lfloor\underline{34{,}992^\circ})^5$

$= 6{,}103^5 \lfloor\underline{34{,}992^\circ \times 5}$

$= 8\ 466{,}752 \lfloor\underline{174{,}960^\circ}$

b) $(-1 + 2i)^4$



$$r(\text{mod}) = \sqrt{(-1)^2 + (2)^2}$$
$$= \sqrt{5}$$

$$\theta(\text{arg}) = 180^\circ - \tan^{-1}\left(\frac{2}{1}\right)$$
$$= 116{,}565^\circ$$

$\therefore\ (-1 + 2i)^4$

$= (\sqrt{5} \lfloor\underline{116{,}565^\circ})^4$

$= \sqrt{5}^4 \lfloor\underline{116{,}565^\circ \times 4}$

$= 25 \lfloor\underline{466{,}26^\circ}$

$= 25 \lfloor\underline{106{,}26^\circ}$

- 446,26° – 360° = 106,26° (more than 1 revolution ≈ 1,6 revolutions)

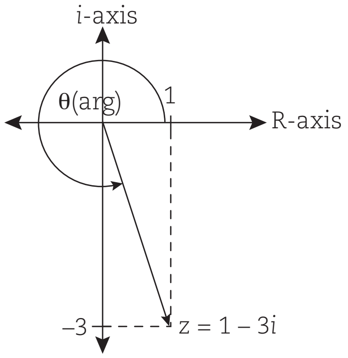
c) $z = [2(\cos -85^\circ + i \sin -85^\circ)]^3$

$= 2^3 \lfloor\underline{-85^\circ \times 3}$

$= 8 \lfloor\underline{-225^\circ}$

$= 16\ (\cos -225^\circ + i \sin -225)$ or $16 \lfloor\underline{-225^\circ}$ or $16 \lfloor\underline{135^\circ}$

2. a) $(1 - 3i)^4$



$$r(\text{mod}) = \sqrt{(1)^2 + (-3)^2}$$
$$= \sqrt{10}$$

$$\theta(\text{arg}) = 360^\circ - \tan^{-1}\left(\frac{3}{1}\right)$$
$$= 288{,}435^\circ$$

$\therefore (1 - 3i)^4$

$= (\sqrt{10}\,\underline{|288{,}435^\circ}\,)^4$

$= \sqrt{10}^4\,\underline{|288{,}435^\circ \times 4}$

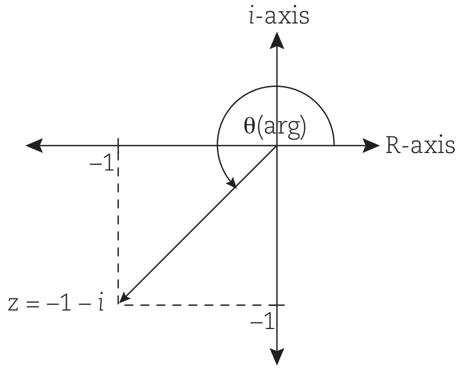
$= 100\,\underline{|1\,153{,}74^\circ}$

$= 100\,\underline{|73{,}74^\circ}$

$= 100\,(\cos 73{,}74^\circ + i \sin 73{,}74^\circ)$
$= 28 + 96i$

- $1153{,}74^\circ \div 360^\circ = 3{,}2$ revolutions
- $\therefore 1153{,}74^\circ - (360^\circ \times 3) = 73{,}74^\circ$
- Negative argument is $73{,}74^\circ - 360^\circ$
  $= -286{,}26^\circ$
  $\therefore 100\,\underline{|73{,}74^\circ} = 100\,\underline{|-286{,}26^\circ}$

b) $(-1 - i)^5$



$$r(\text{mod}) = \sqrt{(-1)^2 + (-1)^2}$$
$$= \sqrt{2}$$

$$\theta(\text{arg}) = 180^\circ + \tan^{-1} 1$$
$$= 225^\circ$$

$(-1 - i)^5$

$= (\sqrt{2}\,\underline{|225^\circ}\,)^5$

$= \sqrt{2}^5\,\underline{|225^\circ \times 5}$

$= 5{,}657\,\underline{|1\,125^\circ}$

$= 5{,}657\,\underline{|45^\circ}$

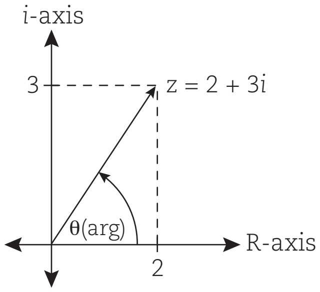
$= 5{,}657(\cos 45^\circ + i \sin 45^\circ)$

$= 4 + 4i$

Assessment activity 1.6

1. a)

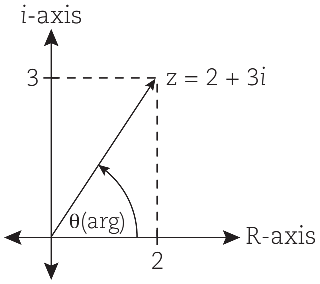
$$(2 + 3i)^4.2\,\underline{|60^\circ}$$
$$= (\sqrt{13}\,\underline{|56{,}31^\circ})^4.2\,\underline{|60^\circ}$$
$$= (\sqrt{13}^4\,\underline{|56{,}31^\circ \times 4}).(2\,\underline{|60^\circ})$$
$$= \sqrt{13}^{\,4}.2\,\underline{|225{,}24^\circ + 60^\circ}$$
$$= 338\,\underline{|285{,}24^\circ}$$



$$r(\text{mod}) = \sqrt{(2)^2 + (3)^2}$$
$$= \sqrt{13}$$
$$\theta(\text{arg}) = \tan^{-1}\frac{3}{2}$$
$$= 56{,}31^\circ$$

b)

$$\frac{(4 - i)^2}{(3{,}2\,\underline{|150^\circ})^3}$$
$$= \frac{(\sqrt{17}\,\underline{|345{,}964^\circ})^2}{(3{,}2\,\underline{|150^\circ})^3}$$
$$= \frac{\sqrt{17}^2\,\underline{|345{,}964^\circ \times 2}}{3{,}2^3\,\underline{|150^\circ \times 3}}$$
$$= \frac{17}{32{,}768}\,\underline{|691{,}928^\circ - 450^\circ}$$
$$= 0{,}519\,\underline{|241{,}928^\circ}$$



$$r(\text{mod}) = \sqrt{(4)^2 + (-1)^2}$$
$$= \sqrt{17}$$
$$\theta(\text{arg}) = 360^\circ - \tan^{-1}\left(\frac{1}{4}\right)$$
$$= 345{,}964^\circ$$

2. a)

$$\frac{-1 + 5i}{3 - 2i} \times \frac{3 + 2i}{3 + 2i}$$
$$= \frac{-3 - 2i + 15i + 10i^2}{9 - 4i}$$
$$= \frac{-3 + 13i + 10(-1)}{9 - 4(-1)}$$
$$= \frac{-13}{13} + \frac{13i}{13}$$
$$= -1 + i$$

b)

$$\frac{(-3 - 2i)^2}{2\,\underline{|145^\circ}} = \frac{(-3 - 2i)^2}{2(\cos 145 + i \sin 145^\circ)}$$
$$= \frac{(-3 - 2i)^2}{-1{,}638 + 1{,}147i}$$
$$= \frac{9 + 12i + 4i^2}{-1{,}638 + 1{,}147i}$$
$$= \frac{5 + 12i}{-1{,}638 + 1{,}147i}$$
$$= \frac{-8{,}19 - 5{,}735i - 19{,}656i - 13{,}764i^2}{2{,}683 - 1{,}316i^2}$$
$$= \frac{5{,}574 - 25{,}391i}{3{,}999}$$
$$= 1{,}394 - 6{,}349i$$

c)

$$\frac{(8 \text{ cis } 180^\circ)(2 \text{ cis } 120^\circ)}{i - 3}$$
$$= \frac{8(\cos 180^\circ + i \sin 180^\circ).2(\cos 120^\circ + i \sin 120^\circ)}{-3 + i}$$
$$= \frac{(-8 + 0i)(-1 + \sqrt{3}i)}{-3 + i}$$
$$= \frac{8 - 8\sqrt{3}\,i}{-3 + i} \cdot \frac{(-3 - i)}{(-3 - i)}$$
$$= \frac{-24 - 8i + 24\sqrt{3}\,i + 8\sqrt{3}\,i^2}{9 - i^2}$$
$$= \frac{-37{,}856 + 33{,}569i}{10}$$
$$= -3{,}786 + 3{,}357i$$

Assessment activity 1.7

1. **a)** $-3 + 2i = x + 4yi$

$\therefore -3 = x; \qquad 2 = 4y$

$\therefore x = -3 \quad \text{and } y = \frac{1}{2}$

b) $3x - 2i = -yi$

$\therefore 3x = 0; \qquad -2 = -y$

$\therefore x = 0 \quad \text{and } y = 2$

c) $-2x - 8yi = -10 + 16i$

$\therefore -2x = -10; \quad -8y = 16$

$\therefore x = 5 \quad \text{and } y = -2$

d) $3x + 2yi - 5 = 4 + 6i$

$(3x - 5) + 2yi = 4 + 6i$

$\therefore 3x - 5 = 4; \qquad 2y = 6$

$3x = 9; \qquad y = 3$

$x = 3$

e) $2x + 3 + i(y + 5) = x - y + ix + iy$

$(2x + 3) + i(y + 5) = (x - y) - i(x + y)$

$\therefore 2x + 3 = x - y; \; y + 5 = x + y$

$x + y = -3; \qquad x = 5$

$\therefore x + y = -3$ ....$(x = 5)$

$5 + y = -3$

$y = -8 \quad \text{and } x = 5$

f) $4 + 5i = x + yi - (1 + i)$

$= x + yi - 1 - i$

$4 + 5i = (x - 1) + i(y - 1)$

$\therefore 4 = x - 1; \qquad 5 = y - 1$

$\therefore x = 5 \qquad \text{and } y = 6$

g) $(3 - 2i)^2 = x - yi$

$(3 - 2i)(3 - 2i) = x - yi$

$9 - 12i + 4i = x - yi$

$9 - 12i + 4(-1) = x - yi$

$5 - 12i = x - yi$

$\therefore 5 = x; \quad -12 = -y$

$\therefore x = 5 \;\text{ and } y = 12$

h) $(i + 1)^2 + (3 + i)i = x + y + 4yi$

$(i + 1)(i + 1) + 3i + i^2 = (x + y) + 4yi$

$i^2 + 2i + 1 + 3i + i^2 = (x + y) + 4yi$

$(-1) + 5i + 1 + (-1) = (x + y) + 4yi$

$5i - 1 = (x + y) + 4yi$

$\therefore 5 = 4y; \qquad -1 = x + y$

$y = \frac{5}{4}; \qquad x = -1 - y$

$\therefore x = -1 - \frac{5}{4}$

$\therefore x = -\frac{9}{4}$

$\therefore y = \frac{5}{4} \text{ or } 1\frac{1}{4} \text{ and } x = -\frac{9}{4} \text{ or } -2\frac{1}{4}$

2. **a)** $i(x - iy) = i(y - i9) - 3x - i$

$xi - yi^2 = yi - 9i^2 - 3x - i$

$xi - y(-1) = yi - 9(-1) - 3x - i$

$xi + y = yi + 9 - 3x - i$

$= (9 - 3x) + i(y - 1)$

$\therefore x = y - 1; \; y = 9 - 3x$ .... ②

$y = x + 1$ .... ①

① = ②: $x + 1 = 9 - 3x$

$4x = 8$

$x = 2$

$\therefore y = x + 1$ .... ①

$y = 2 + 1$

$y = 3$

$\therefore (2; 3)$

b)

$$(1 + i)(x - iy) = (2 + 3i)^2$$
$$x - iy + ix - i^2y = (2 + 3i)(2 + 3i)$$
$$x - iy + ix - (-1)y = 4 + 12i + 9i^2$$
$$x - iy + ix + y = 4 + 12i + 9(-1)$$
$$(x + y) + i(-y + x) = -5 + 12i$$
$$\therefore x + y = -5 \text{ ….. ①; } -y + x = 12 \text{ ….. ②}$$
$$\therefore ① + ②\text{: } 2x = 7$$
$$x = \frac{7}{2} \text{ or } 3\frac{1}{2}$$
$$\therefore x + y = -5$$
$$\frac{7}{2} + y = -5$$
$$y = -5 - \frac{7}{2}$$
$$\therefore y = -\frac{17}{2} \text{ or } -8\frac{1}{2}$$
$$\therefore \left(3\frac{1}{2}\,;\,-8\frac{1}{2}\right)$$

c)

$$(5 - 2i)(x + yi) = \frac{1 + i}{1 - i}$$
$$5x + 5yi - 2xi - 2yi^2 = \frac{1 + i}{1 - i} \times \frac{1 + i}{1 + i}$$
$$5x + 5yi - 2xi - 2y(-1) = \frac{1 + 2i + i^2}{1 - i^2}$$
$$(5x + 2y) + i(5y - 2x) = \frac{1 + 2i + (-1)}{1 - (-1)}$$
$$= \frac{2i}{2}$$
$$\therefore (5x + 2y) + i(5y - 2x) = 0 + i$$
$$\therefore 5x + 2y = 0;\ 5y - 2x = 1$$
$$5x = -2y$$
$$x = -\frac{2}{5}y$$
$$\therefore 5y - 2\left(-\frac{2}{5}\right)y = 1$$
$$5y + \frac{4}{5}y = 1$$
$$\frac{29}{5}y = 1$$
$$y = \frac{5}{29}$$
$$\text{or } 0{,}172$$
$$\therefore x = -\frac{2}{5}\left(\frac{5}{25}\right)$$
$$x = \frac{2}{29}$$
$$= 0{,}69$$

3. a)

$$(3 - 4i)^2 = \frac{a + bi}{i^2}$$
$$(3 - 4i)(3 - 4i) = \frac{a + bi}{(-1)}$$
$$9 - 24i + 16i^2 = -a - bi$$
$$9 - 24i + 16(-1) = -a - bi$$
$$-7 - 24i = -a - bi$$
$$\therefore -7 = -a; \qquad -24 = -b$$
$$\therefore a = 7 \qquad \text{and } b = 24$$

b)

$$(a + bi) = \frac{(3 + 5i)(2 - 5i)}{1 - 3i}$$
$$a + bi = \frac{6 - 15i + 10i - 25i^2}{1 - 3i}$$
$$= \frac{6 - 5i - 25(-1)}{1 - 3i}$$
$$= \frac{6 - 5i + 25}{1 - 3i}$$
$$= \frac{31 - 5i}{1 - 3i} \times \frac{1 + 3i}{1 + 3i}$$
$$= \frac{31 + 93i - 5i - 15i^2}{1 - 9i^2}$$
$$= \frac{31 + 88i - 15(-1)}{1 - 9(-1)}$$
$$a + bi = \frac{46}{10} + \frac{88}{10}i$$
$$\therefore a = \frac{46}{10} \qquad \text{and } b = \frac{88}{10}$$
$$a = 4{,}6 \qquad \text{and } b = 8{,}8$$

c)

$$a - bi = \frac{5 - i^5}{1 + i}$$

$$= \frac{5 - (i^2)^2 i}{1 + i}$$

$$= \frac{5 - (-1)^2\, i}{1 + i}$$

$$= \frac{5 - i}{1 + i} \times \frac{1 - i}{1 - i}$$

$$= \frac{5 - 6i + i^2}{1 - i^2}$$

$$= \frac{5 - 6i + (-1)}{1 - (-1)}$$

$$= \frac{4}{2} - \frac{6}{2}\, i$$

$\therefore\ a - bi = 2 - 3i$

$\therefore\ a = 2; \qquad -b = -3$

$b = 3$

d)

$$\frac{3 - 2i}{1 + i} - \frac{1 - 3i}{1 + 3i} = a + bi$$

$$\left[\frac{1 + i}{1 + i} \times \frac{3 - 2i}{1 - i}\right) - \left[\frac{1 - 3i}{1 + 3i} \times \frac{1 - 3i}{1 - 3i}\right) = a + bi$$

$$\left[\frac{3 + i - 2i^2}{1 - i^2}\right) - \left[\frac{1 - 6i + 9i^2}{1 - 9i^2}\right) = a + bi$$

$$\left[\frac{3 + i - 2(-1)}{1 - (-1)}\right) - \left[\frac{1 - 6i + 9(-1)}{1 - 9(-1)}\right) = a + bi$$

$$\left[\frac{5}{2} + \frac{i}{2}\right) - \left[-\frac{8}{10} - \frac{6i}{10}\right) = a + bi$$

$$\frac{5}{2} + \frac{1}{2}\, i + \frac{8}{10} + \frac{6}{10}\, i = a + bi$$

$$\frac{5}{2} + \frac{1}{2}\, i + \frac{4}{5} + \frac{3}{5}\, i = a + bi$$

$$\therefore\ \frac{33}{10} + \frac{11}{10}\, i = a + bi$$

$\therefore\ \frac{33}{10} = a; \qquad \frac{11}{10} = b$

$\therefore\ a = 3{,}3$ and $b = 1{,}1$

Assessment activity 1.8

1. **a)**

$$x^2 + 8 = 0$$

$$x^2 = -8$$

$$x = \pm\sqrt{-8}$$

$$= \pm\sqrt{8}\, i$$

$$= \pm\, 2{,}828i$$

b)

$$2x^2 - x + 3 = 0$$

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(2)(3)}}{2(2)}$$

$$= \frac{1 \pm \sqrt{-23}}{4}$$

$$= \frac{1}{4} \pm \frac{\sqrt{23}i}{4}$$

$$= 0{,}25 \pm 1{,}199i$$

c)

$$x^2 - 4x = -7$$

$$x^2 - 4x + 7 = 0$$

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(7)}}{2(1)}$$

$$= \frac{4 \pm \sqrt{-12}}{2}$$

$$= \frac{4}{2} \pm \frac{\sqrt{12}i}{2}$$

$$= 2 \pm 1{,}732i$$

d)

$$(x - 2)(x^2 + 2x + 4) = 0$$

$\therefore\ x - 2 = 0;\ x^2 + 2x + 4 = 0$

$\therefore\ x = 2; \qquad x = \frac{-2 \pm \sqrt{(2)^2 - 4(1)(4)}}{2(1)}$

$$= \frac{-2 \pm \sqrt{-12}}{2}$$

$$= \frac{-2}{2} \pm \frac{\sqrt{12}i}{2}$$

$$= -1 \pm 1{,}732i$$

e)

$$x^5 = 16x$$

$$x^5 - 16x = 0$$

$$x(x^4 - 16) = 0$$

$$x(x^2 - 4)(x^2 + 4) = 0$$

$$x(x - 2)(x + 2)(x^2 + 4) = 0$$

$\therefore\ x = 0;\ x = 2;\ x = -2;\ x^2 + 4 = 0$

$$x^2 = -4$$

$$x = \pm\sqrt{-4}$$

$$= \pm\sqrt{4}\, i$$

$$= \pm\, 2i$$

f)

$$
\begin{aligned}
3x(x-6)+39 &= 0\\
3x^2-18x+39 &= 0\\
x^2-6x+13 &= 0\\
x &= \frac{-(-6)\pm\sqrt{(-6)^2-4(1)(13)}}{2(1)}\\
&= \frac{6\pm\sqrt{-16}}{2}\\
&= \frac{6}{2}\pm\frac{\sqrt{16}i}{2}\\
&= 3+2i
\end{aligned}
$$

g)

$$
\begin{aligned}
x+\frac{6}{x} &= -3\\
x^2+6 &= -3x\\
x^2+3x+6 &= 0\\
x &= \frac{-3\pm\sqrt{(3)^2-4(1)(6)}}{2(1)}\\
&= \frac{-3\pm\sqrt{-15}}{2}\\
&= -\frac{3}{2}\pm\frac{\sqrt{15}i}{2}\\
&= -1{,}5\pm 1{,}936i
\end{aligned}
$$

h)

$$
\begin{aligned}
256x^4-1 &= 0\\
(16x^2-1)(16x^2+1) &= 0
\end{aligned}
$$

$\therefore\ 16x^2 = 1$ or $16x^2+1=0$

$$
\begin{aligned}
x^2 &= \frac{1}{16}\\
x &= \pm\sqrt{\frac{1}{16}}\\
\therefore\ x &= \pm\frac{1}{4}
\end{aligned}
\qquad\qquad
\begin{aligned}
16x^2 &= -1\\
x^2 &= -\frac{1}{16}\\
x &= \pm\sqrt{\frac{1}{-16}}\\
&= \pm\sqrt{\frac{1}{16}}\,i\\
\therefore\ x &= \pm\frac{1}{4}i
\end{aligned}
$$

i)

$$
\begin{aligned}
x^3+6x^2 &= -13x\\
x^3+6x^2+13x &= 0\\
x(x^2+6x+13) &= 0
\end{aligned}
$$

$\therefore\ x = 0;\quad x^2+6x+13=0$

$$
\begin{aligned}
x &= \frac{-6\pm\sqrt{(6)^2-4(1)(13)}}{2(1)}\\
&= \frac{-6\pm\sqrt{-16}}{2}\\
&= -\frac{6}{2}\pm\frac{\sqrt{16}i}{2}\\
&= -3\pm 2i
\end{aligned}
$$

2.

$$
\begin{aligned}
34 &= -x^2+3x\\
0 &= -x^2+3x-34\\
x &= \frac{-(3)\pm\sqrt{(3)^2-4(-1)(-34)}}{2(-1)}\\
&= \frac{-3\pm\sqrt{-127}}{-2}\\
&= \frac{-3}{-2}\pm\frac{\sqrt{127}i}{-2}\\
&= 1{,}5\pm 5{,}635i
\end{aligned}
$$

Solutions for summative assessment: Chapter 1

1.1 **1.1.1** $(3i)^5 \times i^7$

$= 243i^5 \times i^7$

$= 243i^{12}$

$= 243(i^2)^6$

$= 243(-1)^6$

$= 243$

(2)

1.1.2 $\sqrt{-1}\,(-\sqrt{-81} + \sqrt{16} - \sqrt{-9}\,)$

$= i(-\sqrt{81}\,.\sqrt{-1} + 4 - \sqrt{9}\,.\sqrt{-1}\,)$

$= i(-9i + 4 - 3i)$

$= i(-12i + 4)$

$= -12i^2 + 4i$

$= -12(-1) + 4i$

$= 12 + 4i$ (3)

1.2 **1.2.1** $-3 - 2i - (2 + 4i)$

$= -3 - 2i - 2 - 4i$

$= -5 - 6i$ (2)

1.2.2 $\dfrac{(-3 - 2i)(-3 + 2i)}{(2 - 4i)}$

$= \dfrac{9 - 4i^2}{2 - 4i}$

$= \dfrac{9 - 4(-1)}{2 - 4i}$

$= \dfrac{13}{2 - 4i} \times \dfrac{2 + 4i}{2 + 4i}$

$= \dfrac{26 + 52i}{4 - 16i^2}$

$= \dfrac{26 + 52i}{4 - 16(-1)}$

$= \dfrac{26 + 52i}{20}$

$= 1{,}3 - 2{,}6i$ (5)

1.3 $(-4 + 5i)(3 + 2i)^2$

$= (-4 + 5i)(3 + 2i)(3 + 2i)$

$= (-4 + 5i)(9 + 12i + 4i^2)$

$= (-4 + 5i)[9 + 12i + 4(-1)]$

$= (-4 + 5i)(5 + 12i)$

$= -20 - 48i + 25i + 60i^2$

$= -20 - 23i + 60(-1)$

$= -80 - 23i$ (4)

[16]

2.1 $\dfrac{3 \text{ cis } 75,4^\circ \times (2 \text{ cis } 24,5^\circ)^2}{4 \text{ cis } 60^\circ}$

$= \dfrac{3\,\underline{|75{,}4^\circ} \times 2^2\,\underline{\underline{|24{,}5^\circ \times 2}}}{4\,\underline{|60^\circ}}$

$= \dfrac{3 \times 4}{4}\,\underline{|75{,}4^\circ + 49 - 60^\circ}$

$= 3\,\underline{|64{,}4^\circ}$ (3)

2.2 **2.2.1**

$$\frac{(3\text{ cis }95^\circ)^5}{(\sqrt{5}\text{ cis }125^\circ)^2}$$

$$= \frac{(3\,\underline{|95^\circ})^5}{(\sqrt{5}\,\underline{|125^\circ})^2}$$

$$= \frac{3^5\,\underline{|95^\circ \times 5}}{\sqrt{5}^2\,\underline{|125^\circ \times 2}}$$

$$= \frac{3^5}{\sqrt{5}^2}\,\underline{|475^\circ - 250^\circ}$$

$$= 48{,}6\,\underline{|225^\circ}$$

(3)

2.2.2

$$\left(\frac{4\text{ cis }75^\circ}{2\text{ cis }-20^\circ}\right)^3 \times \left(\frac{2\text{ cis }20^\circ}{5\text{ cis }110^\circ}\right)^{-2}$$

$$= \left(\frac{4\,\underline{|75^\circ}}{2\text{ cis }\underline{|-20^\circ}}\right)^3 \times \left(\frac{5\,\underline{|110^\circ}}{2\,\underline{|20^\circ}}\right)^2$$

$$= \left(\frac{4}{2}\,\underline{|95^\circ}\right)^3 \times \left(\frac{5}{2}\,\underline{|90^\circ}\right)^2$$

$$= 2^3\,\underline{|95^\circ \times 3} \times \left(\frac{5}{2}\right)^2\underline{|90^\circ \times 2}$$

$$= 8\,\underline{|285^\circ} \times 6{,}25\,\underline{|180^\circ}$$

$$= 50\,\underline{|465^\circ} \text{ or}$$

$$= 50\,\underline{|105^\circ}$$

(5)

2.2.3

$$[\sqrt{3}\,(\cos 25^\circ + i\sin 25^\circ)]^6$$

$$= \sqrt{3}^6\,\underline{|25^\circ \times 6}$$

$$= 27\,\underline{|150^\circ}$$

(2)

2.3 **2.3.1**

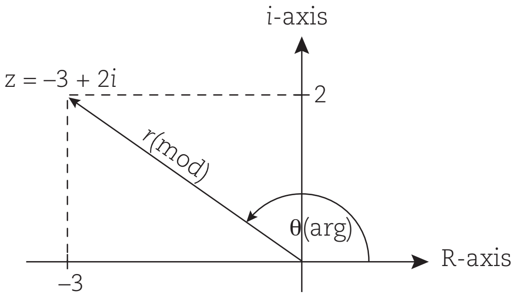
$$r(\text{mod}) = \sqrt{(-3)^2 + (2)^2}$$

$$= \sqrt{13}$$

$$\theta(\text{arg}) = 180^\circ - \tan^{-1}\left(\frac{2}{3}\right)$$

$$= 146{,}31^\circ$$

$$\therefore\ r\,\underline{|\theta} = \sqrt{13}\,\underline{|146{,}31^\circ} \text{ or } 3{,}606\,\underline{|146{,}31^\circ}$$



(2)

2.3.2

$$(\sqrt{13}\,\underline{|146{,}31^\circ})^4$$

$$= \sqrt{13}^4\,\underline{|146{,}31^\circ \times 4}$$

$$= 169\,\underline{|585{,}24^\circ} \text{ or}$$

$$= 169\,\underline{|225{,}24^\circ}$$

$$= 169\,(\cos 225{,}24^\circ + i\sin 225{,}24^\circ)$$

$$= -118{,}999 - 120\,i$$

(4)

[19]

3.1 **3.1.1**

$$x + y + ix - iy = 5 - 3i$$

$$(x + y) + (x - y)i = 5 - 3i$$

$$\therefore\ x + y = 5 \text{ .... ①}; \qquad x - y = -3 \text{ ....②}$$

$$\text{①} + \text{②}: 2x = 2$$

$$x = 1$$

$$\therefore\ x + y = 5 \text{ .... ①}$$

$$1 + y = 5$$

$$y = 4$$

$$(1; 4)$$

(4)

3.1.2 $x^2 + i4x - i2y = 3x - 2 + 4i$

$x^2 + i(4x - 2y) = (3x - 2) + 4i$

$\therefore x^2 = 3x - 2$ .... ①; $(4x - 2y) = 4$

$x^2 - 3x + 2 = 0$ $4x - 2y = 4$ .... ②

$(x - 2)(x - 1) = 0$

$\therefore x = 2; \quad x = 1$

Substitute $x = 2$ into ② and $x = 1$ into ②

$\therefore 4(2) - 2y = 4$ $\therefore 4(1) - 2y = 4$

$-2y = -4$ $-2y = 0$

$y = +2$ $y = 0$

$\therefore$ (2; 2) and (1; 0) are the solution (4)

3.1.3

$$x + yi = \frac{2 - 3i}{1 - i} - \frac{1 - 2i}{1 + 2i}$$

$$x + yi = \left[\frac{1 + i}{1 + i} \times \frac{2 - 3i}{1 - i}\right] - \left[\frac{1 - 2i}{1 + 2i} \times \frac{1 - 2i}{1 - 2i}\right]$$

$$= \left[\frac{2 - i - 3i^2}{1 - i^2}\right] - \left[\frac{1 - 4i + 4i^2}{1 - 4i^2}\right]$$

$$= \left[\frac{2 - i - 3(-1)}{1 - (-1)}\right] - \left[\frac{1 - 4i + 4(-1)}{1 - 4(-1)}\right]$$

$$= \left[\frac{5}{2} - \frac{i}{2}\right] - \left[-\frac{3}{5} - \frac{4i}{5}\right]$$

$$= \frac{5}{2} - \frac{1}{2}i + \frac{3}{5} + \frac{4}{5}i$$

$\therefore x + yi = 3{,}1 + 0{,}3i$

$\therefore x = 3{,}1; y = 0{,}3$

(3,1; 0,3) (4)

3.2 **3.2.1** $5x^2 - 6x = -5$

$5x^2 - 6x + 5 = 0$

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(5)(5)}}{2(5)}$$

$$= \frac{6 \pm \sqrt{36 - 100}}{10}$$

$$= \frac{6 \pm \sqrt{-64}}{10}$$

$$= \frac{6 \pm \sqrt{64}i}{10}$$

$$= \frac{6}{10} \pm \frac{8i}{10}$$

$= \frac{3}{5} \pm \frac{4}{5}i$ or $0{,}6 \pm 0{,}8i$ (4)

3.2.2 $y = \frac{1}{4}x^2 + 1$

Roots: $y = 0$

$\therefore 0 = \frac{1}{4}x^2 + 1$

$\frac{1}{4}x^2 = -1$

$x^2 = -4$

$\therefore x = \pm \sqrt{-4}$

$= \pm\sqrt{4} . \sqrt{-1}$

$\therefore x = \pm 2i$ (4)

[20]

4. $(-1 + 2i)^2(2\underline{|100^\circ})^2$

$= (-1 + 2i)^2\ (2^2\underline{|100^\circ \times 2})$

$= (-1 + 2i)^2[4(\cos 200^\circ + i \sin 200^\circ)]$

$= (-1 + 2i)(-1 + 2i)(-3{,}759 - 1{,}368i)$

$= (1 - 4i + 4i^2)(-3{,}759 - 1{,}368i)$

$= (1 - 4i - 4)(-3{,}759 - 1{,}368i)$

$= (-3 - 4i)(-3{,}759 - 1{,}368i)$

$= 11{,}277 + 4{,}104i + 15{,}036i + 5{,}472i^2$

$= 11{,}277 + 19{,}14i - 5{,}472$

$= 5{,}805 + 19{,}14i$ (5)

Total [60]

Assessment activity 2.1

1. a)

$$\begin{aligned} f(x) &= 2x^3 + 3x^2 - x + 5 \\ f(2) &= 2(2)^3 + 3(2)^2 - (2) + 5 \\ &= 2(8) + 3(4) - 2 + 5 \\ &= 31 \end{aligned}$$

b)

$$\begin{aligned} f(-3) &= 2(-3)^3 + 3(-3)^2 - (-3) + 5 \\ &= 2(-27) + 3(9) + 3 + 5 \\ &= -19 \end{aligned}$$

c)

$$\begin{aligned} f(a-1) &= 2(a-1)^3 + 3(a-1)^2 - (a-1) + 5 \\ &= 2(a-1)(a-1)(a-1) + 3(a-1)(a-1) - a + 1 + 5 \\ &= 2(a^2 - 2a + 1)(a-1) + 3(a^2 - 2a + 1) - a + 1 + 5 \\ &= 2(a^3 - a^2 - 2a^2 + 2a + a - 1) + 3(a^2 - 2a + 1) - a + 1 + 5 \\ &= 2(a^3 - 3a^2 + 3a - 1) + 3a^2 - 6a + 3 - a + 6 \\ &= 2a^3 - 6a^2 + 6a - 2 + 3a^2 - 7a + 9 \\ &= 2a^3 - 3a^2 - a + 7 \end{aligned}$$

2. a)

$$\frac{3x^2 - 5x + 6}{x - 2}$$

$$\begin{aligned} \therefore x - 2 &= 0 \\ x &= 2 \end{aligned}$$

$$\begin{aligned} f(x) &= 3x^2 - 5x + 6 \\ f(2) &= 3(2)^2 - 5(2) + 6 \\ &= 12 - 10 + 6 \\ &= 8 \end{aligned}$$

b)

$$\frac{x^3 - 2x^2 - 4x + 3}{x + 3}$$

$$\begin{aligned} \therefore x + 3 &= 0 \\ x &= -3 \end{aligned}$$

$$\begin{aligned} f(x) &= x^3 - 2x^2 - 4x + 3 \\ f(-3) &= (-3)^3 - 2(-3)^2 - 4(-3) + 3 \\ &= -27 - 2(9) + 12 + 3 \\ &= -30 \end{aligned}$$

c)

$$(-x^3 - x^2 - 10x - 6) \div (3x + 2)$$

$$\begin{aligned} \therefore 3x + 2 &= 0 \\ x &= -\tfrac{2}{3} \end{aligned}$$

$$\begin{aligned} f(x) &= -\left(-\tfrac{2}{3}\right)^3 - \left(-\tfrac{2}{3}\right)^2 - 10\left(-\tfrac{2}{3}\right) - 6 \\ &= \frac{8}{27} - \frac{4}{9} + \frac{20}{3} - 6 \\ &= \frac{8 - 12 + 180 - 162}{27} \\ &= \frac{14}{27} \text{ or } 0{,}519 \end{aligned}$$

d)

$$(2x^3 - x^2 - 8) \div (2x - 1)$$

$$\begin{aligned} \therefore 2x - 1 &= 0 \\ 2x &= 1 \\ x &= \frac{1}{2} \end{aligned}$$

$$\begin{aligned} \therefore f(x) &= 2x^3 - x^2 - 8 \\ f\left(\tfrac{1}{2}\right) &= 2\left(\tfrac{1}{2}\right)^3 - \left(\tfrac{1}{2}\right)^2 - 8 \\ &= 2\left(\tfrac{1}{8}\right) - \tfrac{1}{4} - 8 \\ &= \tfrac{1}{4} - \tfrac{1}{4} - 8 \\ &= -8 \end{aligned}$$

e)

$$\frac{4x^3 + 6x^2b - 5xa + bx}{x - 2a - b}$$

$$\begin{aligned} \therefore x - 2a - b &= 0 \\ x &= 2a + b \end{aligned}$$

$$\begin{aligned} \therefore f(x) &= 4x^3 + 6x^2b - 5xa + bx \\ f(2a+b) &= 4(2a+b)^3 + 6(2a+b)^2b - 5(2a+b)a + b(2a+b) \\ &= 4(2a+b)(2a+b)(2a+b) + 6b(2a+b)(2a+b) - 10a^2 - 5ab + 2ab + b^2 \\ &= 4(4a^2 + 4ab + b^2)(2a+b) + 6b(4a^2 + 4ab + b^2) - 10a^2 - 3ab + b^2 \\ &= (8a + 4b)(4a^2 + 4ab + b^2) + 24a^2b + 24ab^2 + 6b^3 - 10a^2 - 3ab + b^2 \\ &= 32a^3 + 32a^2b + 8ab^2 + 16a^2b + 16ab^2 + 4b^3 + 24a^2b + 24ab^2 + 6b^3 - 10a^2 - 3ab + b^2 \\ &= 32a^3 + 72a^2b + 48ab^2 + 10b^3 + b^2 - 10a^2 - 3ab \end{aligned}$$

f) $\frac{a^3 - 3a^2b + a(b^2 + 3b) - b^2}{a - 1}$

$\therefore a - 1 = 0$

$a = 1$

$f(a) = a^3 - 3a^2b + a(b^2 + 3b) - b^2$

$f(1) = (1)^3 - 3(1)^2b + (1)(b^2 + 3b) - b^2$

$= 1 - 3b + b^2 + 3b - b^2$

$= 1$

g) $(3x^3 - x^2 - 20x + 5) \div (x - 4)$

$\therefore x - 4 = 0$

$x = 4$

$f(x) = 3x^3 - x^2 - 20x + 5$

$f(4) = 3(4)^3 - (4)^2 - 20(4) + 5$

$= 192 - 16 - 80 + 5$

$= 101$

3. $f(x) = x^3 + mx^2 - x + 5$

$f(2) = (2)^3 + m(2)^2 - (2) + 5$

$23 = 8 + 4m - 2 + 5$

$12 = 4m$

$\therefore m = 3$

- and $x - 2 = 0$

  $x = 2$
- Remainder = 23

4. $\frac{2x^3 + 4px^2 - 3p^2x - 2}{x - p}$

$x - p = 0$

$\therefore x = p$

$f(x) = 2x^3 + 4px^2 - 3p^2x - 2$

$f(p) = 2(p)^3 + 4p(p)^2 - 3p^2(p) - 2$

$12 = 2p^3 + 4p^3 - 3p^3 - 2$

$12 = 3p^3 - 2$

$3p^3 = 14$

$p^3 = \frac{14}{3}$

$p = 1{,}671$

- Remainder = 12

5. $f(x) = 2x^2 + ax - 5$

$-8 = 2\left(\frac{1}{2}\right)^2 + a\left(\frac{1}{2}\right) - 5$

$-8 = 2\left(\frac{1}{4}\right) + \frac{1}{2}a - 5$

$-8 = \frac{1}{2} + \frac{1}{2}a - 5$

$\frac{1}{2}a = -3\frac{1}{2}$

$a = -7$

- $2x - 1 = 0$

  $x = \frac{1}{2}$

Assessment activity 2.2

1. **a)** $f(x) = x^3 + 8$

- $x = -2$

$f(-2) = (-2)^3 + 8$

$= 0$

$\therefore x + 2$ is a factor

b) $f(x) = x^3 - 7x^2 - 6x + 72$

- $x = 4$

$f(4) = (4)^3 - 7(4)^2 - 6(4) + 72$

$= 64 - 112 - 24 + 72$

$= 0$

$\therefore x - 4$ is a factor

c) $f(x) = 2x^3 - x^2 + 2x - 1$

- $x = \frac{1}{2}$

$f\left(\frac{1}{2}\right) = 2\left(\frac{1}{2}\right)^3 - \left(\frac{1}{2}\right)^2 + 2\left(\frac{1}{2}\right) - 1$

$= 2\left(\frac{1}{8}\right) - \frac{1}{4} + 1 - 1$

$= \frac{1}{4} - \frac{1}{4}$

$= 0$

$\therefore 2x - 1$ is a factor

d) $f(a) = 2a^3 - 7a^2b + 7ab^2 - 2b^3$ • $a = 2b$

$f(2b) = 2(2b)^3 - 7(2b)^2b + 7(2b)b^2 - 2b^3$

$= 16b^3 - 28b^3 + 14b^3 - 2b^3$

$= 0$

$\therefore a - 26$ is a factor

e) $f(x) = x^3 + 4x^2y - xy + 4xy^2 - 2y^2$ • $x = -2y$

$f(-2y) = (-2y)^3 + 4(-2y)^2y - (-2y)y + 4(-2y)y^2 - 2y^2$

$= -8y^3 + 16y^3 + 2y^2 - 8y^3 - 2y^2$

$= 0$

$\therefore x + 2y$ is a factor

2. $f(x) = 2x^3 + x^2a - 8xa^2 - 4a^3$ • $x = 2a$

$f(2a) = 2(2a)^3 + (2a)^2a - 8(2a)a^2 - 4a^3$

$= 16a^3 + 4a^3 - 16a^3 - 4a^3$

$= 0$

$\therefore x - 2a$ is a factor

3. $f(x) = 2x^3 + a^2x + 81$ • $x = a$

$f(a) = 2(a)^3 + a^2(a) + 81$

$0 = 2a^3 + a^3 + 81$

$0 = 3a^3 + 81$

$3a^3 = -81$

$a^3 = -27$

$a = -3$

4. $f(x) = mx^3 + nx^2 - 4x + 6$ • $x = 1$

$f(1) = m(1)^3 + n(1)^2 - 4(1) + 6$

$0 = m + n - 4 + 6$

$-2 = m + n$ .... ①

$f(2) = m(2)^3 + n(2)^2 - 4(2) + 6$ • $x = 2$

$0 = 8m + 4n - 8 + 6$

$2 = 8m + 4n$

$1 = 4m + 2n$ .... ②

$\therefore$ ③: $m = -2 - n$ .... ①

Substitute ① into ②:

$1 = 4(-2 - n) + 2n$

$1 = -8 - 4n + 2n$

$9 = -2n$

$n = -\frac{9}{2}$ or $-4{,}5$

Substitute $n = -\frac{9}{2}$ into ①:

$-2 = m + n$

$-2 = m - \frac{9}{2}$

$-2 + \frac{9}{2} = m$

$m = \frac{5}{2}$ or $2{,}5$

$\therefore m = \frac{5}{2} = 2\frac{1}{2}$ and $n = -\frac{9}{2} = -4\frac{1}{2}$

5. $f(x) = 2x^3 - 5x^2 - 4x + 3$

$f(-1) = 2(-1)^3 - 5(-1)^2 - 4(-1) + 3$

$= -2 - 5 + 4 + 3$

$= 0$

$\therefore$ $x + 1$ is a factor

$f\left(\frac{1}{2}\right) = 2\left(\frac{1}{2}\right)^3 - 5\left(\frac{1}{2}\right)^2 - 4\left(\frac{1}{2}\right) + 3$

$= \frac{1}{4} - \frac{5}{4} - 2 + 3$

$= -1 - 2 + 3$

$= 0$

$\therefore$ $2x - 1$ is a factor

$\therefore$ $(x + 1)(2x - 1)$ are factors

- $x = -1; x = \frac{1}{2}$

6. $f(x) = x^3 + 4x^2 + x - 6$

$f(-2) = (-2)^3 + 4(-2)^2 + (-2) - 6$

$= -8 + 16 - 2 - 6$

$= 0$

$\therefore$ $x + 2$ is a factor

$f(1) = (1)^3 + 4(1) + (1) - 6$

$= 1 + 4 + 1 - 6$

$= 0$

$\therefore$ $x - 1$ is a factor

$\therefore$ $(x + 2)(x - 1) = x^2 + x - 2$ is a factor

- $x^2 + x - 2$
  $= (x + 2)(x - 1)$
  $\therefore x = -2; x = 1$

Assessment activity 2.3

1. $x - 3 = 0$

$x = 3$

$f(x) = x^3 - x^2 - x - 15$

$f(3) = (3)^3 - (3)^2 - (3) - 15$

$= 27 - 9 - 3 - 15$

$= 0$

$\therefore$ $x - 3$ is a factor of $f(x)$

2. $f(x) = 2x^3 - x^2 - 13x - 6$

$f(-2) = 2(-2)^3 - (-2)^2 - 13(-2) - 6 = 2(-8) - 4 + 26 - 6$

$= -16 - 4 + 26 - 6$

$= 0$

$\therefore$ $x + 2$ is a factor

$f(x) = 2x^3 - x^2 - 13x - 6 = (x + 2)(2x^2 + bx - 3)$

$4x^2$

$+$

bx^2

$\therefore 4x^2 + bx^2 = -x^2$

$bx^2 = -5x^2$

$\therefore b = -5$

$\therefore f(x) = (x + 2)(2x^2 - 5x - 3)$

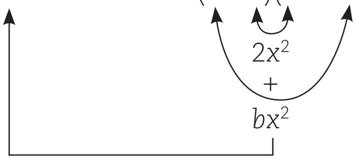
$= (x + 2)(2x + 1)(x - 3)$

3. **a)**

$$f(x) = x^3 + 9x^2 + 26x + 24$$

$$\begin{aligned} f(-2) &= (-2)^3 + 9(-2)^2 + 26(-2) + 24 \\ &= -8 + 36 - 52 + 24 \\ &= 0 \end{aligned}$$

$$\therefore\ x^3 + 9x^2 + 26x + 24 = (x + 2)(x^2 + bx + 12)$$



$$\begin{aligned} \therefore\ 2x^2 + bx^2 &= 9x^2 \\ bx^2 &= 7x^2 \\ \therefore\ b &= 7 \end{aligned}$$

$$\begin{aligned} \therefore\ x^3 + 9x^2 + 26x + 24 &= (x + 2)(x^2 + 7x + 12) \\ &= (x + 2)(x + 4)(x + 3) \end{aligned}$$

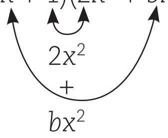
b)

$$f(x) = 2x^3 + x^2 - 7x - 6$$

$$\begin{aligned} f(1) &= 2(1)^3 + (1)^2 - 7(1) - 6 \\ &= 2 + 1 - 7 - 6 \\ &\neq 0 \end{aligned}$$

$$\begin{aligned} f(-1) &= 2(-1)^3 + (-1)^2 - 7(-1) - 6 \\ &= -2 + 1 + 7 - 6 \\ &= 0 \end{aligned}$$

$$\therefore\ (x + 1)(2x^2 + bx - 6)$$



$$\begin{aligned} \therefore\ 2x^2 + bx^2 &= x^2 \\ bx^2 &= -x^2 \\ b &= -1 \end{aligned}$$

$$\therefore\ (x + 1)(2x^2 - x - 6) = (x + 1)(2x + 3)(x - 2)$$

c)

$$f(x) = 3x^3 - 7x^2 + 4$$

$$\begin{aligned} f(1) &= 3(1)^3 - 7(1)^2 + 4 \\ &= 3 - 7 + 4 \\ &= 0 \end{aligned}$$

$$\begin{aligned} \therefore\ (x - 1)(3x^2 + bx - 4) &= (x - 1)(3x^2 - 4x - 4) \\ &= (x - 1)(3x + 2)(x - 2) \end{aligned}$$

- $$\begin{aligned} -3x^2 + bx^2 &= -7x^2 \\ bx^2 &= -4x^2 \\ \therefore\ b &= -4 \end{aligned}$$

d)

$$f(x) = x^3 - 12x - 16$$

$$\begin{aligned} f(1) &= (1)^3 - 12(1) - 16 \neq 0 \\ f(-1) &= (-1)^3 - 12(-1) - 16 \neq 0 \\ f(2) &= (2)^3 - 12(2) - 16 \neq 0 \\ f(-2) &= (-2)^3 - 12(-2) - 16 = -8 + 24 - 16 = 0 \end{aligned}$$

$$\begin{aligned} \therefore\ f(x) &= (x + 2)(x^2 + bx - 8) \\ &= (x + 2)(x^2 - 2x - 8) \\ &= (x + 2)(x - 4)(x + 2) \end{aligned}$$

- $$\begin{aligned} 2x^2 + bx^2 &= 0 \\ bx^2 &= -2x^2 \\ \therefore\ b &= -2 \end{aligned}$$

e) $f(x) = x^3 - 8$

$f(2) = (2)^3 - 8 = 0$

$\therefore f(x) = (x - 2)(x^2 + bx + 4)$

$= (x - 2)(x^2 + 2x + 4)$

- $-2x^2 + bx^2 = 0$
  $\therefore bx^2 = 2x^2$
  $b = 2$

f) $f(x) = x^3 + 3x^2 - 16x - 48$

$f(-3) = (-3)^3 + 3(-3)^2 - 16(-3) - 48$

$= -27 + 27 + 48 - 48$

$= 0$

$\therefore f(x) = (x + 3)(x^2 + bx - 16)$

$= (x + 3)(x^2 - 16)$

$= (x + 3)(x - 4)(x + 4)$

- $3x^2 + bx^2 = 3x^2$
  $bx^2 = 0$
  $b = 0$

g) $f(x) = -x^3 + 6x^2 - 9x + 4$

$f(1) = -(1)^3 + 6(1)^2 - 9(1) + 4 = 0$

$f(x) = (x - 1)(-x^2 + bx - 4)$

$= (x - 1)(-x^2 + 5x - 4)$

$= -(x - 1)(x^2 - 5x + 4)$

$= -(x - 1)(x - 4)(x - 1)$

- $x^2 + bx^2 = 6x^2$
  $bx^2 = 5x^2$
  $\therefore b = 5$

4. **a)** $f(x) = 2x^3 + x^2 - 5x + 2$

$f(1) = 2(1)^3 + (1)^2 - 5(1) + 2 = 0$

$f(x) = (x - 1)(2x^2 + bx - 2)$

$= (x - 1)(2x^2 + 3x - 2)$

$= (x - 1)(2x - 1)(x + 2)$

- $-2x^2 + bx^2 = x^2$
  $bx^2 = 3x^2$
  $b = 3$

$0 = (x - 1)(2x - 1)(x + 2)$

$\therefore x - 1 = 0 \text{ or } 2x - 1 = 0 \text{ or } x + 2 = 0$

$x = 1 \text{ or } \quad x = \frac{1}{2} \text{ or } \quad x = -2$

- $f(x) = 0$

b) $f(x) = 4x^3 - 4x^2 - 23x + 30$

$f(-1) = 4(-1)^3 - 4(-1)^2 - 23(-1) + 30$

$= -4 - 4 + 23 + 30 \neq 0$

$f(2) = 4(2)^3 - 4(2)^2 - 23(2) + 30$

$= 4(8) - 4(4) - 46 + 30$

$= 32 - 16 - 46 + 30$

$= 0$

$\therefore f(x) = (x - 2)(4x^2 + bx - 15)$

$0 = (x - 2)(4x^2 + 4x - 15)$

$0 = (x - 2)(2x - 3)(2x + 5)$

- $-8x^2 + bx^2 = -4x^2$
  $bx^2 = 4x^2$
  $\therefore b = 4$

$\therefore x - 2 = 0 \text{ or } 2x - 3 = 0 \text{ or } 2x + 5 = 0$

$x = 2 \text{ or } \quad x = \frac{3}{2} \text{ or } \quad x = \frac{-5}{1}$

5. $f(x) = (x - 2)(2x^2 + bx - 2)$

$= (x - 2)(2x^2 + 3x - 2)$

$0 = (x - 2)(2x - 1)(x + 2)$

$\therefore x - 2 = 0$ or $2x - 1 = 0$ or $x + 2 = 0$

$x = 2$ $\quad x = \frac{1}{2}$ $\quad x = -2$

- $-4x^2 + bx^2 = -x^2$

  $bx^2 = 3x^2$

  $\therefore b = 3$

Assessment activity 2.4

1. $\{(5; -1); (4; 0); (3; 1); (2; 2)\}$

2. **a)** $f(x)$: $y = 2x - 5$

$f^{-1}(x)$: $x = 2y - 5$

$2y = x + 5$

$\therefore y = \frac{1}{2}x + \frac{5}{2}$

$\therefore f^{-1}(x) = \frac{1}{2}x + \frac{5}{2}$

b) $f(x)$: $y = \frac{1}{2}x + 2$

$f^{-1}(x)$: $x = \frac{1}{2}y + 2$

$\frac{1}{2}y = x - 2$

$y = 2x - 4$

$\therefore f^{-1}(x) = 2x - 4$

c) $f(x)$: $y = -\frac{2}{3}x - 1$

$f^{-1}(x)$: $x = -\frac{2}{3}y - 1$

$\frac{2}{3}y = -x - 1$

$y = -\frac{3}{2}x - \frac{3}{2}$

$\therefore f^{-1}(x) = -\frac{3}{2}x - \frac{3}{2}$

d) $f(x)$: $y = 4x^2$

$f'(x)$: $x = 4y^2$

$4y^2 = x$

$y^2 = \frac{x}{4}$

$y = \pm\frac{1}{2}\sqrt{x}$

$\therefore f^{-1}(x) = \pm\frac{1}{2}\sqrt{x}$

e) $f(x)$: $y = -x^2$

$f^{-1}(x)$: $x = -y^2$

$y^2 = x$

$y = \pm\sqrt{x}$

$\therefore f^{-1}(x) = \pm\sqrt{x}$

f) $f(x)$: $y = \frac{1}{2}x^2$

$f^{-1}(x)$: $x = \frac{1}{2}y^2$

$\frac{1}{2}y^2 = x$

$\therefore y^2 = 2x$

$y = \pm\sqrt{2x}$

$y = \pm\sqrt{2}\sqrt{x}$ or $y = \pm 1{,}414\sqrt{x}$

g) $f(x)$: $y = 3x - \frac{2}{3}$

$f^{-1}(x)$: $x = 3y - \frac{2}{3}$

$3y = x + \frac{2}{3}$

$\therefore y = \frac{1}{3}x + \frac{2}{9}$

$\therefore f^{-1}(x) = \frac{1}{3}x + \frac{2}{9}$

Assessment activity 2.5

1. $f(x) = -2x - 4$

a) $f(2) = -2(2) - 4$

$= -8$

b) $f\left(-\frac{1}{4}\right) = -2\left(-\frac{1}{4}\right) - 4$

$= \frac{1}{2} - 4$

$= -3\frac{1}{2}$

c) $f(a - b) = -2(a - b) - 4$

$= -2a + 2b - 4$

2.
- **a)** f is a function.
- **b)** g is a function.
- **c)** h is not a function; vertical line crosses twice.
- **d)** i is a function.
- **e)** j is a function.

3. **a)** $f(x) = 2x - 5$

- x-intercept: $y = 0$

  $\therefore 0 = 2x - 5$

  $2x = 5$

  $\therefore x = \frac{5}{2}$ or $2\frac{1}{2}$
- y-intercept: $x = 0$

  $\therefore y = 2(0) - 5$

  $\therefore y = -5$

$f(x)$: $y = 2x - 5$

$f^{-1}(x)$: $x = 2y - 5$

$2y = x + 5$

$y = \frac{1}{2}x + \frac{5}{2}$

$\therefore f^{-1}(x) = \frac{1}{2}x + \frac{5}{2}$

- x-intercept: $y = 0$

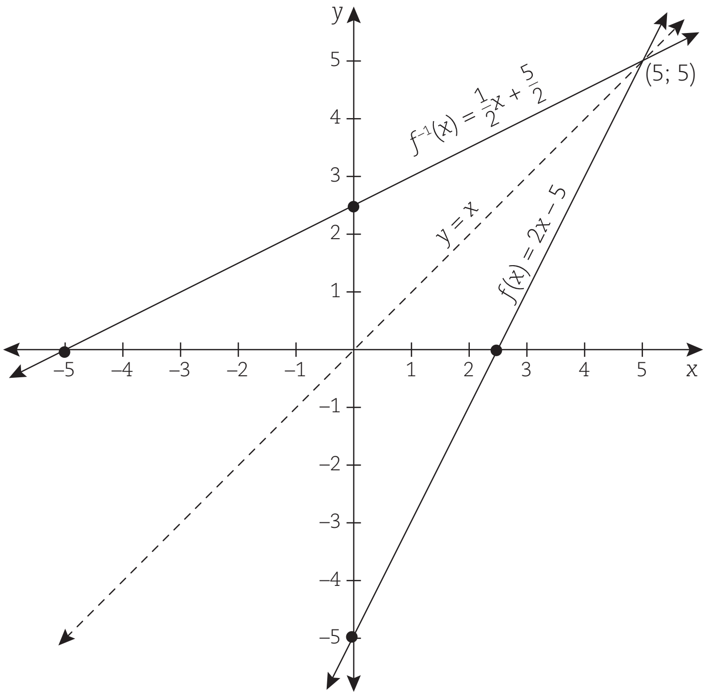
  $\therefore 0 = \frac{1}{2}x + \frac{5}{2}$

  $\frac{1}{2}x = -\frac{5}{2}$

  $x = -5$
- y-intercept: $x = 0$

  $\therefore y = \frac{1}{2}(0) + \frac{5}{2}$

  $\therefore y = \frac{5}{2}$ or $2\frac{1}{2}$



b) $2x - 5 = \frac{1}{2}x + \frac{5}{2}$

$4x - 10 = x + 5$

$3x = 15$

$x = 5$

- $\therefore y = 2(5) - 5$
  $= 5$
- $\therefore (x; y) = (5; 5)$

c) Yes. The vertical line crosses $f(x)$ only once.
One-to-one function

d) Yes. The vertical line crosses $f^{-1}(x)$ only once.
One-to-one function

e) $f(x)$: $\{x: x \in \mathbb{R}\}$

f) $f^{-1}(x)$: $\{y: y \in \mathbb{R}\}$

g) Increasing function. y increases as x increases.

4. **a)** $f(x) = -\frac{1}{2}x + 3$

- x-intercept: $y = 0$
  $0 = -\frac{1}{2}x + 3$
  $\frac{1}{2}x = 3$
  $\therefore x = 6$
- y-intercept: $x = 0$
  $\therefore y = 3$

$f(x)$: $y = -\frac{1}{2}x + 3$

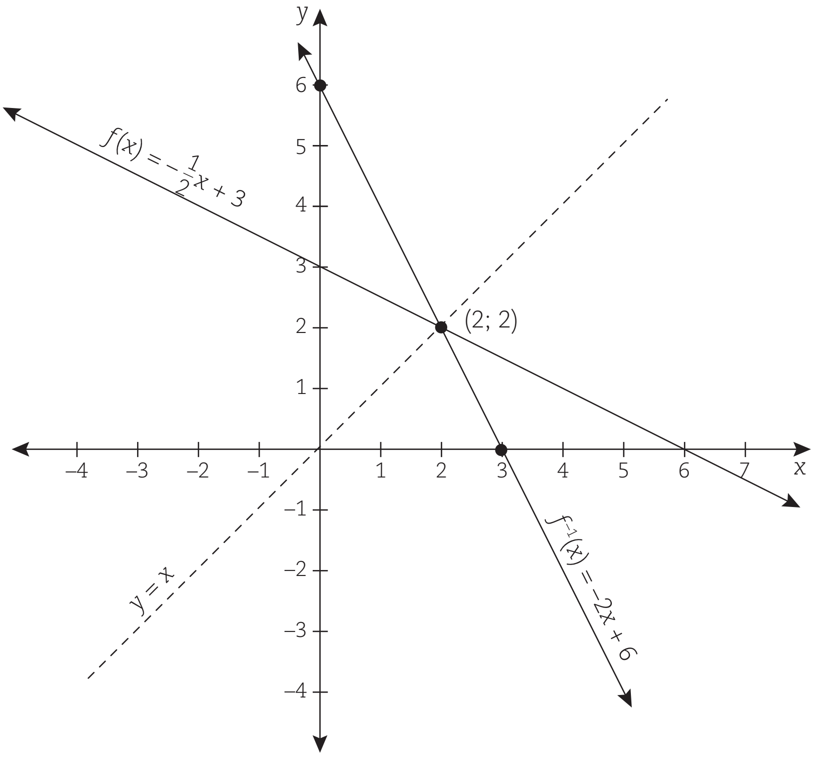
$f^{-1}(x)$: $x = -\frac{1}{2}y + 3$

$\frac{1}{2}y = -x + 3$

$y = -2x + 6$

$\therefore f^{-1}(x) = -2x + 6$

- x-intercept: $y = 0$
  $0 = -2x + 6$
  $2x = 6$
  $\therefore x = 3$
- y-intercept: $x = 0$
  $\therefore y = 6$



b) $-\frac{1}{2}x + 3 = -2x + 6$
$-x + 6 = -4x + 12$
$3x = 6$
$x = 2$

- $\therefore y = -2x + 6$
$= -2(2) + 6$
$\therefore y = 2$
$\therefore (x; y) = (2; 2)$

c) Yes. The vertical line crosses $f(x)$ only once.
One-to-one function

d) Yes. The vertical line crosses $f^{-1}(x)$ only once.
One-to-one function

e) $f(x)$: Domain = $\{x: x \in \mathbb{R}\}$
Range = $\{y: y \in \mathbb{R}\}$

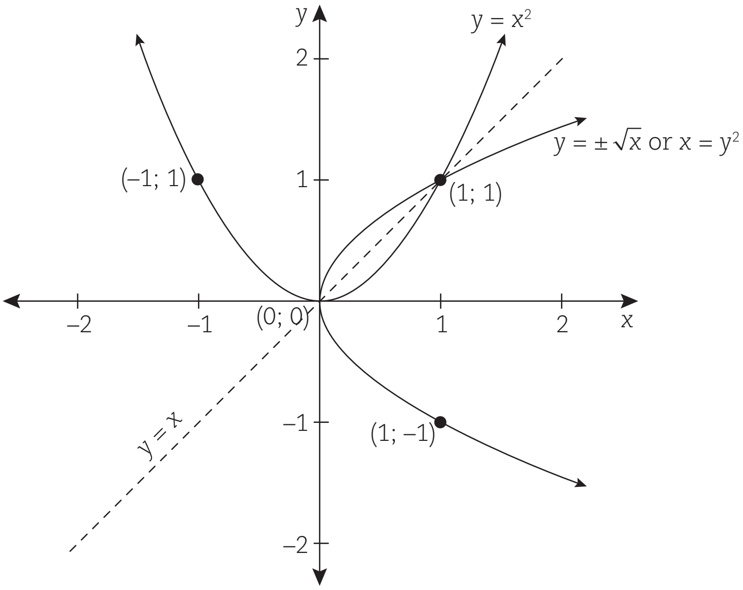
f) $f^{-1}(x)$: Domain = $\{x: x \in \mathbb{R}\}$
Range = $\{y: y \in \mathbb{R}\}$

g) Decreasing function. y decreases as x increases.

Assessment activity 2.6

1. a) $f(x)$: $y = x^2$
$x = y^2$
$\therefore y = \pm\sqrt{x}$
$\therefore f^{-1}(x) = \pm\sqrt{x}$

b)



c) Yes, a vertical line through the graph crosses graph once.
Many-to-one function

d) No, a vertical line through the graph crosses graph more than once.
The inverse is a one-to-many relation.

e) Domain: $\{x: x \in \mathbb{R}\}$
Range: $\{y: y \geq 0; y \in \mathbb{R}\}$

f) Domain: $\{x: x \geq 0; x \in \mathbb{R}\}$
Range: $\{y: y \in \mathbb{R}\}$

g) Continuous

h)
1. $f(x) = x^2$ where $x \geq 0$
2. $f(x) = x^2$ where $x \leq 0$

2. $f(x) = \frac{1}{2}x^2$

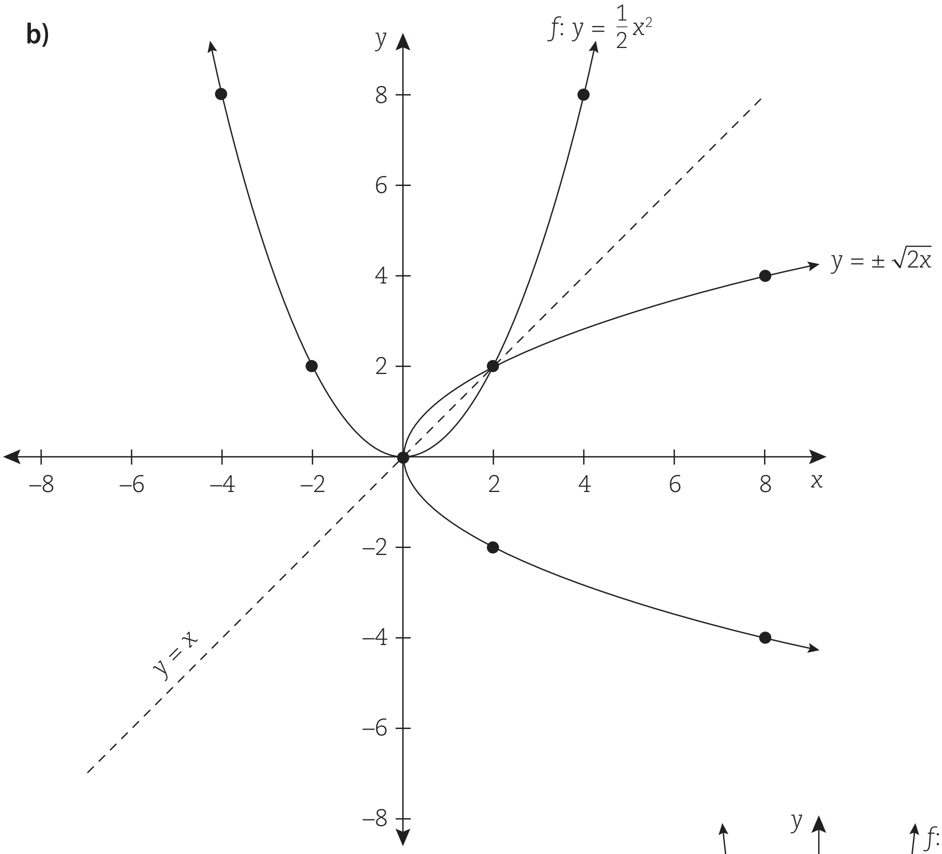
a) $f(x)$: $y = \frac{1}{2}x^2$

$x = \frac{1}{2}y^2$

$y^2 = 2x$

$\therefore\ y = \pm\sqrt{2x}$

b)



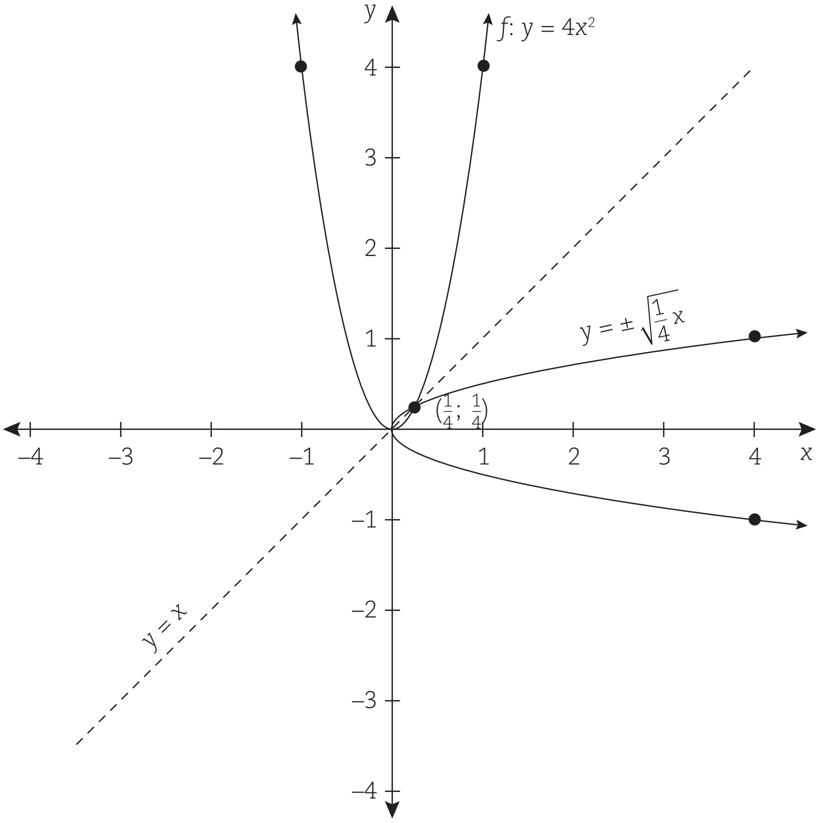
3. f: $y = 4x^2$

$x = 4y^2$

$y^2 = \frac{1}{4}x$

$y = \pm\sqrt{\frac{1}{4}x}$

$\therefore\ y = \pm\frac{1}{2}\sqrt{x}$



b) $4x^2 = \pm\sqrt{\tfrac{1}{4}x}$

$\therefore\ (4x^2)^2 = \left(\pm\sqrt{\tfrac{1}{4}x}\right)^2$

$16x^4 = \frac{1}{4}x$

$64x^4 = x$

$64x^4 - x = 0$

$x(64x^3 - 1) = 0$

$x = 0 \quad \text{or} \quad 64x^3 - 1 = 0$

$64x^3 = 1$

$x^3 = \frac{1}{64}$

$x = \sqrt[3]{\frac{1}{64}}$

$x = \frac{1}{4}$

Substitute $x = 0$ into $y = 4x^2$ Substitute $x = \frac{1}{4}$ into $y = 4x^2$

$\therefore\ f(0) = 4(0)^2 = 0$ $\therefore\ f\left(\frac{1}{4}\right) = 4\left(\frac{1}{4}\right)^2 = \frac{1}{4}$

$\therefore$ Points of intersection are $(0; 0)$ and $\left(\frac{1}{4}; \frac{1}{4}\right)$

4. $f(x) = -2x^2$

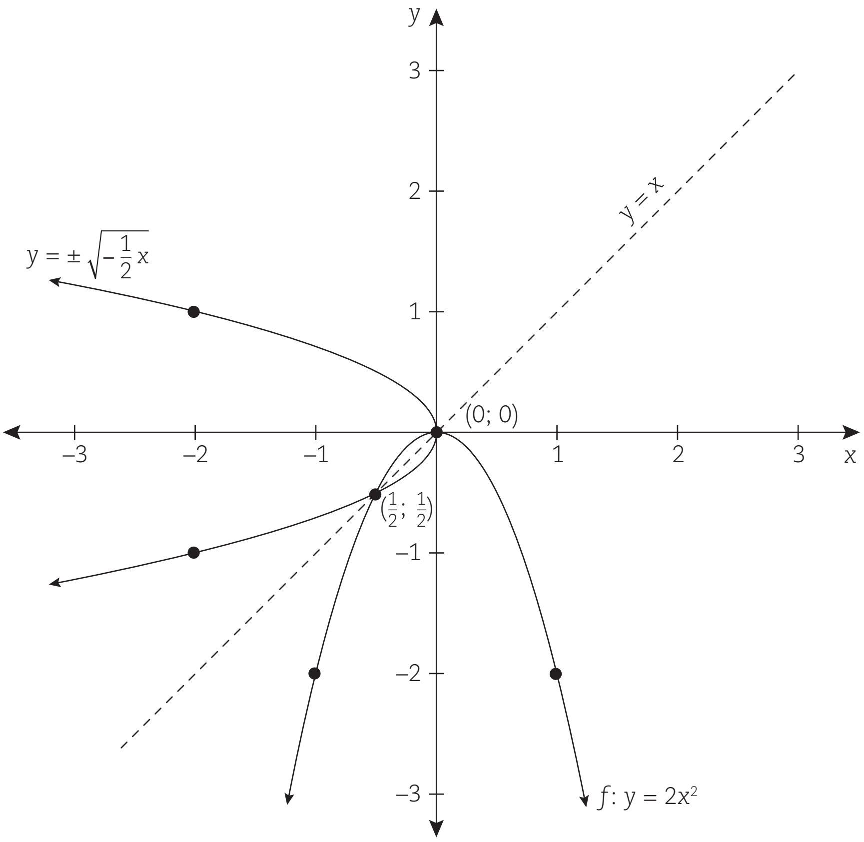
a) $f(x)\colon y = -2x^2$

$x = -2y^2$

$y^2 = -\frac{1}{2}x$

$\therefore\ y = \pm\sqrt{-\frac{1}{2}x}$

b)



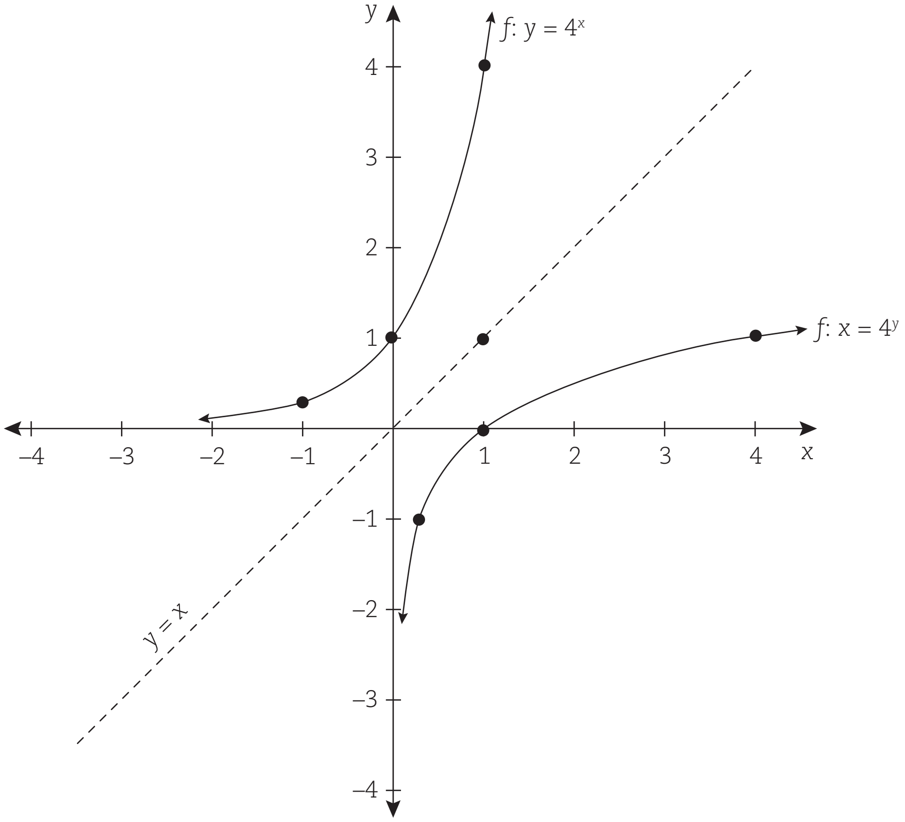
Assessment activity 2.7

1. f: $y = 4^x$

x	−1	0	1
y	$\frac{1}{4}$	1	4

f^{-1}: $x = 4^y$

y	−1	0	1
x	$\frac{1}{4}$	1	4



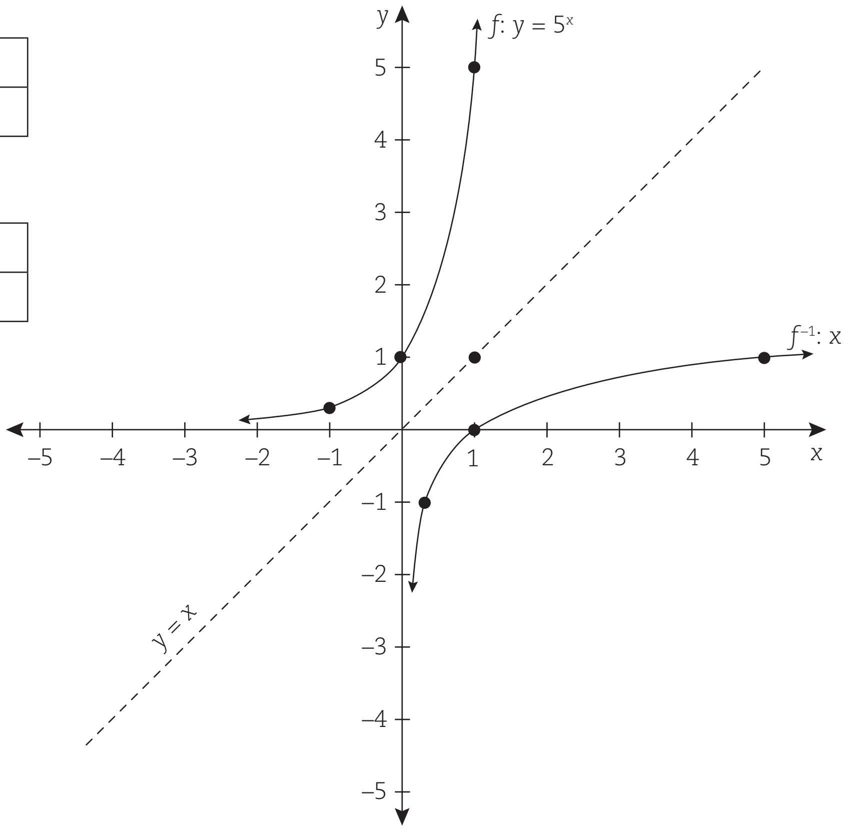
2. **a)** f^{-1}: $x = 5^y$

b) f: $y = 5^x$

x	−1	0	1
y	$\frac{1}{5}$	1	5

$x = 5^y$

y	−1	0	1
x	$\frac{1}{5}$	1	5



c) $y = 0$ (x-axis)

d) $x = 0$ (y-axis)

e) $\{x: x \in \mathbb{R}\}$

f) $\{x: x > 0; x \in \mathbb{R}\}$

3. $f(x) = 3^{-x}$

$\therefore f(x) = \left(\frac{1}{3}\right)^x$

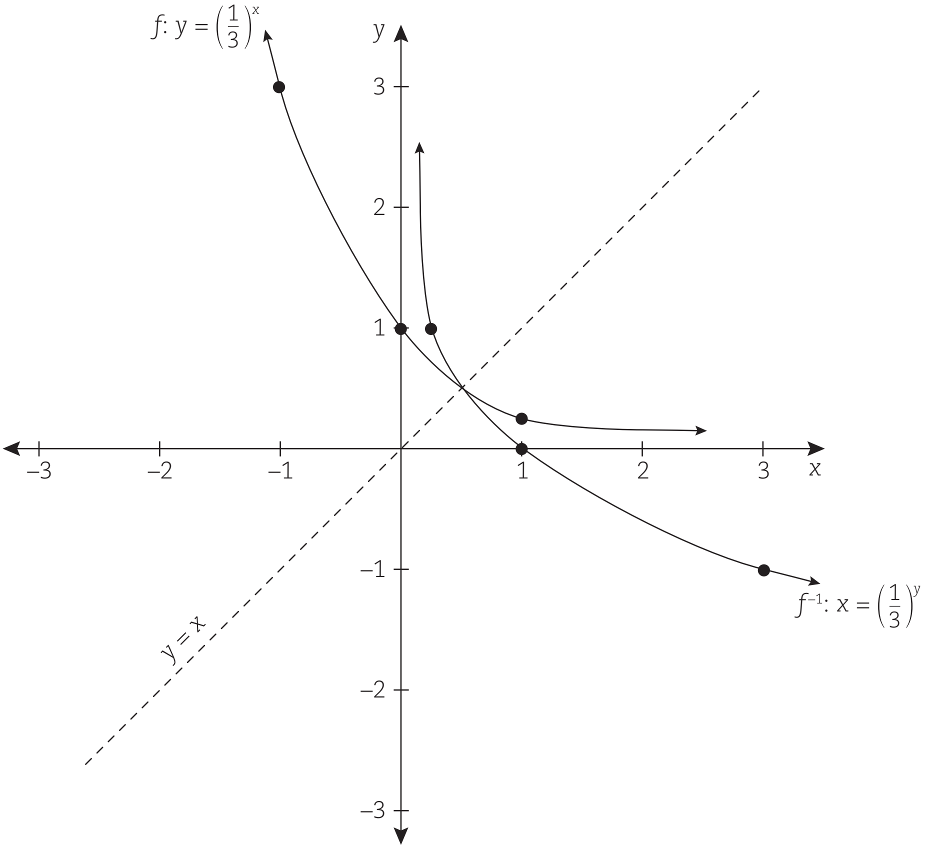
a) $f^{-1}: x = \left(\frac{1}{3}\right)^y$

b) $y = \left(\frac{1}{3}\right)^x$

x	–1	0	1
y	3	1	$\frac{1}{3}$

$x = \left(\frac{1}{3}\right)^y$

y	–1	0	1
x	3	1	$\frac{1}{3}$



c) Yes, there is only one y-value for each x-value.
Any vertical line cuts the graph only once.

d) Decreasing. If y decreases, x increases.

e) Domain: $\{x: x \in \mathbb{R}\}$
Range: $\{y: y > 0; y \in \mathbb{R}\}$

f) Domain: $\{x: x > 0; x \in \mathbb{R}\}$
Range: $\{y: y \in \mathbb{R}\}$

g) Continuous

Assessment activity 2.8

1. **a)** $x = 0$

b) Yes. Many-to-one mapping

c) $f(x)$: $y = -x^2$

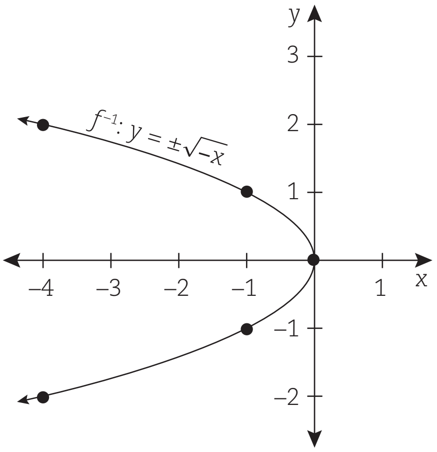
$f^{-1}(x)$: $x = -y^2$

$\therefore y^2 = -x$

$y = \pm\sqrt{-x}$

$\therefore f^{-1}(x) = \pm\sqrt{-x}$

d)



e) (0; 0)

f) Domain: $\{x: x \in \mathbb{R}\}$

g) Domain: $\{x: x \leq 0; x \in \mathbb{R}\}$ or $(-\infty; 0]$

h) No. The inverse is a one-to-two relation.
A vertical line, drawn to the left of the y-axis, cuts the graph twice.

i) Continuous

2. **a)** y-intercept: where $x = 0$

$\therefore y = 5^0 \qquad y = 3^0$

$y = 1 \qquad y = 1$

$\therefore Q = (0; 1)$

b) $y = 0$

c) $g(x) = 5^x$

$\therefore y = 5^x$

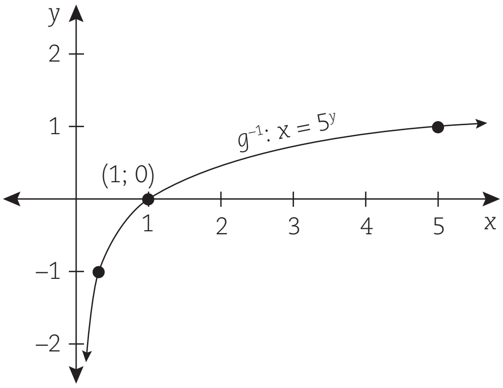
g^{-1}: $x = 5^y$

$y = 5^x$

x	–1	0	1
y	$\frac{1}{5}$	1	5

$x = 5^y$

y	–1	0	1
x	$\frac{1}{5}$	1	5



d) $y \in \mathbb{R}$

e) $f(x) = 3^x$

$\therefore f^{-1}$: $x = 3^y$

f) $x = 0$

g) Yes, the graph is a one-to-one function.

h) Yes, the graph is a one-to-one function.

3. **a)** f: $y = mx + c$

$\therefore f$: $y = 2x + 6$

b) f^{-1}: $x = 2y + 6$

$2y = x - 6$

$\therefore f^{-1}$: $y = \frac{1}{2}x - 3$

c) $2x + 6 = \frac{1}{2}x - 3$

$4x + 12 = x - 6$

$3x = -18$

$x = -6$

$\therefore y = 2x + 6$

$y = 2(-6) + 6$

$y = -6$

$\therefore$ Point of intersection: A(–6; –6)

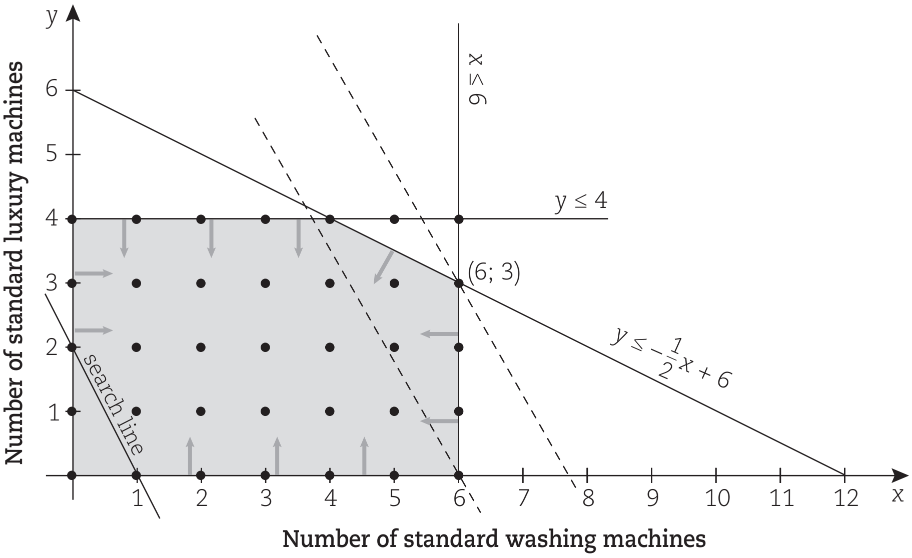
d) Yes. One-to-one function. (No vertical line crosses f^{-1} more than once.)

e) Increasing function. The value of y increases as x increases.

f) Increasing function. The inverse of an increasing function is an increasing function.

Assessment activity 2.9

1. a)



$x + 2y \leq 12$

$2y \leq -x + 12$

$\therefore y \leq -\frac{1}{2}x + 6$

b) $P = 600x + 300y$

$300y = P - 600x$

$y = -\frac{600}{300}x + \frac{P}{300}$

$\therefore m = -2$

$\therefore$ 6 standard washing machines and 3 luxury washing machines must be manufactured per day.

c) Maximum profit:

$P = 600(6) + 300(3)$ • (6 ; 3)

$\therefore P = R4\ 500$

2. Let x be the laser printers.
Let y be the inkjet printers.

a) $x + y \leq 26$

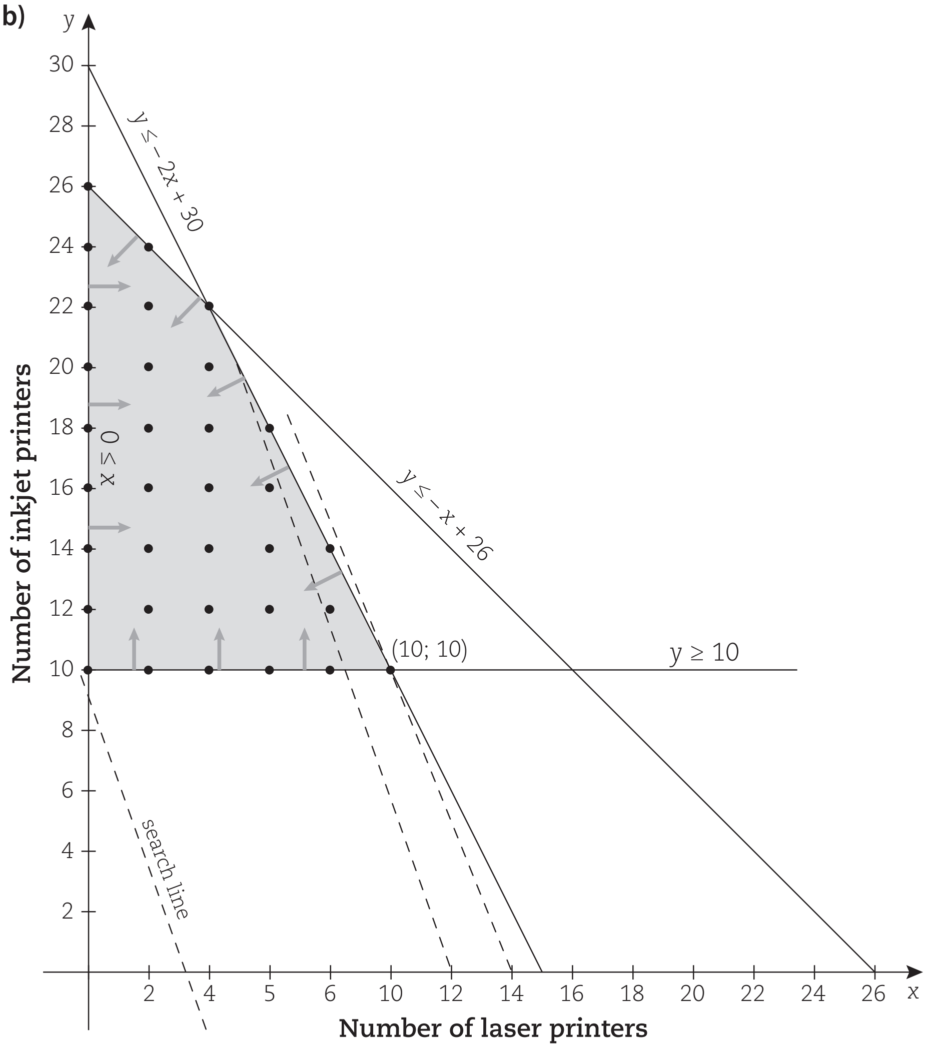
$y \geq 10$

$70x + 35y \leq 1\ 050$

$x \geq 0$

$x, y \in \mathbb{Z}$

b)



$70x + 35y \leq 1\,050$

$$\frac{35y}{35} \leq -\frac{70x}{35} + \frac{1\,050}{35}$$

$y \leq -2x + 30$

c) $P = 2\,700x + 900y$

$$y = -\frac{2\,700}{900}x + \frac{P}{900}$$

$$\therefore y = -3x + \frac{P}{900}$$

$x = 10$ = laser printers

$y = 10$ = inkjet printers

d) $P = 2\,700x + 900y$

$= 2\,700(10) + 900(10)$

$\therefore P = R36\,000$

- Maximum profit

3. Hamburgers = x

Chicken burgers = y

a) $x \leq 400$

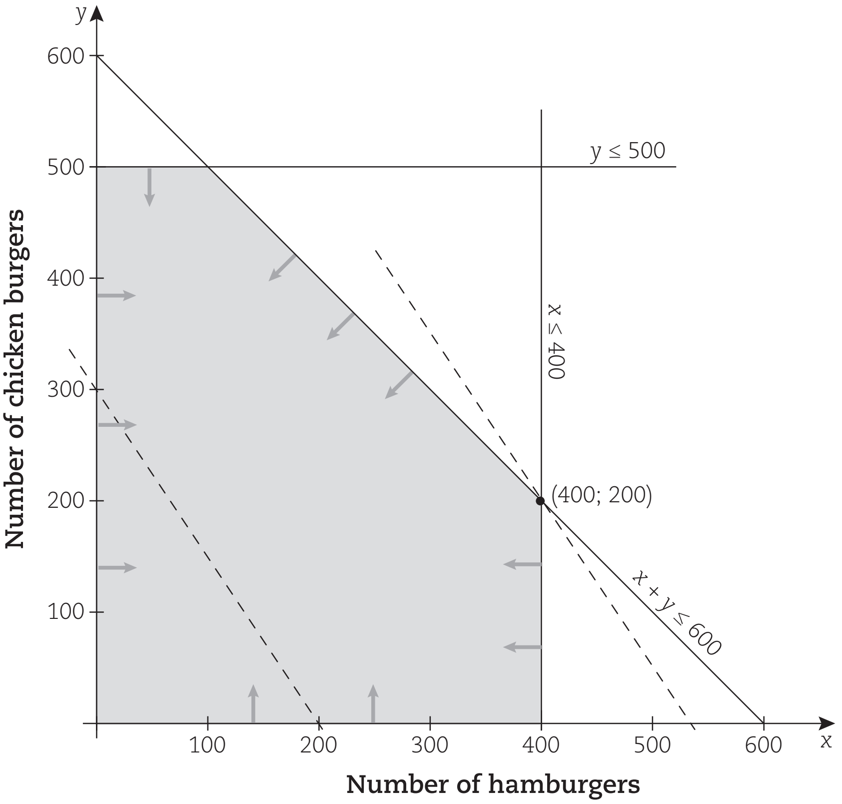
$y \leq 500$

$x + y \leq 600$

$x, y \in \mathbb{N}_0$

Given: $x \geq 0$; $y \geq 0$

b)



c) $P = 6x + 4y$

$$\frac{4y}{4} = -\frac{6}{4}x + \frac{P}{4}$$

$$\therefore y = -\frac{3}{2}x + \frac{P}{4}$$

d) 400 hamburgers and 200 chicken burgers must be sold.

e) $\therefore P = 6(400) + 4(200)$

$\therefore P = R3\ 200$

Maximum profit is R3 200

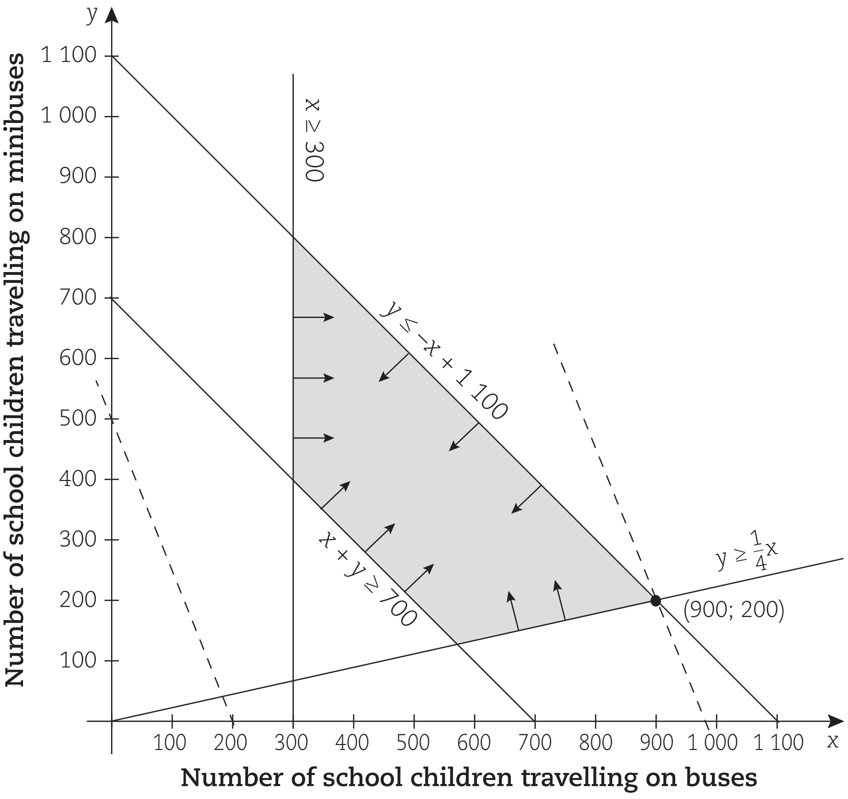
4. **a)** $x \geq 300$

$x + y \leq 1\ 100 \Rightarrow y \leq -x + 1\ 100$

$x + y \geq 700 \Rightarrow y \geq -x + 700$

$x \leq 4y \Rightarrow y \geq \frac{1}{4}x$

b)



c)

$$P = 1{,}5x + 0{,}60y$$

$$0{,}60y = -1{,}5x + P$$

$$y = -\frac{1{,}5}{0{,}6}x + \frac{P}{0{,}6}$$

$$= \frac{-\frac{3}{2}}{\frac{6}{10}}x + \frac{P}{0{,}6}$$

$$\therefore y = -\frac{5}{2}x + \frac{P}{0{,}6}$$

$$\therefore m = -\frac{5}{2}$$

$$\frac{1}{4}x = -2 + 1\,100$$

$$\frac{5}{4}x = 1\,100$$

$$x = 800$$

$$y = \frac{1}{4}(880)$$

$$y = 220$$

Maximum profit is at $x = 880$ and $y = 220$.

d)

$$P = 1{,}5x + 0{,}6y$$

$$P = 1{,}5(880) + 0{,}6(220)$$

$\therefore$ P = R1 452

$\therefore$ Maximum profit is R1 452.

Assessment activity 2.10

1. **a)**

$$f(x) = 5x - 3$$

$$f(x + h) = 5(x + h) - 3$$

$$= 5x + 5h - 3$$

$$f'(x) = \lim_{h\to 0} \frac{5x + 5h - 3 - (5x - 3)}{h}$$

$$= \lim_{h\to 0} \frac{5h}{h}$$

$$= \lim_{h\to 0} 5$$

$$= 5$$

b)

$$f(x) = -3x^2 - 2$$

$$f(x + h) = -3(x + h)^2 - 2$$

$$= -3(x + h)(x + h) - 2$$

$$= -3(x^2 + 2xh + h^2) - 2$$

$$= -3x^2 - 6xh - 3h^2 - 2$$

$$f'(x) = \lim_{h\to 0} \frac{-3x^2 - 6xh - 3h^2 - 2 - (-3x^2 - 2)}{h}$$

$$= \lim_{h\to 0} \frac{-6xh - 3h^2}{h}$$

$$= \lim_{h\to 0} \frac{h(-6x - 3h)}{h}$$

$$= \lim_{h\to 0} (-6x - 3h)$$

$$= -6x$$

c)

$$f(x) = 4x^0$$

$$f(x + h) = 4(x + h)^0$$

$$f'(x) = \lim_{h\to 0} \frac{4(x + h)^0 - 4x^0}{h}$$

$$= \lim_{h\to 0} \frac{0}{h}$$

$$= \lim_{h\to 0} 0$$

$$= 0$$

d)

$$f(x) = -x^3$$

$$f(x + h) = -(x + h)^3$$

$$= -(x + h)(x + h)(x + h)$$

$$= -(x^2 + 2xh + h^2)(x + h)$$

$$= -(x^3 + x^2h + 2x^2h + 2xh^2 + xh^2 + h^3)$$

$$= -x^3 - x^2h - 2x^2h - 2xh^2 - xh^2 - h^3$$

$$= -x^3 - 3x^2h - 3xh^2 - h^3$$

$$f'(x) = \lim_{h\to 0} \frac{-x^3 - 3x^2h - 3xh^2 - h^3 - (-x^3)}{h}$$

$$= \lim_{h\to 0} \frac{-3x^2h - 3xh^2 - h^3}{h}$$

$$= \lim_{h\to 0} \frac{h(-3x^2 - 3xh - h^2}{h}$$

$$= \lim_{h\to 0} -3x^2 - 3xh - h^2$$

$$= -3x^2$$

e)

$$\begin{aligned} f(x) &= -\frac{1}{x} \\ f(x+h) &= -\frac{1}{x+h} \end{aligned}$$

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{-\frac{1}{x+h} - (-\frac{1}{x})}{h} \\ &= \lim_{h\to 0} \frac{-\frac{1}{x+h} + \frac{1}{x}}{h} \\ &= \lim_{h\to 0} \frac{\frac{-x+(x+h)}{x(x+h)}}{h} \\ &= \lim_{h\to 0} \frac{-x+x+h}{x(x+h)} \times \frac{1}{h} \\ &= \lim_{h\to 0} \frac{1}{x(x+h)} \\ &= \frac{1}{x^2} \end{aligned}$$

f)

$$\begin{aligned} y &= \frac{4}{x} \\ f(x) &= \frac{4}{x} \\ f(x+h) &= \frac{4}{x+h} \end{aligned}$$

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{\frac{4}{x+h} - \frac{4}{x}}{h} \\ &= \lim_{h\to 0} \frac{\frac{4x-4(x+h)}{x(x+h)}}{h} \\ &= \lim_{h\to 0} \frac{4x-4x-4h}{x(x+h)} \times \frac{1}{h} \\ &= \lim_{h\to 0} \frac{-4}{x(x+h)} \\ &= -\frac{4}{x^2} \end{aligned}$$

g)

$$\begin{aligned} y &= \frac{1}{2}x^3 \\ f(x) &= \frac{1}{2}x^3 \\ f(x+h) &= \frac{1}{2}(x+h)^3 \\ &= \frac{1}{2}(x+h)(x+h)(x+h) \\ &= \frac{1}{2}(x^2+2xh+h^2)(x+h) \\ &= \frac{1}{2}(x^3+x^2h+2x^2h+2xh^2+xh^2+h^3) \\ &= \frac{1}{2}x^3+\frac{1}{2}x^2h+x^2h+xh^2+\frac{1}{2}xh^2+\frac{1}{2}h^3 \\ &= \frac{1}{2}x^3+\frac{3}{2}x^2h+\frac{3}{2}xh^2+\frac{1}{2}h^3 \end{aligned}$$

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{\frac{1}{2}x^3+\frac{3}{2}x^2h+\frac{3}{2}xh^2+\frac{1}{2}h^3-\frac{1}{2}x^3}{h} \\ &= \lim_{h\to 0} \frac{h(\frac{3}{2}x^2+\frac{3}{2}xh+\frac{1}{2}h^2)}{h} \\ &= \lim_{h\to 0} \left(\frac{3}{2}x^2+\frac{3}{2}xh+\frac{1}{2}h^2\right) \\ &= \frac{3}{2}x^2 \end{aligned}$$

2. **a)**

$$\begin{aligned} f(x) &= -4x^3 \\ f(x+h) &= -4(x+h)^3 \\ &= -4(x^2+2xh+h^2)(x+h) \\ &= -4(x^3+3x^2h+3xh^2+h^3) \\ &= -4x^3-12x^2h-12xh^2-4h^3 \end{aligned}$$

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{-4x^3-12x^2h-12xh^2-4h^3-(-4x^3)}{h} \\ &= \lim_{h\to 0} \frac{h(-12x^2-12xh-4h^2)}{h} \\ &= \lim_{h\to 0} (-12x^2-12xh-4h^2) \\ &= -12x^2 \end{aligned}$$

b)

$$\begin{aligned} f'(x) &= -12x^2 \\ \therefore f'(-2) &= -12(-2)^2 \\ &= -48 \end{aligned}$$

3. **a)**

$$\begin{aligned}
f(x) &= 3x^3\\
f(x+h) &= 3(x+h)^3\\
&= 3(x^2+2xh+h^2)(x+h)\\
&= 3(x^3+3x^2h+3xh^2+h^3)\\
&= 3x^3+9x^2h+9xh^2+3h^2
\end{aligned}$$

$$\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{3x^3+9x^2h+9xh^2+3h^2-3x^3}{h}\\
&= \lim_{h\to 0}\frac{h(9x^2+9xh+3h)}{h}\\
&= \lim_{h\to 0}(9x^2+9xh+3h)\\
&= 9x^2
\end{aligned}$$

$$\begin{aligned}
\therefore f'(x) &= 9x^2\\
f'\left(-\frac{1}{3}\right) &= 9\left(-\frac{1}{3}\right)^2\\
&= 9\left(\frac{1}{9}\right)\\
&= 1
\end{aligned}$$

b)

$$\begin{aligned}
f(x) &= -2x^2-4\\
f(x+h) &= -2(x+h)^2-4\\
&= -2(x^2+2xh+h^2)-4\\
&= -2x^2-4xh-2h^2-4
\end{aligned}$$

$$\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{-2x^2-4xh-2h^2-4-(-2x^2-4)}{h}\\
&= \lim_{h\to 0}\frac{-4xh-2h^2}{h}\\
&= \lim_{h\to 0}\frac{h(-4x-2h)}{h}\\
&= \lim_{h\to 0}(-4x-2h)\\
&= -4x
\end{aligned}$$

$$\begin{aligned}
\therefore f'(x) &= -4x\\
f'(2) &= -4(2)\\
&= -8
\end{aligned}$$

c)

$$\begin{aligned}
f(x) &= -\frac{1}{4x}\\
f(x+h) &= -\frac{1}{4(x+h)}
\end{aligned}$$

$$\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{-\frac{1}{4(x+h)}-(-\frac{1}{4x})}{h}\\
&= \lim_{h\to 0}\frac{-\frac{1}{4(x+h)}+\frac{1}{4x}}{h}\\
&= \lim_{h\to 0}\frac{\frac{-x+(x+h)}{4x(x+h)}}{h}\\
&= \lim_{h\to 0}\frac{-x+x+h}{4x(x+h)}\times\frac{1}{h}\\
&= \lim_{h\to 0}\frac{1}{4x(x+h)}\\
&= \frac{1}{4x^2}
\end{aligned}$$

$$\begin{aligned}
\therefore f'(x) &= -\frac{1}{4x^2}\\
f'(-3) &= -\frac{1}{4(-3)^2}\\
&= -\frac{1}{36}
\end{aligned}$$

4.
$$\begin{aligned}
f(t) &= -5t^2 + 3\\
f(t+h) &= -5(t+h)^2 + 3\\
&= -5(t^2 + 2th + h^2) + 3\\
&= -5t^2 - 10th - 5h^2 + 3
\end{aligned}$$

$$\begin{aligned}
f'(t) &= \lim_{h\to 0} \frac{-5t^2 - 10th - 5h^2 + 3 + 5t^2 - 3}{h}\\
&= \lim_{h\to 0} \frac{h(-10t - 5h)}{h}\\
&= \lim_{h\to 0} (-10t - 5h)\\
&= -10t
\end{aligned}$$

$$\begin{aligned}
\therefore f'(t) &= -10t\\
f'(-1) &= -10(-1)\\
&= 10
\end{aligned}$$

5. **a)**
$$\begin{aligned}
f(x) &= -2ax^2 + 2\\
f(x+h) &= -2a(x+h)^2 + 2\\
&= -2a(x^2 + 2xh + h^2) + 2\\
&= -2ax^2 - 4axh - 2ah^2 + 2
\end{aligned}$$

$$\begin{aligned}
&\lim_{h\to 0} \frac{-2ax^2 - 4axh - 2ah^2 + 2 + 2ax^2 - 2}{h}\\
&= \lim_{h\to 0} \frac{h(-4ax - 2ah)}{h}\\
&= \lim_{h\to 0} (-4ax - 2ah)\\
&= -4ax
\end{aligned}$$

b) **(i)**
$$\begin{aligned}
f(x) &= -2ax^2 + 2\\
f(4) &= -2a(4)^2 + 2\\
&= -32a + 2
\end{aligned}$$

(ii)
$$\begin{aligned}
f'(x) &= -4ax\\
f'(4) &= -4a(4)\\
&= -16a
\end{aligned}$$

(iii)
$$\begin{aligned}
f'(x) &= -4ax\\
f'(-3) &= -4a(-3)\\
&= 12a
\end{aligned}$$

Assessment activity 2.11

1. **a)**
$$\begin{aligned}
y &= -4x^2\\
\frac{dy}{dx} &= -8x
\end{aligned}$$

b)
$$\begin{aligned}
f(x) &= \frac{3}{x}\\
f(x) &= 3x^{-1}\\
f'(x) &= -3x^{-2}\\
&= -\frac{3}{x^2}
\end{aligned}$$

c)
$$\begin{aligned}
f(x) &= -7\\
f'(x) &= 0
\end{aligned}$$

d) $\frac{d}{dx}(4a^2) = 0$

e)
$$\begin{aligned}
&\frac{d}{dp}\left(p^{-3} - \frac{p^2}{2} + \pi^2\right)\\
&= \frac{d}{dp}\left(p^{-3} - \frac{1}{2}p^2 + \pi^2\right)\\
&= -3p^{-4} - 2\left(\frac{1}{2}\right)p + 0\\
&= -\frac{3}{p^4} - p
\end{aligned}$$

2. **a)**

$$\begin{aligned} y &= -3x^{-3} + x^2 + 1 - 2a + \frac{x^2}{4} \\ y &= -3x^{-3} + x^2 + 1 - 2a + \frac{1}{4}x^2 \\ \frac{dy}{dx} &= 9x^{-4} + 2x + 0 - 0 + \frac{1}{2}x \\ &= \frac{9}{x^4} + 2x + \frac{1}{2}x \end{aligned}$$

b)

$$\begin{aligned} y &= 2x - \frac{4}{x^2} + x^2m - \frac{7}{2x} + m^2x \\ y &= 2x - 4x^{-2} + x^2m - \frac{7}{2}x^{-1} + m^2x \\ \frac{dy}{dx} &= 2 + 8x^{-3} + 2xm + \frac{7}{2}x^{-2} + m^2 \\ &= 2 + \frac{8}{x^3} + 2xm + \frac{7}{2x^2} + m^2 \end{aligned}$$

c)

$$\begin{aligned} f(x) &= 6x(3x^2 - 4) \\ f(x) &= 18x^3 - 24x \\ f'(x) &= 54x^2 - 24 \end{aligned}$$

d)

$$\begin{aligned} f(x) &= \frac{x^2 + x - 6}{x - 2} \\ f(x) &= \frac{(x + 3)(x - 2)}{(x - 2)} \\ f(x) &= x + 3 \\ f'(x) &= 1 \end{aligned}$$

e)

$$\begin{aligned} f(x) &= \sqrt{x}\left(x^4 - \frac{1}{3x^2}\right) \\ f(x) &= x^{\frac{1}{2}}\left(x^4 - \frac{1}{3}x^{-2}\right) \\ f(x) &= x^{\frac{9}{2}} - \frac{1}{3}x^{-\frac{3}{2}} \\ f'(x) &= \frac{9}{2}x^{\frac{9}{2}-1} - \frac{1}{3}\left(-\frac{3}{2}\right)x^{-\frac{3}{2}-1} \\ &= \frac{9}{2}x^{\frac{7}{2}} + \frac{1}{2}x^{-\frac{5}{2}} \\ &= \frac{9}{2}x^{\frac{7}{2}} + \frac{1}{2x^{\frac{5}{2}}} \\ &= \frac{9}{2}\sqrt{x^7} + \frac{1}{2\sqrt{x^5}} \end{aligned}$$

f)

$$\begin{aligned} y &= \frac{1}{3\sqrt[3]{x}} - \frac{1}{x\sqrt{x}} + x^0 \\ y &= \frac{1}{3}x^{-\frac{1}{3}} - \frac{1}{x^{\frac{3}{2}}} + 1 \\ y &= \frac{1}{3}x^{-\frac{1}{3}} - x^{-\frac{3}{2}} + 1 \\ \frac{dy}{dx} &= -\frac{1}{9}x^{-\frac{4}{3}} + \frac{3}{2}x^{\frac{5}{2}} \\ &= -\frac{1}{9\sqrt[3]{x^4}} + \frac{3}{2\sqrt{x^5}} \\ &= -\frac{1}{9x\sqrt[3]{x}} + \frac{3}{2x^2\sqrt{x}} \end{aligned}$$

g)

$$\begin{aligned} f(x) &= \frac{2x^2 - 5x + 1}{5x^2} \\ f(x) &= \frac{2}{5} - x^{-1} + \frac{1}{5}x^{-2} \\ f'(x) &= x^{-2} - \frac{2}{5}x^{-3} \\ &= \frac{1}{x^2} - \frac{2}{5x^3} \end{aligned}$$

h)

$$\begin{aligned} f(x) &= \frac{(x + 1)(x + 3)}{x^2} \\ &= \frac{x^2 + 4x + 3}{x^2} \\ &= 1 + 4x^{-1} + 3x^{-2} \\ \therefore f'(x) &= -4x^{-2} - 6x^{-3} \\ &= -\frac{4}{x^2} - \frac{6}{x^3} \end{aligned}$$

Assessment activity 2.12

1. **a)**

$$\begin{aligned} y &= 2e^x + e^{4x} \\ \frac{dy}{dx} &= 2e^x + 4e^{4x} \end{aligned}$$

b)

$$\begin{aligned} y &= e^{-\frac{x}{4}} \\ \frac{dy}{dx} &= -\frac{1}{4}e^{-\frac{x}{4}} \end{aligned}$$

c)

$$\begin{aligned} y &= e^{3x} - e^{-\frac{x}{2}} \\ \frac{dy}{dx} &= 3e^{3x} + \frac{1}{2}e^{-\frac{x}{2}} \end{aligned}$$

d)

$$\begin{aligned} y &= -3e^{-2x} + 3x^5 - \frac{1}{\sqrt{x}} - \frac{x^2}{4} \\ y &= -3e^{-2x} + 3x^5 - x^{-\frac{1}{2}} - \frac{1}{4}x^2 \\ \frac{dy}{dx} &= -2(-3e^{-2x}) + 15x^4 + \frac{1}{2}x^{-\frac{3}{2}} - \frac{1}{2}x \\ &= \frac{6}{e^{2x}} + 15x^4 + \frac{1}{2\sqrt{x^3}} - \frac{1}{2}x \end{aligned}$$

e)

$$\begin{aligned} y &= x^e + \pi x^2 - ae^{4x} + e^2 \\ \frac{dy}{dx} &= ex^{e-1} + 2\pi x^2 - 4ae^{4x} \end{aligned}$$

f)

$$\begin{aligned} y &= e^{5x} + \frac{1}{3e^{-x}} - \frac{x}{3} + 6 \\ y &= e^{5x} + \frac{1}{3}e^x - \frac{1}{3}x + 6 \\ \frac{dy}{dx} &= 5e^{5x} + \frac{1}{3}e^x - \frac{1}{3} \end{aligned}$$

2. **a)**
$$\begin{aligned} y &= 3 \sin x \\ \frac{dy}{dx} &= 3 \cos x \end{aligned}$$

b)
$$\begin{aligned} y &= 2 \cos 3x \\ \frac{dy}{dx} &= -6 \sin 3x \end{aligned}$$

c)
$$\begin{aligned} y &= 5 \sin 2x - 5 \cos 3x \\ \frac{dy}{dx} &= 10 \cos 2x + 15 \sin 3x \end{aligned}$$

d)
$$\begin{aligned} y &= -\frac{1}{2} \sin 2x + 3 \cos \pi x - \sin \frac{x}{4} \\ \frac{dy}{dx} &= -\cos 2x - 3\pi \sin \pi x - \frac{1}{4} \cos \frac{x}{4} \end{aligned}$$

e)
$$\begin{aligned} y &= \frac{2}{x} + \sqrt[3]{x} + ae^{-2x} - 3 \cos \frac{x}{2} \\ y &= 2x^{-1} + x^{\frac{1}{3}} + ae^{-2x} - 3 \cos \frac{1}{2}x \\ \frac{dy}{dx} &= -2x^{-2} + \frac{1}{3}x^{-\frac{2}{3}} - 2ae^{-2x} + \frac{3}{2} \sin \frac{1}{2}x \\ &= \frac{-2}{x^2} + \frac{1}{3\sqrt[3]{x^2}} - 2a^{-2x} + \frac{3}{2} \sin \frac{x}{2} \end{aligned}$$

f)
$$\begin{aligned} y &= x^{\pi} + e^{-\frac{x}{2}} - \cos ax \\ \frac{dy}{dx} &= \pi x^{\pi - 1} - \frac{1}{2}e^{-\frac{x}{2}} + a \sin ax \\ &= \pi x^{\pi - 1} - \frac{1}{2e^{\frac{x}{2}}} + a \sin ax \end{aligned}$$

3. **a)**
$$\begin{aligned} f(x) &= 2 \ln x \\ f'(x) &= \frac{2}{x} \end{aligned}$$

b)
$$\begin{aligned} f(x) &= -3 \ln 6x \\ f'(x) &= -\frac{3}{x} \end{aligned}$$

c)
$$\begin{aligned} f(x) &= -e^{4x} + 4 \ln \frac{x}{2} \\ f'(x) &= -4e^{4x} + \frac{4}{x} \end{aligned}$$

d)
$$\begin{aligned} f(x) &= 2 \tan x \\ f'(x) &= 2 \sec^2 x \end{aligned}$$

e)
$$\begin{aligned} f(x) &= -2 \tan 3x + 3 \cos 2x \\ f'(x) &= -6 \sec^2 3x - 6 \sin 2x \end{aligned}$$

f)
$$\begin{aligned} f(x) &= \ln ax - 3 \tan \frac{x}{2} + 2\sqrt{x} - 4e^{-2x} \\ f(x) &= \ln ax - 3 \tan \frac{1}{2}x + 2x^{\frac{1}{2}} - 4e^{-2x} \\ f'(x) &= \frac{1}{x} - \frac{3}{2} \sec^2 \frac{x}{2} + \frac{1}{2} .2x^{-\frac{1}{2}} + 8e^{-2x} \\ &= \frac{1}{x} - \frac{3}{2} \sec^2 \frac{x}{2} + \frac{1}{\sqrt{x}} + \frac{8}{e^{2x}} \end{aligned}$$

g)
$$\begin{aligned} f(x) &= \frac{1}{2} \sin \frac{3bx}{a} + \frac{1}{2} \ln x - 1 + \frac{2}{x} - 3 \ln a \\ f(x) &= \frac{1}{2} \sin \frac{3b}{a}x + \frac{1}{2} \ln x - 1 + 2x^{-1} - 3 \ln a \\ f'(x) &= \frac{3b}{a} . \frac{1}{2} \cos \frac{3b}{a}x + \frac{\frac{1}{2}}{x} + (-1)2x^{-2} \\ &= \frac{3b}{2a} \cos \frac{3bx}{a} + \frac{1}{2x} - \frac{2}{x^2} \end{aligned}$$

Assessment activity 2.13

1. **a)**
$$\begin{aligned} y &= -3x^2(2x - 1) \\ \therefore \frac{dy}{dx} &= -3x^2(2) + (-6x)(2x - 1) \\ &= -6x^2 - 12x^2 + 6x \\ &= -18x^2 + 6x \end{aligned}$$

b)
$$\begin{aligned} y &= (4x^2 - 4)(2x + 1) \\ \therefore \frac{dy}{dx} &= (4x^2 - 4)(2) + (8x)(2x + 1) \\ &= 8x^2 - 8 + 16x^2 + 8x \\ &= 24x^2 + 8x - 8 \end{aligned}$$

c)
$$\begin{aligned} y &= \sqrt{x}(6x^2 - 3x + 2) \\ &= x^{\frac{1}{2}}(6x^2 - 3x + 2) \\ \therefore \frac{dy}{dx} &= x^{\frac{1}{2}}(12x - 3) + \frac{1}{2}x^{-\frac{1}{2}}(6x^2 - 3x + 2) \\ &= 12x^{\frac{3}{2}} - 3x^{\frac{1}{2}} + 3x^{\frac{3}{2}} - \frac{3}{2}x^{\frac{1}{2}} + x^{-\frac{1}{2}} \\ &= 15x^{\frac{3}{2}} - 4\frac{1}{2}x^{\frac{1}{2}} + x^{-\frac{1}{2}} \\ &= 15\sqrt{x^3} - \frac{9}{2}\sqrt{x} + \frac{1}{\sqrt{x}} \end{aligned}$$

d)
$$\begin{aligned} y &= e^{-x}.e^{2x} \\ \therefore \frac{dy}{dx} &= e^{-x}(2e^{2x}) + (-e^{-x})(e^{2x}) \\ &= 2e^{x} - e^{x} \\ &= e^{x} \end{aligned}$$

e)
$$\begin{aligned} y &= 3 \sin x.4 \cos 2x \\ \therefore \frac{dy}{dx} &= 3 \sin x\,(-8 \sin 2x) + (3 \cos x)(4 \cos 2x) \\ &= -24 \sin x.\sin 2x + 12 \cos x.\cos 2x \end{aligned}$$

f)

$$\begin{aligned} y &= x \cos x \\ \therefore \frac{dy}{dx} &= x(-\sin x) + (1)(\cos x) \\ &= -x \sin x + \cos x \end{aligned}$$

g)

$$\begin{aligned} y &= \frac{4x^2 - 2x + 3}{x^4} \\ y &= x^{-4}(4x^2 - 2x + 3) \\ \therefore \frac{dy}{dx} &= x^{-4}(8x - 2) + (-4x^{-5})(4x^2 - 2x + 3) \\ &= 8x^{-3} - 2x^{-4} - 16x^{-3} + 8x^{-4} - 12x^{-5} \\ &= -12x^{-5} + 6x^{-4} - 8x^{-3} \\ &= -\frac{12}{x^5} + \frac{6}{x^4} - \frac{8}{x^3} \end{aligned}$$

h)

$$\begin{aligned} y &= \frac{4}{5}\pi x^2 \left(x^2 + \frac{1}{2}\right) \\ \therefore \frac{dy}{dx} &= \frac{4}{5}\pi x^2(2x) + \left(\frac{8}{5}\pi x\right)\left(x^2 + \frac{1}{2}\right) \\ &= \frac{8}{5}\pi x^3 + \frac{8}{5}\pi x^3 + \frac{4}{5}\pi x \\ &= \frac{16}{5}\pi x^3 + \frac{4}{5}\pi x \end{aligned}$$

i)

$$\begin{aligned} y &= x \sin x \\ \therefore \frac{dy}{dx} &= x(\cos x) + 1(\sin x) \\ &= x \cos x + \sin x \end{aligned}$$

j)

$$\begin{aligned} y &= 2 \tan \frac{x}{5}.\ \ln 3x \\ \therefore \frac{dy}{dx} &= \left(2 \tan \frac{x}{5}\right)\left(\frac{1}{x}\right) + \left(\frac{2}{5}\sec^2 \frac{x}{5}\right)(\ln 3x) \\ \text{or} \quad &= \frac{2 \tan \frac{x}{5}}{x} + \left(\frac{2}{5}\sec^2 \frac{x}{5}\right)(\ln 3x) \end{aligned}$$

k)

$$\begin{aligned} y &= e^x.\tan x \\ \therefore \frac{dy}{dx} &= e^x(\sec^2 x) + e^x(\tan x) \end{aligned}$$

l)

$$\begin{aligned} y &= 3 \ln 5x.e^{-\frac{x}{2}} \\ \therefore \frac{dy}{dx} &= 3 \ln 5x\left(-\frac{1}{2}e^{-\frac{x}{2}}\right) + \left(\frac{3}{x}\right)\left(e^{-\frac{x}{2}}\right) \\ &= \frac{-\frac{3}{2}\ln 5x}{e^{\frac{x}{2}}} + \frac{\frac{3}{x}}{e^{\frac{x}{2}}} \\ \text{or} \quad &= \frac{-3 \ln 5x}{2e^{\frac{x}{2}}} + \frac{3}{xe^{\frac{x}{2}}} \end{aligned}$$

2. a)

$$\begin{aligned} y &= \sqrt[3]{t}\,(t^2 - 4t + 1) \\ y &= t^{\frac{1}{3}}(t^2 - 4t + 1) \\ \therefore \frac{dy}{dt} &= t^{\frac{1}{3}}(2t - 4) + \left(\frac{1}{3}t^{-\frac{2}{3}}\right)(t^2 - 4t + 1) \\ &= 2t^{\frac{4}{3}} - 4t^{\frac{1}{3}} + \frac{1}{3}t^{\frac{4}{3}} - \frac{4}{3}t^{\frac{1}{3}} + \frac{1}{3}t^{-\frac{2}{3}} \\ &= \frac{7}{3}t^{\frac{4}{3}} - \frac{16}{3}t^{\frac{1}{3}} + \frac{1}{3\sqrt[3]{t^2}} \\ &= \frac{7}{3}\sqrt[3]{t^4} - \frac{16}{3}\sqrt[3]{t} + \frac{1}{3\sqrt[3]{t^2}} \end{aligned}$$

b)

$$\begin{aligned} f(z) &= (3 - 4z)(4 - z + 3z^2) \\ f'(z) &= (3 - 4z)(-1 + 6z) + (-4)(4 - z + 3z^2) \\ &= -3 + 18z + 4z - 24z^2 - 16 + 4z - 12z^2 \\ &= -36z^2 + 26z - 19 \end{aligned}$$

c)

$$\begin{aligned} f(x) &= e^{-3x}.x^2 \\ f'(x) &= e^{-3x}(2x) + (-3e^{-3x})(x^2) \\ &= 2xe^{-3x} - 3x^2e^{-3x} \end{aligned}$$

d)

$$\begin{aligned} y &= e^{-t}.\sin 3t \\ \therefore \frac{dy}{dt} &= e^{-t}(3 \cos 3t) + (-e^{-t})(\sin 3t) \\ &= 3e^{-t}.\cos 3t - e^{-t}.\sin 3t \\ &= \frac{3 \cos 3t}{e^t} - \frac{\sin 3t}{e^t} \\ \text{or} \quad &= \frac{3 \cos 3t - \sin 3t}{e^t} \end{aligned}$$

e)

$$\begin{aligned} y &= \sin x(x^2 - 1) \\ \therefore \frac{dy}{dx} &= \sin x(2x) + (\cos x)(x^2 - 1) \\ &= 2x \sin x + x^2 \cos x - \cos x \end{aligned}$$

f)

$$\begin{aligned} y &= 3x^2.\cos 2x \\ \therefore \frac{dy}{dx} &= 3x^2(-2 \sin 2x) + (6x)(\cos 2x) \\ &= -6x^2 \sin 2x + 6x \cos 2x \end{aligned}$$

g)

$$r = t \ln t$$
$$\therefore \frac{dr}{dt} = t\left(\frac{1}{t}\right) + 1(\ln t)$$
$$= 1 + \ln t$$

h)

$$s = r^4.\tan 2r$$
$$\therefore \frac{ds}{dr} = r^4(2\sec^2 2r) + (4r^3)(\tan 2r)$$
$$= 2r^4 \sec^2 2r + 4r^3 \tan 2r$$

i)

$$y = r \ln x$$
$$\therefore \frac{dy}{dx} = \frac{r}{x}$$

j)

$$y = r \ln x$$
$$\therefore y = (\ln x)r$$
$$\therefore \frac{dy}{dr} = \ln x$$

k)

$$y = (x^2 - 24)(2 \ln 2x)$$
$$\therefore \frac{dy}{dx} = (x^2 - 24)\left(\frac{2}{x}\right) + 2x(2 \ln x)$$
$$= (x^2 - 24)\left(\frac{2}{x}\right) + 4x \ln x$$
$$= 2x - \frac{48}{x} + 4x \ln x$$

3.

$$y = x^3e^{2x}$$
$$\therefore \frac{dy}{dx} = 2x^3(2e^{2x}) + (3x^2)(e^{2x})$$
$$= 2x^3e^{2x} + 3x^2e^{2x}$$
$$= 2\left(\frac{1}{2}\right)^3 e^{2\left(\frac{1}{2}\right)} + 3\left(\frac{1}{2}\right)^2 e^{2\left(\frac{1}{2}\right)}$$
$$= \frac{1}{4}e + 3\left(\frac{1}{4}\right).e$$
$$= \frac{1}{4}e + \frac{3}{4}e$$
$$= e\left(\frac{1}{4} + \frac{3}{4}\right)$$
$$= e \text{ or } 2{,}718$$

Assessment activity 2.14

1. a)

$$y = \frac{3x - 2}{3x^2 - 4x}$$
$$\frac{dy}{dx} = \frac{(3x^2 - 4x)(3) - (3x - 2)(6x - 4)}{(3x^2 - 4x)^2}$$
$$= \frac{9x^2 - 12x - (18x^2 - 12x - 12x + 8)}{(3x^2 - 4x)^2}$$
$$= \frac{9x^2 - 12x - 18x^2 + 24x - 8}{(3x^2 - 4x)^2}$$
$$= \frac{-9x^2 + 12x - 8}{(3x^2 - 4x)^2}$$

b)

$$y = \frac{x^2 - x + 1}{\sqrt[3]{x}}$$
$$y = \frac{x^2 - x + 1}{x^{\frac{1}{3}}}$$
$$\frac{dy}{dx} = \frac{x^{\frac{1}{3}}(2x - 1) - (x^2 - x + 1)(\frac{1}{3}x^{-\frac{2}{3}})}{(x^{\frac{1}{3}})^2}$$
$$= \frac{2x^{\frac{4}{3}} - x^{\frac{1}{3}} - \frac{1}{3}x^{\frac{4}{3}} + \frac{1}{3}x^{\frac{1}{3}} - \frac{1}{3}x^{-\frac{2}{3}}}{x^{\frac{2}{3}}}$$
$$= \frac{\frac{5}{3}x^{\frac{4}{3}} - \frac{2}{3}x^{\frac{1}{3}} - \frac{1}{3}x^{-\frac{2}{3}}}{x^{\frac{2}{3}}}$$
$$= \frac{5}{3}x^{\frac{2}{3}} - \frac{2}{3}x^{-\frac{1}{3}} - \frac{1}{3}x^{-\frac{4}{3}}$$
$$= \frac{5}{3}\sqrt[3]{x^2} - \frac{2}{3\sqrt[3]{x}} - \frac{1}{3\sqrt[3]{x^4}}$$

c)

$$y = \frac{x^3 + 2x^2 - 2}{e^x}$$
$$\frac{dy}{dx} = \frac{e^x(3x^2 + 4x) - (x^3 + 2x^2 - 2)e^x}{(e^x)^2}$$
$$= \frac{e^x(3x^2 + 4x - x^3 - 2x^2 + 2)}{e^{2x}}$$
$$= \frac{-x^3 + x^2 + 4x + 2}{e^x}$$

d)

$$y = \frac{ax + b}{cx + d}$$
$$\frac{dy}{dx} = \frac{(cx + d)(a) - (ax + b)(c)}{(cx + d)^2}$$
$$= \frac{acx + ad - acx - bc}{(cx + d)^2}$$
$$= \frac{ad - bc}{(cx + d)^2}$$

e) $y = \frac{x}{\cos 2x}$

$$\begin{aligned}\frac{dy}{dx} &= \frac{(\cos 2x)(1) - x(-2\sin 2x)}{(\cos\ 2x)^2}\\ &= \frac{\cos 2x + 2x\sin 2x}{\cos^2 2x}\end{aligned}$$

f) $y = \frac{e^x - 2}{e^x + 2}$

$$\begin{aligned}\frac{dy}{dx} &= \frac{(e^x + 2)(e^x) - (e^x - 2)(e^x)}{(e^x + 2)^2}\\ &= \frac{e^{2x} + 2e^x - e^{2x} + 2e^x}{(e^x + 2)^2}\\ &= \frac{4e^x}{(e^x + 2)^2}\end{aligned}$$

g) $y = \frac{e^{2x}}{x^4}$

$$\begin{aligned}\frac{dy}{dx} &= \frac{x^4(2e^{2x}) - (e^{2x})(4x^3)}{(x^4)^2}\\ &= \frac{2x^4.e^{2x} - 4x^3.e^{2x}}{x^8}\\ &= \frac{x^3(2x.e^{2x} - 4e^{2x})}{x^8}\\ &= \frac{2xe^{2x} - 4e^{2x}}{x^5}\end{aligned}$$

h) $y = \frac{\pi^2}{x^3 - 2}$

$$\begin{aligned}\frac{dy}{dx} &= \frac{(x^3 - 2)(0) - \pi^2(3x^2)}{(x^3 - 2)^2}\\ &= \frac{0 - 3\pi^2x^2}{(x^3 - 2)^2}\\ &= \frac{-3\pi^2x^2}{(x^3 - 2)^2}\end{aligned}$$

i) $y = \frac{3\ln x}{x^3}$

$$\begin{aligned}\frac{dy}{dx} &= \frac{x^3(\frac{3}{x}) - (3\ln x)(3x^2)}{(x^3)^2}\\ &= \frac{3x^2 - 9x^2.\ln x}{x^6}\\ &= \frac{x^2(3 - 9\ln x)}{x^6}\\ &= \frac{3 - 9\ln x}{x^4}\end{aligned}$$

j) $y = \frac{-\tan 3x}{\ln 3x}$

$$\begin{aligned}\frac{dy}{dx} &= \frac{(\ln 3x)(-3\sec^2 3x) - (-\tan 3x)(\frac{1}{x})}{(\ln 3x)^2}\\ &= \frac{-3\ln 3x.\sec^2 3x + (\tan 3x)(\frac{1}{x})}{(\ln 3x)^2}\end{aligned}$$

2. **a)** $z = \frac{1}{\sin x}$

$$\begin{aligned}\frac{dz}{dx} &= \frac{\sin x(0) - (1)\cos x}{(\sin x)^2}\\ &= \frac{-\cos x}{\sin^2 x}\end{aligned}$$

b) $p = \frac{e^{-2u}}{1 + u}$

$$\begin{aligned}\frac{dp}{du} &= \frac{(1 + u)(-2e^{-2u}) - (e^{-2u})(1)}{(1 + u)^2}\\ &= \frac{-2e^{-2u} - 2ue^{-2u} - e^{-2u}}{(1 + u)^2}\\ &= \frac{-3e^{-2u} - 2ue^{-2u}}{(1 + u)^2}\end{aligned}$$

c) $s = \frac{t^2 - 2t + 3}{e^{-t}}$

$$\begin{aligned}\frac{ds}{dt} &= \frac{e^{-t}(2t - 2) - (t^2 - 2t + 3)(-e^{-t})}{(e^{-t})^2}\\ &= \frac{2te^{-t} - 2e^{-t} + t^2e^{-t} - 2te^{-t} + 3e^{-t}}{e^{-2t}}\\ &= \frac{e^{-t} + t^2e^{-t}}{e^{-2t}}\\ &= e^t + t^2e^t\end{aligned}$$

d) $f(x) = \frac{3x - 1}{-3\cos\frac{x}{2}}$

$$\begin{aligned}f'(x) &= \frac{-3\cos\frac{x}{2}(3) - (3x - 1)(\frac{3}{2}\sin\frac{x}{2})}{(-3\cos\frac{x}{2})^2}\\ &= \frac{-9\cos\frac{x}{2} - \frac{9}{2}x\sin\frac{x}{2} + \frac{3}{2}\sin\frac{x}{2}}{9\cos^2\frac{x}{2}}\end{aligned}$$

e) $f(x) = \frac{e^{2x} + 1}{2x^2 - 4}$

$$\begin{aligned}f'(x) &= \frac{(2x^2 - 4)(2e^{2x}) - (e^{2x} + 1)(4x)}{(2x^2 - 4)^2}\\ &= \frac{4x^2e^{2x} - 8e^{2x} - 4xe^{2x} - 4x}{(2x^2 - 4)^2}\end{aligned}$$

f) $y = \frac{4\ln 3t}{t}$

$$\begin{aligned}\frac{dy}{dt} &= \frac{t(\frac{4}{t}) - (4\ln 3t)(1)}{t^2}\\ &= \frac{4 - 4\ln 3t}{t^2}\end{aligned}$$

g)

$$y = \frac{4}{3\tan 4x}$$

$$\frac{dy}{dx} = \frac{(3\tan 4x)(0) - 4(12\sec^2 4x)}{(3\tan 4x)^2}$$

$$= \frac{-48\sec^2 4x}{9\tan^2 4x}$$

3.

$$y = \frac{x^2}{1-3x}$$

$$\frac{dy}{dx} = \frac{(1-3x)(2x) - x^2(-3)}{(1-3x)^2}$$

$$= \frac{2x - 6x^2 + 3x^2}{(1-3x)^2}$$

$$= \frac{2x - 3x^2}{(1-3x)^2}$$

$$\therefore f'(-2) = \frac{2(-2) - 3(-2)^2}{[1-3(-2)]^2}$$

$$= \frac{-4-12}{[7]^2}$$

$$= \frac{-16}{49}$$

$$= -0{,}327$$

Assessment activity 2.15

1. a)

$$y = (x^3 + x^2 - 1)^5$$

$$\therefore y = u^5 \quad \text{where} \quad u = x^3 + x^2 - 1$$

$$\frac{dy}{du} = 5u^4 \qquad \frac{du}{dx} = 3x^2 + 2x$$

$$\therefore \frac{dy}{dx} = 5u^4.(3x^2 + 2x)$$

$$= 5(x^3 + x^2 - 1)^4(3x^2 + 2x)$$

b)

$$y = \sqrt{-x^2 + 2x - 4}$$

$$y = (-x^2 + 2x - 4)^{\frac{1}{2}}$$

$$\therefore y = u^{\frac{1}{2}} \quad \text{where} \quad u = -x^2 + 2x - 4$$

$$\frac{dy}{du} = \frac{1}{2}u^{-\frac{1}{2}} \qquad \frac{du}{dx} = -2x + 2$$

$$\therefore \frac{dy}{dx} = \frac{1}{2}u^{-\frac{1}{2}}.\ (-2x + 2)$$

$$= \frac{1}{2}(-x^2 + 2x - 4)^{-\frac{1}{2}}(-2x + 2)$$

$$= \frac{-\frac{1}{2}(-2x + 2)}{(-x^2 + 2x - 4)^{\frac{1}{2}}}$$

$$= \frac{x-1}{\sqrt{-x^2 + 2x - 4}}$$

c)

$$y = 3e^{-2x}$$

$$y = 3e^{u} \quad \text{where} \quad u = -2x$$

$$\frac{dy}{du} = 3e^{u} \qquad \frac{du}{dx} = -2$$

$$\therefore \frac{dy}{dx} = 3e^{u}.(-2)$$

$$= 3e^{-2x}(-2)$$

$$= -6e^{-2x}$$

d)

$$y = e^{2x-3}$$

$$y = e^{u} \quad \text{where} \quad u = 2x - 3$$

$$\frac{dy}{du} = e^{u} \qquad \frac{du}{dx} = 2$$

$$\therefore \frac{dy}{dx} = e^{u}.2$$

$$= e^{2x-3}.2$$

$$= 2e^{2x-3}$$

e)

$$y = \frac{3}{1-6x}$$

$$y = 3(1-6x)^{-1}$$

$$y = 3u^{-1} \quad \text{where} \quad u = 1 - 6x$$

$$\frac{dy}{du} = -3u^{-2} \qquad \frac{du}{dx} = -6$$

$$\therefore \frac{dy}{dx} = -3u^{-2}(-6)$$

$$= -3(1-6x)^{-2}(-6)$$

$$= \frac{18}{(1-6x)^2}$$

f)

$$\begin{aligned}
y &= \sqrt[3]{(1-3x^3)^2} \\
y &= (1-3x^3)^{\frac{2}{3}} \\
y &= u^{\frac{2}{3}} \qquad \text{where} \quad u = 1-3x^3 \\
\frac{dy}{du} &= \frac{2}{3}u^{-\frac{1}{3}} \qquad \frac{du}{dx} = -9x^2 \\
\therefore \frac{dy}{dx} &= \frac{2}{3}u^{-\frac{1}{3}}.(-9x^2) \\
&= \frac{2}{3}(1-3x^3)^{-\frac{1}{3}}.(-9x^2) \\
&= \frac{-6x^2}{\sqrt[3]{1-3x^3}}
\end{aligned}$$

g)

$$\begin{aligned}
y &= \frac{1}{\sqrt{4x^3}} \\
y &= (4x^3)^{-\frac{1}{2}} \\
y &= u^{-\frac{1}{2}} \qquad \text{where} \quad u = 4x^3 \\
\frac{dy}{du} &= -\frac{1}{2}u^{-\frac{3}{2}} \qquad \frac{du}{dx} = 12x^2 \\
\therefore \frac{dy}{dx} &= -\frac{1}{2}u^{-\frac{3}{2}}.(12x^2) \\
&= -\frac{1}{2}(4x^3)^{-\frac{3}{2}}.12x^2 \\
&= \frac{-6x^2}{\sqrt{(4x^3)^3}} \\
&= \frac{-6x^2}{\sqrt{4^3}\,x^{\frac{9}{2}}} \\
&= \frac{-6x^2}{8x^{\frac{9}{2}}} \\
&= -\frac{3}{4}x^{2-\frac{9}{2}} \\
&= -\frac{3}{4}x^{-\frac{5}{2}} \\
&= \frac{-3}{4\sqrt{x^5}}
\end{aligned}$$

h)

$$\begin{aligned}
y &= \cos^2 x \\
y &= (\cos x)^2 \\
y &= u^2 \qquad \text{where} \quad u = \cos x \\
\frac{dy}{du} &= 2u \qquad \frac{du}{dx} = -\sin x \\
\therefore \frac{dy}{dx} &= 2u.(-\sin x) \\
&= 2\cos x(-\sin x) \\
&= -2\sin x\cos x \\
\text{or} \quad &= -\sin 2x
\end{aligned}$$

i)

$$\begin{aligned}
y &= \sqrt{\sin 3x} \\
y &= (\sin 3x)^{\frac{1}{2}} \\
y &= u^{\frac{1}{2}} \qquad \text{where} \quad u = \sin 3x \\
\frac{dy}{du} &= \frac{1}{2}u^{-\frac{1}{2}} \qquad \frac{du}{dx} = 3\cos 3x \\
\therefore \frac{dy}{dx} &= \frac{1}{2}u^{-\frac{1}{2}}.3\cos 3x \\
&= \frac{1}{2}(\sin 3x)^{-\frac{1}{2}}(3\cos 3x) \\
&= \frac{\frac{3}{2}\cos 3x}{\sqrt{\sin 3x}} \\
\text{or} \quad &= \frac{3\cos 3x}{2\sqrt{\sin 3x}}
\end{aligned}$$

j)

$$\begin{aligned}
y &= \frac{1}{\sqrt[3]{3\cos 2x}} \\
y &= (3\cos 2x)^{-\frac{1}{3}} \\
y &= u^{-\frac{1}{3}} \qquad \text{where} \quad u = 3\cos 2x \\
\frac{dy}{du} &= \frac{-1}{3}u^{-\frac{4}{3}} \qquad \frac{du}{dx} = -6\sin 2x \\
\therefore \frac{dy}{dx} &= \frac{-1}{3}u^{-\frac{4}{3}}.(-6\sin 2x) \\
&= \frac{-1}{3}(3\cos 2x)^{-\frac{4}{3}}.(-6\sin 2x) \\
&= \frac{+2\sin 2x}{\sqrt[3]{(3\cos 2x)^4}}
\end{aligned}$$

k)

$$\begin{aligned}
y &= \ln(6x - x^3) \\
y &= \ln u \qquad u = 6x - x^3 \\
\therefore \frac{dy}{du} &= \frac{1}{u} \qquad \frac{du}{dx} = 6 - 3x^2 \\
\therefore \frac{dy}{dx} &= \frac{1}{u}(6-3x^2) \\
&= \left(\frac{1}{6x-6x^3}\right)(6-3x^2) \\
&= \frac{6-3x^2}{6x-x^3}
\end{aligned}$$

l)

$$\begin{aligned} y &= (\tan 3x)^4 \\ y &= u^4 \qquad u = \tan 3x \\ \frac{dy}{du} &= 4u^3 \qquad \frac{du}{dx} = 3\sec^2 3x \\ \therefore \frac{dy}{dx} &= 4u^3.3\sec^2 3x \\ &= 4(\tan 3x)^3.3\sec^2 3x \\ &= 12\tan^3 3x.\sec^2 3x \end{aligned}$$

2. **a)**

$$\begin{aligned} p &= 2e^{1-3x} \\ p &= 2e^u \quad \text{where } u = 1 - 3x \\ \frac{dp}{du} &= 2e^u \qquad \frac{du}{dx} = -3 \\ \therefore \frac{dp}{dx} &= 2e^u.(-3) \\ &= 2e^{1-3x}(-3) \\ &= -6e^{1-3x} \end{aligned}$$

b)

$$\begin{aligned} z &= \frac{1}{4w^2 - 3w + 1} \\ &= (4w^2 - 3w + 1)^{-1} \\ z &= u^{-1} \quad \text{where } u = 4w^2 - 3w + 1 \\ \frac{dz}{du} &= -u^{-2} \qquad \frac{du}{dw} = 8w - 3 \\ \therefore \frac{dz}{dw} &= -u^{-2}.(8w - 3) \\ &= -(4w^2 - 3w + 1)^{-2}(8w - 3) \\ &= \frac{-(8w - 3)}{(4w^2 - 3w + 1)^2} \end{aligned}$$

c)

$$\begin{aligned} y &= 3\sin^3\frac{x}{2} \\ y &= 3\left(\sin\frac{x}{2}\right)^3 \\ y &= 3u^3 \quad \text{where } u = \sin\frac{x}{2} \\ \frac{dy}{du} &= 9u^2 \qquad \frac{du}{dx} = \frac{1}{2}\cos\frac{x}{2} \\ \therefore \frac{dy}{dx} &= 9u^2\left(\frac{1}{2}\cos\frac{x}{2}\right) \\ &= 9\left(\sin\frac{x}{2}\right)^2\left(\frac{1}{2}\cos\frac{x}{2}\right) \\ &= \frac{9}{2}\sin^2\frac{x}{2}.\cos\frac{x}{2} \end{aligned}$$

d)

$$\begin{aligned} y &= \frac{2}{3\sqrt[3]{\cos x}} \\ y &= \frac{2}{3}(\cos x)^{-\frac{1}{3}} \\ y &= \frac{2}{3}u^{-\frac{1}{3}} \quad \text{where } u = \cos x \\ \frac{dy}{du} &= -\frac{2}{9}u^{-\frac{4}{3}} \qquad \frac{du}{dx} = -\sin x \\ \therefore \frac{dy}{dx} &= -\frac{2}{9}u^{-\frac{4}{3}}(-\sin x) \\ &= -\frac{2}{9}(\cos x)^{-\frac{4}{3}}.(-\sin x) \\ &= \frac{2\sin x}{9\sqrt[3]{(\cos x)^4}} \end{aligned}$$

e)

$$\begin{aligned} y &= \sin x^4 \\ y &= \sin u \qquad u = x^4 \\ \frac{dy}{du} &= \cos u \qquad \frac{du}{dx} = 4x^3 \\ \therefore \frac{dy}{dx} &= (\cos u).(4x^3) \\ &= (\cos x^4).(4x^3) \\ &= 4x^3.\cos x^4 \end{aligned}$$

f)

$$\begin{aligned} s &= (\ln 2t)^3 \\ s &= u^3 \qquad u = \ln 2t \\ \frac{ds}{du} &= 3u^2 \qquad \frac{du}{dt} = \frac{1}{t} \\ \therefore \frac{ds}{dt} &= 3u^2.\frac{1}{t} \\ &= 3(\ln 2t).\frac{1}{t} \\ &= \frac{3\ln 2t}{t} \end{aligned}$$

g)

$$\begin{aligned} r &= \sqrt[3]{\tan x} \\ r &= (\tan x)^{\frac{1}{3}} \\ r &= u^{\frac{1}{3}} \qquad u = \tan x \\ \frac{dr}{du} &= \frac{1}{3}u^{-\frac{2}{3}} \qquad \frac{du}{dx} = \sec^2 x \\ \therefore \frac{dr}{dx} &= \frac{1}{3}u^{-\frac{2}{3}}(\sec^2 x) \\ &= \frac{1}{3}(\tan x)^{-\frac{2}{3}}(\sec^2 x) \\ &= \frac{\sec^2 x}{3\sqrt[3]{\tan^2 x}} \\ \text{or} \quad &= \frac{\sec^2 x}{3\sqrt[3]{(\tan x)^2}} \end{aligned}$$

h)

$$\begin{aligned} x &= \ln(2y^2 - 3y) \\ x &= \ln u \qquad u = 2y^2 - 3y \\ \frac{dx}{du} &= \frac{1}{u} \qquad \frac{du}{dy} = 4y - 3 \\ \therefore \frac{dx}{dy} &= \frac{1}{u}(4y - 3) \\ &= \left(\frac{1}{2y^2 - 3y}\right)(4y - 3) \\ &= \frac{4y - 3}{2y^2 - 3y} \end{aligned}$$

3. a) $y = \frac{x}{\ln 3x}$

$$\frac{dy}{dx} = \frac{\ln 3x(1) - x(\frac{1}{x})}{(\ln 3x)^2}$$

$$= \frac{\ln 3x - 1}{(\ln 3x)^2}$$

b) $y = x^3.\cos 2x$

$$\frac{dy}{dx} = x^3(-2 \sin 2x) - (3x^2) \cos 2x$$

$$= -2x^3 \sin 2x - 3x^2 \cos 2x$$

c) $y = (-x^3 - 4x)^5$

$$\frac{dy}{dx} = 5(-x^3 - 4x)^4 (-3x^2 - 4)$$

d) $y = 2 \cos x^5$

$y = 2 \cos u$ $\qquad u = x^5$

$\frac{dy}{du} = -2 \sin u$ $\qquad \frac{dy}{du} = 5x^4$

$$\frac{dy}{dx} = -2 \sin u(5x^4)$$

$$= -2 \sin x^5 (5x^4)$$

$$= -10x^4 \sin x^5$$

e) $y = (-3x^2 - 1) \ln x$

$$\frac{dy}{dx} = (-3x^2 - 1)\left(\frac{1}{x}\right) - (-6x)(\ln x)$$

$$= -\frac{3}{x} - \frac{1}{x} + 6x \ln x$$

$$= \frac{-3 - 1}{x} + 6x \ln x$$

$$= \frac{-4}{x} + 6x \ln x$$

f) $y = x^6 \tan 3x$

$$\frac{dy}{dx} = x^6(3 \sec^2 3x) + 6x^5(\tan 3x)$$

$$= 3x^6 \sec^2 3x + 6x^5 \tan 3x$$

g) $y = \frac{6x^2 - 4x + 1}{x^4}$

$$y = \frac{6}{x^2} - \frac{4}{x^3} + \frac{1}{x^4}$$

$$y = 6x^{-2} - 4x^{-3} + x^{-4}$$

$$\frac{dy}{dx} = -12x^{-3} + 12x^{-4} - 4x^{-5}$$

$$= -\frac{12}{x^3} + \frac{12}{x^4} - \frac{4}{x^5}$$

- (or use quotient rule)

h) $y = e^{-3x}.2 \ln 4x$

$$\frac{dy}{dx} = e^{-3x}.\frac{2}{x} + (-3)e^{-3x}.2 \ln 4x$$

$$= e^{-3x}.\frac{2}{x} - 6e^{-3x}.\ 2 \ln 4x$$

$$= \frac{2}{x.e^{3x}} - \frac{6.\ln 4x}{e^{3x}}$$

Assessment activity 2.16

1. **a)**

$y = -x^2 + 3x - 6$

$f'(x) = -2x + 3$

$f'(2) = -2(2) + 3$

$= -1$

$f(x) = -x^2 + 3x - 6$

$f(2) = -(2)^2 + 3(2) - 6$

$= -4 + 6 - 6$

$= -4$

$\therefore (2; -4)$

$y = mx + c$

$y = -x + c$

$-4 = -(2) + c$

$c = -2$

$\therefore y = -x - 2$

b)

$y = (x - 1)^2 - 9$

$y = (x - 1)(x - 1) - 9$

$y = x^2 - 2x + 1 - 9$

$y = x^2 - 2x - 8$

$f'(x) = 2x - 2$

$f'(-2) = 2(-2) - 2$

$= -6$

$f(-2) = (-2)^2 -2(-2) - 8$

$= 0$

$\therefore (-2; 0)$

$y = -6x + c$

$0 = -6(-2) + c$

$c = -12$

$\therefore y = -6x - 12$

c)

$y = 2x^3 - 21x^2 + 59x + 20$

$f'(x) = 6x^2 - 42x + 59$

$f'(1) = 6(1)^2 - 42(1) + 59$

$= 23$

$f(x) = 2x^3 - 21x^2 + 59x + 20$

$f(1) = 2(1)^3 - 21(1)^2 + 59(1) + 20$

$= 60$

$\therefore (1; 60)$

$y = mx + c$

$y = 23x + c$

$60 = 23(1) + c$

$c = 37$

$\therefore y = 23x + 37$

d)

$y = \frac{9x^2 - 1}{3x - 1}$

$y = \frac{(3x - 1)(3x + 1)}{(3x - 1)}$

$\therefore y = (3x + 1)$

$f'(x) = 3$

$f(2) = 3(2) + 1$

$= 7$

$\therefore (2; 7)$

$y = 3x + c$

$7 = 3(2) + c$

$c = 1$

$\therefore y = 3x + 1$

e)

$y = 4 - x^2$

$f'(x) = -2x$

$f'\left(\frac{1}{2}\right) = -2\left(\frac{1}{2}\right)$

$= -1$

$f(x) = 4 - x^2$

$f\left(\frac{1}{2}\right) = 4 - \left(\frac{1}{2}\right)^2$

$= 3\frac{3}{4} = \frac{15}{4}$

$\therefore \left(\frac{1}{2}; \frac{15}{4}\right)$

$\therefore y = -x + c$

$\frac{15}{4} = -\frac{1}{2} + c$

$c = 4\frac{1}{4} = 4{,}25$

$\therefore y = -x + 4\frac{1}{4}$

2. a)

$$\begin{aligned} y &= 2x^3 - 3x^2 + x - 1 \\ f'(x) &= 6x^2 - 6x + 1 \\ f'(-1) &= 6(-1)^2 - 6(-1) + 4 \\ &= 6 + 6 + 1 \\ &= 13 \end{aligned}$$

b)

$$\begin{aligned} f(x) &= 2x^3 - 3x^2 + x - 1 \\ f(-1) &= 2(-1)^3 - 3(-1)^2 + (-1) - 1 \\ &= -2 - 3 - 1 - 1 \\ &= -7 \end{aligned}$$

$\therefore (-1; -7)$

$$\begin{aligned} y &= 13x + c \\ -7 &= 13(-1) + c \\ c &= 6 \end{aligned}$$

$\therefore y = 13x + 6$

3. a)

$$\begin{aligned} f(x) &= 4x^2 - 3 \\ f'(x) &= 8x \\ f'(3) &= 8(3) \\ &= 24 \end{aligned}$$

$$\begin{aligned} f(x) &= 4x^2 - 3 \\ f(3) &= 4(3)^2 - 3 \\ &= 33 \end{aligned}$$

$\therefore (3; 33)$

$$\begin{aligned} y &= 24x + c \\ 33 &= 24(3) + c \\ c &= -39 \end{aligned}$$

$\therefore y = 24x - 39$

b)

$$\begin{aligned} f'(x) &= 8x \\ 32 &= 8x \\ x &= 4 \end{aligned}$$

$$\begin{aligned} \therefore f(x) &= 4x^2 - 3 \\ f(4) &= 4(4)^2 - 3 \\ &= 61 \end{aligned}$$

$\therefore$ At $(4; 61)$

4. a)

$$\begin{aligned} y - 3x &= 4 \\ y &= 3x + 4 \rightarrow m = 3 \end{aligned}$$

$$\begin{aligned} f(x) &= x^2 - 3x + 7 \\ f'(x) &= 2x - 3 \\ 3 &= 2x - 3 \\ 6 &= 2x \\ x &= 3 \end{aligned}$$

$$\begin{aligned} f(x) &= x^2 - 3x + 7 \\ f(3) &= (3)^2 - 3(3) + 7 \\ &= 7 \end{aligned}$$

$\therefore (3; 7)$

$$\begin{aligned} \therefore y &= mx + c \\ y &= 3x + c \\ 7 &= 3(3) + c \\ c &= -2 \end{aligned}$$

b)

$$\begin{aligned} f(x) &= 2x^2 - 4x + 2 \\ f'(x) &= 4x - 4 \\ \tfrac{1}{2} &= 4x - 4 \\ 4x &= \tfrac{9}{2} \\ x &= \tfrac{9}{8} \\ &= 1\tfrac{1}{8} \end{aligned}$$

$$\begin{aligned} 3y + 6x - 4 &= 0 \\ 3y &= -6x + 4 \\ y &= -2x + \tfrac{4}{3} \\ m_1 m_2 &= -1 \\ -2; \left(\tfrac{1}{2}\right) &= -1 \\ \therefore m_2 &= \tfrac{1}{2} \end{aligned}$$

$$f\left(\frac{9}{8}\right) = 2\left(\frac{9}{8}\right)^2 - 4\left(\frac{9}{8}\right) + 2$$

$$= 2\left(\frac{81}{64}\right) - \frac{9}{2} + 2$$

$$= \frac{81}{32} - \frac{9}{2} + 2$$

$$= \frac{81 - 144 + 64}{32}$$

$$= \frac{1}{32}$$

$$\therefore \left(\frac{9}{8}; \frac{1}{32}\right)$$

$$\therefore y = mx + c$$

$$y = \frac{1}{2}x + c$$

$$\frac{1}{32} = \frac{1}{2}\left(\frac{9}{8}\right) + c$$

$$\frac{1}{32} = \frac{9}{16} + c$$

$$c = \frac{1}{32} - \frac{9}{16}$$

$$= \frac{1 - 18}{32}$$

$$= -\frac{17}{32}$$

$$\therefore y = \frac{1}{2}x - \frac{17}{32}$$

Assessment activity 2.17

1. **a)** $v = \frac{ds}{dt} = s'(t)$

$= 3t^2 - 6t - 4$

At $t = 3$s: $\therefore v = 3(3)^2 - 6(3) - 4$

$= 5$ m/s

b) $a = \frac{d^2s}{dt^2} = s''(t)$

$\therefore a = 6t - 6$

$= 6(3) - 6$

$\therefore a = 12$ m/s^2

2. **a)** $h(t) = 20t - 3t^2$

$h(2) = 20(2) - 3(2)^2$

$= 28$ m

b) $h(t) = 20t - 3t^2$

$\frac{dh}{dt} = 20 - 6t$

$0 = 20 - 6t$

$6t = 20$

$t = 3{,}333$ s

c) $h\left(3\frac{1}{3}\right) = 20\left(\frac{10}{3}\right) = 3\left(\frac{10}{3}\right)^2$

$= 33\frac{1}{3}$ m

d) $8 = 20t - 3t^2$

$0 = 3t^2 - 20t + 8$

$$t = \frac{-(-20) \pm \sqrt{(-20)^2 - 4(3)(8)}}{2(3)}$$

$$= \frac{20 \pm \sqrt{304}}{6}$$

$\therefore t = 6{,}239$ s or $t = 0{,}427$ s

e) $v = \frac{dh}{dt} = 20 - 6t$

$= 20 - 6(3)$

$= 2$ m/s

f) $a = \frac{d^2h}{dt^2}$

$= -6$ m/s^2

3. **a)** $s = 112t + 12{,}3t^2 - t^3$

$s = 112(4) + 12{,}3(4)^2 - (4)^3$

$= 580{,}8$ m

b) $v = \frac{ds}{dt} = 112 + 24{,}6t - 3t^2$

$= 112 + 24{,}6(4) - 3(4)^2$

$= 162{,}4$ m/s

c) $a = \frac{d^2s}{dt^2} = 24{,}6 - 6t$

$= 24{,}6 - 6(4)$

$= 0{,}6 \text{ m/s}^2$

Assessment activity 2.18

1. **a)** $f(x) = x^3 - 3x - 2$

$\therefore f(x) = -2$ or $y = -2$

b) $f(x) = x^3 - 3x - 2$

$0 = x^3 - 3x - 2$

$\therefore f(1) = (1)^3 - 3(1) - 2 \neq 0$

$f(-1) = (-1)^3 - 3(-1) - 2 = 0$

$\therefore (x + 1)(x^2 + bx - 2)$

$x^2 + bx^2 = 0$

$bx^2 = -x^2$

$\therefore b = -1$

$\therefore (x + 1)(x^2 - x - 2) = 0$

$(x + 1)(x - 2)(x + 1) = 0$

$\therefore x = -1; x = 2$

c) $f(x) = x^3 - 3x - 2$

$f'(x) = 3x^2 - 3$

$0 = 3(x^2 - 1)$

$0 = 3(x - 1)(x + 1)$

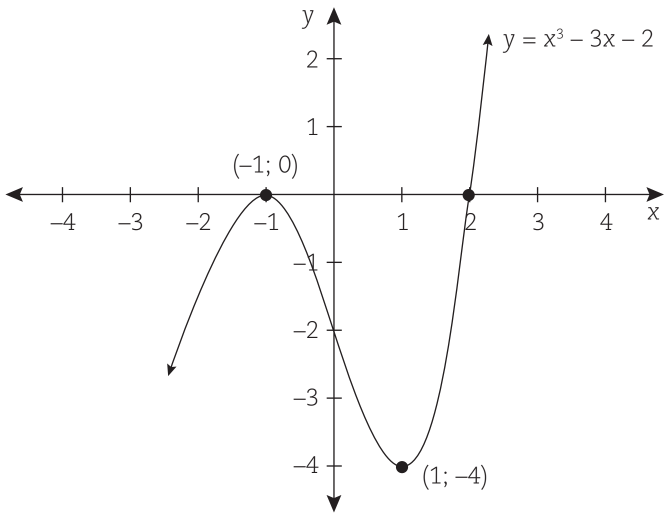
$\therefore x = 1; x = -1$

$f(x) = x^3 - 3x - 2$

$f(1) = (1)^3 - 3(1) - 2 = -4$

$f(-1) = (-1)^3 - 3(-1) - 2 = 0$

TP: (1; –4) and (–1; 0)



2. $f(x) = 2x^3 - 3x^2$

a) **(i)** $f(x) = 2x^3 - 3x^2$ $\quad f(x) = 0$

$0 = 2x^3 - 3x^2$

$0 = x^2(2x - 3)$

$\therefore x^2 = 0$ or $2x - 3 = 0$

$x = 0 \qquad x = \frac{3}{2}$

(ii) $f'(x) = 6x^2 - 6x$

$0 = 6x(x - 1)$

$\therefore 6x = 0$ or $x - 1 = 0$

$x = 0$ or $x = 1$

$f(x) = 2x^3 - 3x^2$

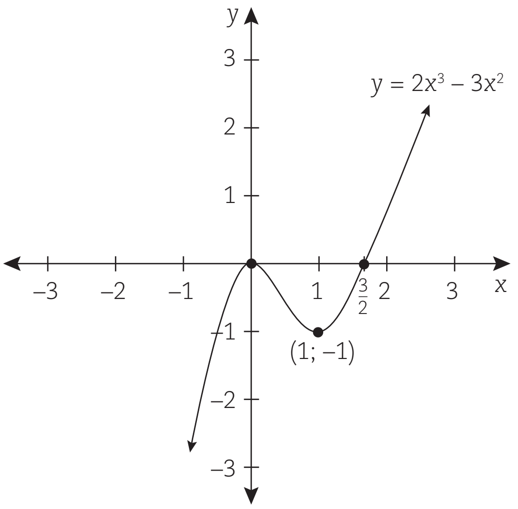
$f(0) = 0$

$f(1) = 2(1)^3 - 3(1)^2$

$= -1$

$\therefore$ TP: (0; 0) and (1; –1)

2. b)



3. a) $f(x) = x^3 - 6x^2 + 11x - 6$

y-intercept: $y = -6$

x-intercepts: $f(x) = x^3 - 6x^2 + 11x - 6$

$$f(2) = (2)^3 - 6(2)^2 + 11(2) - 6$$
$$= 0$$

$$\therefore (x-2)(x^2 + bx + 3) = 0$$

$$\underbrace{-2x^2}$$
$$\underbrace{bx^2}$$

$$-2x^2 + bx^2 = -6x^2$$
$$bx^2 = -4x^2$$
$$\therefore b = -4$$

$$(x-2)(x^2 - 4x + 3) = 0$$
$$\therefore (x-2)(x-3)(x-1) = 0$$

$x = 2$; $x = 3$; $x = 1$

TP: $f'(x) = 3x^2 - 12x + 11$

$$0 = 3x^2 - 12x + 11$$

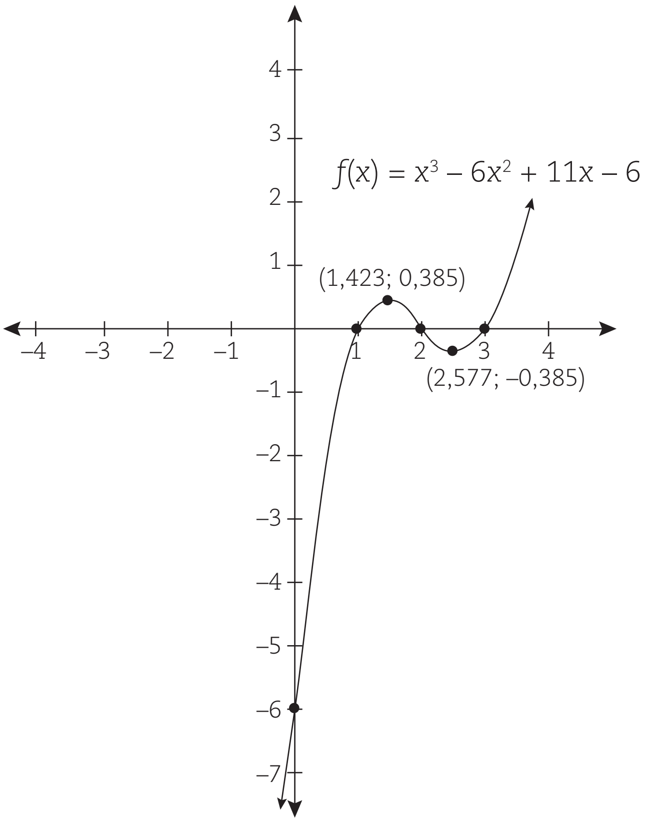
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-(-12) \pm \sqrt{(-12)^2 - 4(3)(11)}}{2(3)}$$
$$= \frac{12 \pm \sqrt{12}}{6}$$

$x = 2{,}577$ or $x = 1{,}423$

$$f(2{,}577) = (2{,}577)^3 - 6(2{,}577)^2 + 11(2{,}577) - 6$$
$$= -0{,}385$$

$$f(1{,}423) = (1{,}423)^3 - 6(1{,}423)^2 + 11(1{,}423) - 6$$
$$= 0{,}385$$

$\therefore$TP: $(2{,}577; -0{,}385)$ and $(1{,}423; 0{,}385)$



b) $f(x) = -x^3 + 2x^2 + 4x - 8$

y-intercept: $y = -8$

x-intercepts: $f(x) = -x^3 + 2x^2 + 4x - 8$

$f(2) = -(2)^3 + 2(2)^2 + 4(2) - 8 = 0$

$\therefore (x - 2)(-x^2 + bx + 4) = 0$

$2x^2$

bx^2

$\therefore 2x^2 + bx^2 = 2x^2$

$bx^2 = 0$

$b = 0$

$\therefore (x - 2)(-x^2 + 4) = 0$

$\therefore x = 2 \quad \text{or } -x^2 + 4 = 0$

$x^2 = 4$

$\therefore x = \pm 2$

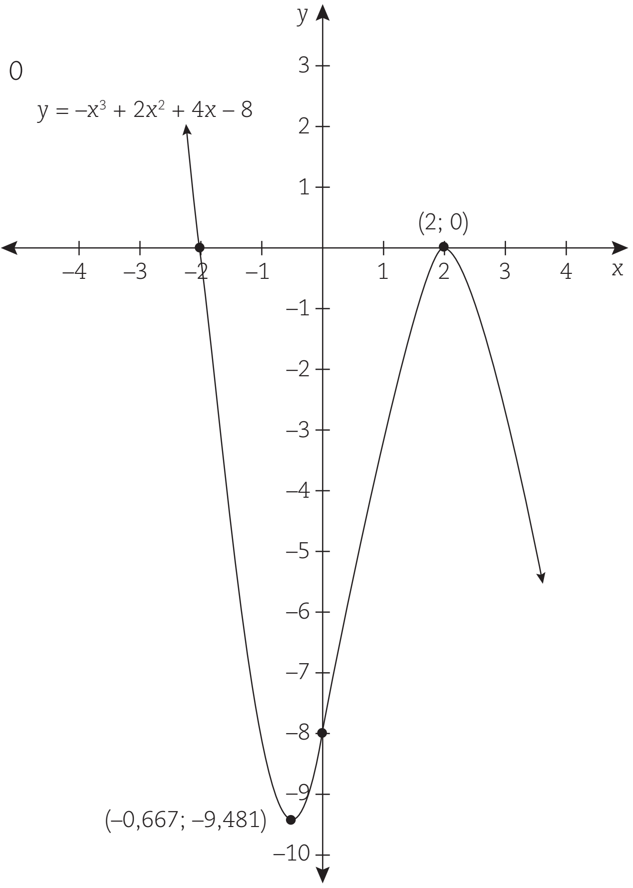
TP: $f'(x) = -3x^2 + 4x + 4$

$0 = 3x^2 - 4x - 4$

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(3)(-4)}}{2(3)}$$

$$= \frac{4 \pm \sqrt{64}}{6}$$

$\therefore x = 2 \text{ or } -0{,}667$



$f(2) = -(2)^3 + 2(2)^2 + 4(2) - 8 = 0$

$f(-0{,}667) = -(-0{,}667)^3 + 2(-0{,}667)^2 + 4(-0{,}667) - 8 = -9{,}481$

$\therefore$ TP: (2; 0) and (–0,667; -9,481)

c) $f(x) = x^3 - 3x^2 + 4$

y-intercept: $y = 4$

x-intercepts: $f(x) = x^3 - 3x^2 + 4$

$f(-1) = (-1)^3 - 3(-1)^2 + 4 = 0$

$\therefore (x + 1)(x^2 + bx + 4) = 0$

x^2

bx^2

$\therefore x^2 + bx^2 = -3x^2$

$bx^2 = -4x^2$

$b = -4$

$\therefore (x + 1)(x^2 - 4x + 4) = 0$

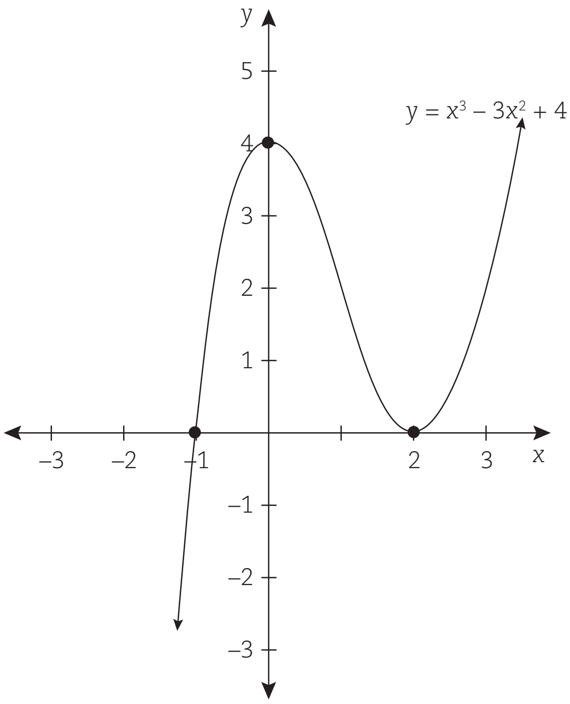
$(x + 1)(x - 2)(x - 2) = 0$

$\therefore x = -1 \quad \text{or } x = 2$

$\therefore$ TP: $f'(x) = 3x^2 - 6x$

$0 = 3x(x - 2)$

$\therefore x = 0 \text{ or } x = 2$



$f(0) = (0)^3 - 3(0)^2 + 4 = 4$

$f(2) = (2)^3 - 3(2)^2 + 4 = 8 - 12 + 4 = 0$

$\therefore$ TP: (0; 4) and (2; 0)

d) $f(x) = x^3 - 8$

$f(x) = x^3 - 8$

y-intercept: $y = -8$

x-intercept: $f(x) = x^3 - 8$

$$f(2) = (2)^3 - 8 = 0$$

$$(x - 2)(x^2 + 2x) = 0$$

$$x = 2$$

or

$$x = \frac{-2 \pm \sqrt{(2)^3 - 4(1)(4)}}{2(1)}$$

$$= \frac{-2 \pm \sqrt{-12}}{2}$$

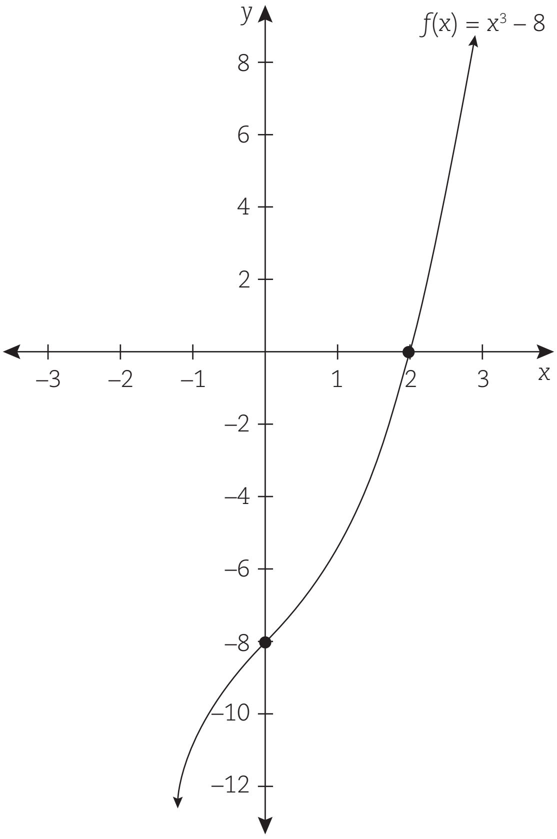
$\therefore$ non-real roots

$f\,'(x) = 3x^2 = 0$

$\therefore x = 0$

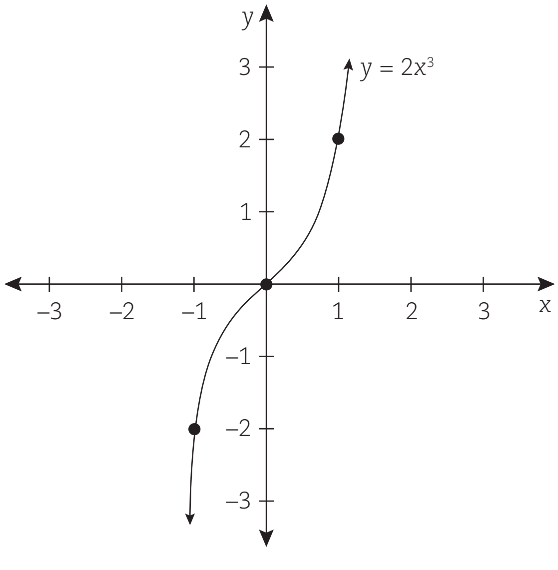
$f(0) = -8$

TP: $(0; -8)$



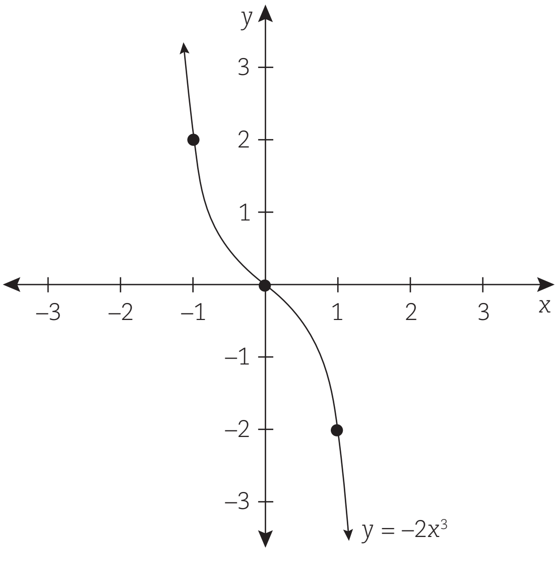
e) $f(x) = 2x^3$

x	–1	0	1
y	–2	0	2



f) $y = -2x^3$

x	–1	0	1
y	2	0	–2



Assessment activity 2.19

1. **a)** $f(x) = -x^2 + 6x - 8$

$f'(x) = -2x + 6$

$0 = -2x + 6$

$2x = 6$

$\therefore x = 3$

$f(3) = -(3)^2 + 6(3) - 8$

$= -9 + 18 - 8$

$= 1$

$\therefore$ TP: $(3; 1)$

Max/min TP:

$f''(x) = -2$

$\therefore f''(x) < 0$

$\therefore (3; 1)$ is a maximum TP.

b) $f(x) = 2x^3 - 9x^2 - 24x$

$f'(x) = 6x^2 - 18x - 24$

$0 = x^2 - 3x - 4$

$0 = (x - 4)(x + 1)$

$\therefore x = 4; x = -1$

$\therefore f(4) = 2(4)^3 - 9(4)^2 - 24(4) = -112$

$f(-1) = 2(-1)^3 - 9(-1)^2 - 24(-1) = 13$

$\therefore$ TP: $(4; -112)$ and $(-1; 13)$

Max/min TP: $f''(x) = 12x - 18$

$f''(4) = 12(4) - 18 > 0 \therefore (4; -112)$ min TP

$f''(-1) = 12(-1) - 1 < 0 \therefore (-1; 13)$ max TP

c) $f(x) = x^3 - 8x^2 + 5x + 14$

$f'(x) = 3x^2 - 16x + 5$

$0 = (3x - 1)(x - 5)$

$\therefore x = \frac{1}{3}; x = 5$

$f\left(\frac{1}{3}\right) = \left(\frac{1}{3}\right)^3 - 8\left(\frac{1}{3}\right)^2 + 5\left(\frac{1}{3}\right) + 14 = 14{,}815$

$f(5) = (5)^3 - 8(5)^2 + 5(5) + 14 = -36$

$\therefore$ TP: $\left(\frac{1}{3}; 14{,}815\right)$ and $(5; -36)$

Max/min TP: $\frac{d^2y}{dx^2} = 6x - 16$

$\therefore \frac{d^2y}{dx^2} = 6\left(\frac{1}{3}\right) - 16 < 0 \therefore \left(\frac{1}{3}; 14{,}815\right)$ max TP.

$\frac{d^2y}{dx^2} = 6(5) - 16 > 0 \therefore (5; -36)$ min TP.

2. **a)** $y = x^3 - 6x^2 + 11x - 6$

$\frac{dy}{dx} = 3x^2 - 12x + 11$

$0 = 3x^2 - 12x + 11$

$\therefore x = \frac{-(-12) \pm \sqrt{(-12)^2 - 4(3)(11)}}{2(3)}$

$= \frac{12 \pm \sqrt{12}}{6}$

$\therefore x = 2{,}577$ or $x = 1{,}423$

$y = x^3 - 6x^2 + 11x - 6$

$\therefore f(2{,}577) = (2{,}577)^3 - 6(2{,}577)^2 + 11(2{,}577) - 6 = -0{,}385$

$f(1{,}423) = (1{,}423)^3 - 6(1{,}423)^2 + 11(1{,}423) - 6 = 0{,}385$

$\therefore$TP: $(2{,}577; -0{,}385)$ and $(1{,}423; 0{,}385)$

b) $\frac{d^2y}{dx^2} = 6x - 12$

$\therefore \frac{d^2y}{dx^2} = 6(2{,}577) - 12 > 0 \therefore (2{,}577; -0{,}385)$ min TP

$\frac{d^2y}{dx^2} = 6(1{,}423) - 12 < 0 \therefore (1{,}423; 0{,}385)$ max TP

c) x-intercepts: $y = 0$

$0 = x^3 - 6x^2 + 11x - 6$

$f(1) = (1)^3 - 6(1)^2 + 11(1) - 6 = 0$

$\therefore (x - 1)(x^2 + bx + 6) = 0$

$(x - 1)(x^2 - 5x + 6) = 0$

$(x - 1)(x - 3)(x - 2) = 0$

$\therefore x = 1; \quad x = 3; \quad x = 2$

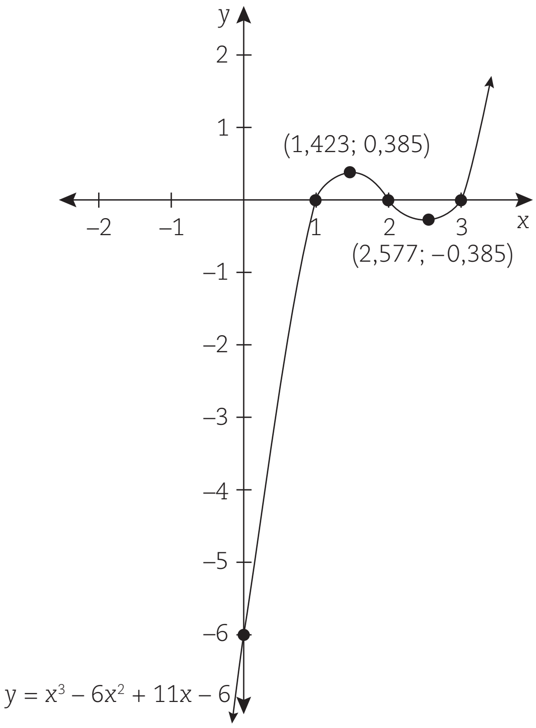
- $-x^2 + bx^2 = -6x^2$

  $bx^2 = -5x^2$

  $\therefore b = -5$

y-intercept: $x = 0$

$\therefore y = -6$



3. **a)** $y = (4 - x)(1 - x)^2$

$y = (4 - x)(1 - 2x + x^2)$

$= 4 - 9x + 6x^2 - x^3$

$\therefore \frac{dy}{dx} = -9 + 12x - 3x^2$

TP: $0 = -9 + 12x - 3x^2$

$0 = 3 - 4x + x^2$

$\therefore x^2 - 4x + 3 = 0$

$(x - 3)(x - 1) = 0$

$\therefore x = 3; \quad x = 1$

$y = f(3) = 4 - 9(3) + 6(3)^2 - (3)^3$

$= 4$

$y = f(1) = 4 - 9(1) + 6(1)^2 - (1)^3$

$= 0$

$\therefore$ TP: (3; 4) and (1; 0)

b) $\frac{d^2y}{dx^2} = 12 - 6x$

$= 12 - 6(3)$.................... $x = 3$

$= -6$

$\therefore \frac{d^2y}{dx^2} < 0$; (3; 4) is a maximum turning point.

$\frac{d^2y}{dx^2} = 12 - 6x$

$= 12 - 6(1)$ .... $x = 1$

$= 6$

$\therefore \frac{d^2y}{dx^2} > 0$; (1; 0) is a minimum turning point.

Assessment activity 2.20

1. **a)** $f(x) = x^3 - 6x^2 + 9x$

$f'(x) = 3x^2 - 12x + 9$

$f''(x) = 6x - 12$

$0 = 6x - 12$

$6x = 12$

$x = 2$

Substitute $x = 2$ in:

$f(x) = x^3 - 6x^2 + 9x$

$\therefore f(x) = (2)^3 - 6(2)^2 + 9(2)$

$= 8 - 24 + 18$

$= 2$

$\therefore$ Point of inflection: (2; 2)

b) $y = x^3 + 9x^2 - 3x - 18$

$\frac{dy}{dx} = 3x^2 + 18x - 3$

$\frac{d^2y}{dx^2} = 6x + 18$

$0 = 6x + 18$

$\therefore 6x = -18$

$x = -3$

Substitute $x = -3$ in:

$y = x^3 + 9x^2 - 3x - 18$

$\therefore y = (-3)^3 + 9(-3)^2 - 3(-3) - 18$

$= -27 + 81 + 9 - 18$

$\therefore y = 45$

$\therefore$ Point of inflection: (–3; 45)

c) $y = -2x^3 + 7x^2 + 5x - 4$

$\frac{dy}{dx} = -6x^2 + 14x + 5$

$\frac{d^2y}{dx^2} = -12x + 14$

$0 = -12x + 14$

$12x = 14$

$x = \frac{14}{12}$

$= \frac{7}{6}$ or 1,667

Substitute $x = \frac{7}{6}$ in:

$y = -2x^3 + 7x^2 + 5x - 4$

$\therefore y = -2\left(\frac{7}{6}\right)^3 + 7\left(\frac{7}{6}\right)^2 + 5\left(\frac{7}{6}\right) - 4$

$\therefore y = 8{,}185$

$\therefore$ Point of inflection: (1,667; 8,185)

2. $f(x) = 2x^3 + 3x^2 - x + 5$

TP: $f'(x) = 6x^2 + 6x - 1$

$0 = 6x^2 + 6x - 1$

$$x = \frac{-6 \pm \sqrt{(6)^2 - 4(6)(-1)}}{2(6)}$$

$$= \frac{-6 \pm \sqrt{60}}{12}$$

$\therefore x = 0{,}145;\ x = -1{,}145$

$\therefore f(0{,}145) = 2(0{,}145)^3 + 3(0{,}145)^2 - 0{,}145 + 5 = 4{,}924$

$f(-1{,}145) = 2(-1{,}145)^3 + 3(-1{,}145)^2 - (-1{,}145) + 5 = 7{,}076$

$\therefore$ TP: (0,145; 4,924) and (–1,145; 7,076)

Point of inflection:

$$\frac{d^2y}{dx^2} = 12x + 6$$

$$0 = 12x + 6$$

$$12x = -6$$

$$x = -\frac{1}{2}$$

$$\therefore f\left(-\frac{1}{2}\right) = 2\left(-\frac{1}{2}\right)^3 + 3\left(-\frac{1}{2}\right)^2 - \left(-\frac{1}{2}\right) + 5$$

$$= 2\left(-\frac{1}{8}\right) + \frac{3}{4} + \frac{1}{2} + 5$$

$$= 5$$

Point of inflection: $(-\frac{1}{2}; 5)$

3. **a)** TP: $\frac{dy}{dx} = -3x^2 + 4x + 5$

$$0 = -3x^2 + 4x + 5$$

$$0 = 3x^2 - 4x - 5$$

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(3)(-5)}}{2(3)}$$

$$= \frac{4 \pm \sqrt{76}}{6}$$

$\therefore x = 2{,}12; x = -0{,}79$

$\therefore f(2{,}12) = -(2{,}12)^3 + 2(2{,}12)^2 + 5(2{,}12) - 6 = 4{,}06$

$f(-0{,}79) = -(-0{,}79)^3 + 2(-0{,}79)^2 + 5(-0{,}79) - 6 = -8{,}21$

$\therefore$ TP: (2,12; 4,06) and (–0,79; -8,21)

b) $\frac{d^2y}{dx^2} = -6x + 4$

$$= -6(2{,}12) + 4$$

$$= -8{,}72$$

$\therefore \frac{d^2y}{dx^2} < 0$; maximum TP = (2,12; 4,06)

and

$$\frac{d^2y}{dx^2} = -6x + 4$$

$$= -6(-0{,}79) + 4$$

$$= 8{,}74$$

$\therefore \frac{d^2y}{dx^2} > 0$; minimum TP = (–0,79; –8,21)

c) $\frac{d^2y}{dx^2} = -6x + 4$

$$0 = -6x + 4$$

$$6x = 4$$

$$x = \frac{2}{3}$$

$$\therefore y = -\left(\frac{2}{3}\right)^3 + 2\left(\frac{2}{3}\right)^2 + 5\left(\frac{2}{3}\right) - 6$$

$$= -\frac{8}{27} + \frac{8}{9} + \frac{10}{3} - 6$$

$$= -2{,}074$$

$\therefore$ Point of inflection: $\left(\frac{2}{3}; -2{,}074\right)$

or (0,667; -2,074)

Assessment activity 2.21

1. $\int x^5\,dx = \frac{x^6}{6} + c$ or $\frac{1}{6}x^6 + c$

2. $\int 4x^2\,dx = \frac{4x^3}{3} + c$ or $\frac{4}{3}x^3 + c$

3. $\int 2\,dx = 2x + c$

4.
$$\begin{aligned}\int \frac{2}{x^2}\,dx &= \int 2x^{-2}\,dx\\ &= \frac{2x^{-1}}{-1} + c\\ &= -\frac{2}{x} + c\end{aligned}$$

5. $\int dy = y + c$

6. $\int 0{,}23\,dr = 0{,}23r + c$

7.
$$\begin{aligned}\int \frac{dx}{x^3} &= \int \frac{1}{x^3}\,dx\\ &= \int x^{-3}\,dx\\ &= \frac{x^{-2}}{-2} + c\\ &= -\frac{1}{2x^2} + c\end{aligned}$$

8.
$$\begin{aligned}&\int \left(-3x^4 + \frac{1}{3x^3} - 4\pi\right)dx\\ &= \int \left(-3x^4 + \frac{1}{3}x^{-3} - 4\pi\right)dx\\ &= \frac{-3x^5}{5} + \frac{1}{3}.\frac{x^{-2}}{-2} - 4\pi x + c\\ &= -\frac{3}{5}x^5 - \frac{1}{6x^2} - 4\pi x + c\end{aligned}$$

9.
$$\begin{aligned}&\int (x^2 + 3)(x - 4)\,dx\\ &= \int (x^3 - 4x^2 + 3x - 12)\,dx\\ &= \frac{x^4}{4} - \frac{4x^3}{3} + \frac{3x^2}{2} - 12x + c\\ &= \frac{1}{4}x^4 - \frac{4}{3}x^3 + \frac{3}{2}x^2 - 12x + c\end{aligned}$$

10.
$$\begin{aligned}&\int \frac{2x^3 - x^2 + 2x - 6}{x^6}\,dx\\ &= \int (2x^{-3} - x^{-4} + 2x^{-5} - 6x^{-6})\,dx\\ &= \frac{2x^{-2}}{-2} - \frac{x^{-3}}{-3} + \frac{2x^{-4}}{-4} - \frac{6x^{-5}}{-5} + c\\ &= -\frac{1}{x^2} + \frac{1}{3x^3} - \frac{1}{2x^4} + \frac{6}{5x^5} + c\end{aligned}$$

11.
$$\begin{aligned}&\int \left(\sqrt{x^3} + \sqrt{5x} - \frac{1}{\sqrt{x}}\right)dx\\ &= \int \left(x^{\frac{3}{2}} + \sqrt{5}\,x^{\frac{1}{2}} - x^{-\frac{1}{2}}\right)dx\\ &= \frac{x^{\frac{5}{2}}}{\frac{5}{2}} + \frac{\sqrt{5}\,x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c\\ &= \frac{2}{5}\sqrt{x^5} + \frac{2}{3}\sqrt{5}\sqrt{x^3} - 2\sqrt{x} + c\\ &= \frac{2}{5}\sqrt{x^5} + \frac{2}{3}\sqrt{5x^3} - 2\sqrt{x} + c\end{aligned}$$

12.
$$\begin{aligned}&\int \frac{t^5 + 3t^3 - 1}{t^5}\,dt\\ &= \int (1 + 3t^{-2} - t^{-5})\,dt\\ &= t + \frac{3t^{-1}}{-1} - \frac{t^{-4}}{-4} + c\\ &= t - \frac{3}{t} + \frac{1}{4t^4} + c\end{aligned}$$

13.
$$\begin{aligned}&\int \left(4x^{\frac{1}{4}} + 2\sqrt{x} - \sqrt[3]{x}\right)dx\\ &= \int \left(4x^{\frac{1}{4}} + 2x^{\frac{1}{2}} - x^{\frac{1}{3}}\right)dx\\ &= \frac{4x^{\frac{5}{4}}}{\frac{5}{4}} - \frac{2x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^{\frac{4}{3}}}{\frac{4}{3}} + c\\ &= \frac{16}{5}\sqrt[4]{x^5} - \frac{4}{3}\sqrt{x^3} - \frac{3}{4}\sqrt[3]{x^4} + c\end{aligned}$$

14.
$$\begin{aligned}&\int (ax^2 + bx + c)\,dx\\ &= \frac{ax^3}{3} + \frac{bx^2}{2} + cx + d\end{aligned}$$

15. $$\begin{aligned}&\int\left(3-\sqrt{y}\right)^2\,dy\\&=\int\left(3-\sqrt{y}\right)\left(3-\sqrt{y}\right)\,dy\\&=\int(9-3\sqrt{y}\,-3\sqrt{y}\,+y)\,dy\\&=\int(9-6y^{\frac{1}{2}}+y)\,dy\\&=9y-\frac{6y^{\frac{3}{2}}}{\frac{3}{2}}+\frac{y^2}{2}+c\\&=9y-\frac{2}{3}.6\sqrt{y^3}+\frac{1}{2}y^2+c\\&=9y-4\sqrt{y^3}+\frac{1}{2}y^2+c\end{aligned}$$

16. $$\begin{aligned}&\int\frac{x^2-x-6}{x-3}\,dx\\&=\int\frac{(x-3)(x+2)}{(x-3)}\,dx\\&=\int(x+2)\,dx\\&=\frac{x^2}{2}+2x+c\text{ or }\frac{1}{2}x^2+2x+c\end{aligned}$$

Assessment activity 2.22

1. $$\begin{aligned}&\int\frac{7}{x}\,dx\\&=7\ln x+c\end{aligned}$$

2. $$\begin{aligned}&\int(x^{-1}+\frac{1}{x^2}-3)\,dx\\&=\int(x^{-1}+x^{-2}-3)\,dx\\&=\ln x+\frac{x^{-1}}{-1}-3x+c\\&=\ln x-\frac{1}{x}-3x+c\end{aligned}$$

3. $$\begin{aligned}&\int\left(\frac{1}{2x^2}-\frac{1}{3x}+6x-\frac{2}{x}\right)dx\\&=\int\left(\frac{1}{2}x^{-2}-\frac{1}{3}x^{-1}+6x-2x^{-1}\right)dx\\&=\frac{1}{2}.\frac{x^{-1}}{-1}-\frac{1}{3}\ln x+\frac{6x^2}{2}-2\ln x+c\\&=-\frac{1}{2x}-\frac{1}{3}\ln x+3x^2-2\ln x+c\\&=-\frac{1}{2x}+3x^2-\frac{7}{3}\ln x+c\end{aligned}$$

4. $$\begin{aligned}&\int\frac{5x^2+2x-1}{x}\,dx\\&=\int\left(5x+2-\frac{1}{x}\right)dx\\&=\frac{5x^2}{2}+2x-\ln x+c\\&\text{or }\frac{5}{2}x^2+2x-\ln x+c\end{aligned}$$

5. $$\begin{aligned}&\int\left(bx+\frac{1}{bx}\right)dx\\&=\frac{bx^2}{2}+\frac{1}{b}\ln x+c\end{aligned}$$

Assessment activity 2.23

1. $$\begin{aligned}&\int 2e^x\,dx\\&=2e^x+c\end{aligned}$$

2. $$\begin{aligned}&\int -3e^{-2x}\,dx\\&=\frac{-3e^{-2x}}{-2}+c\\&=\frac{3}{2e^{2x}}+c\end{aligned}$$

3. $$\begin{aligned}&\int(\sqrt[4]{x^2}-e^x+\pi x-x^e)\,dx\\&=\int(x^{\frac{1}{2}}-e^x+\pi x-x^e)\,dx\\&=\frac{x^{\frac{3}{2}}}{\frac{3}{2}}+e^x+\frac{\pi x^2}{2}-\frac{x^{e+1}}{e+1}+c\\&=\frac{2}{3}\sqrt{x^3}+e^x+\frac{\pi}{2}x^2+\frac{x^{e+1}}{e+1}+c\end{aligned}$$

4. $$\begin{aligned}&\int\left(\frac{e^{\frac{x}{2}}}{\frac{1}{2}}+\frac{1}{2x}-\frac{1}{x^3}+\frac{x}{3}\right)dx\\&=\int(e^{\frac{x}{2}}+\frac{1}{2}x^{-1}-x^{-3}+\frac{1}{3}x)\,dx\\&=e^{\frac{x}{2}}+\frac{1}{2}\ln x-\frac{x^{-2}}{-2}+\frac{1}{3}.\frac{x^2}{2}+c\\&=2e^{\frac{x}{2}}+\frac{1}{2}\ln x+\frac{1}{2x^2}+\frac{1}{6}x^2+c\end{aligned}$$

5. $\int (2e^{3x})(3e^{x})\ dx$

$= \int 6e^{4x}\ dx$

$= \frac{6e^{4x}}{4} + c$

$= \frac{3}{2}e^{4x} + c$

6. $\int (e^{x} + e^{-x})^{2}\ dx$

$= \int (e^{x} + e^{-x})(e^{x} + e^{-x})\ dx$

$= \int (e^{2x} + e^{0} + e^{0} + e^{-2x})\ dx$

$= \int (e^{2x} + 2 + e^{-2x})\ dx$

$= \frac{e^{2x}}{2} + 2x + \frac{e^{-2x}}{-2} + c$

$= \frac{1}{2}e^{2x} + 2x - \frac{1}{2e^{2x}} + c$

Assessment activity 2.24

1. $\int 5 \sin 3x\ dx$

$= \frac{-5 \cos 3x}{3} + c$

or $-\frac{5}{3} \cos 3x + c$

2. $\int 2 \cos \frac{x}{3}\ dx$

$= 2.\frac{\sin \frac{x}{3}}{\frac{1}{3}} + c$

$= 6 \sin \frac{x}{3} + c$

3. $\int 4 \sec^{2} 3x\ dx$

$= 4\frac{\tan 3x}{3} + c$

$= \frac{4}{3} \tan 3x + c$

4. $\int (-4 \cos 3x + 2 \sin 6x)\ dx$

$= -\frac{4 \sin 3x}{3} - \frac{2 \cos 6x}{6} + c$

$= -\frac{4}{3} \sin 3x - \frac{1}{3} \cos 6x + c$

5. $\int (\cos 5x + \frac{2}{x} - 3x^{2})\ dx$

$= \frac{\sin 5x}{5} + 2 \ln x - \frac{3x^{3}}{3} + c$

$= \frac{1}{5} \sin 5x + 2 \ln x - x^{3} + c$

6. $\int (2 \cos \pi x - 3 \sin 5x - e^{-2x} + 5x^{-1})\ dx$

$= \frac{2 \sin \pi x}{\pi} + \frac{3 \cos 5x}{5} - \frac{e^{-2x}}{-2} + 5 \ln x + c$

$= \frac{2}{\pi} \sin \pi x + \frac{3}{5} \cos 5x + \frac{1}{2e^{2x}} + 5 \ln x + c$

7. $\int \left(\frac{4}{3} \sec^{2} \frac{x}{2}\right) dx$

$= \frac{4}{3}\ \frac{\tan \frac{x}{2}}{\frac{1}{2}} + c$

$= \frac{8}{3} \tan \frac{x}{2} + c$

Assessment activity 2.25

1. $\int_{1}^{3} (5x^{2} - 1)\ dx$

$= \left[\frac{5x^{3}}{3} - x\right]_{1}^{3}$

$= \left[\frac{5(3)^{3}}{3} - (3)\right] - \left[\frac{5(1)^{3}}{3} - (1)\right]$

$= [42] - \left[\frac{2}{3}\right]$

$= 41\frac{1}{3}$

2. $\int_{2}^{4} \left(\sqrt{x} - \frac{1}{x^{3}} + 4x^{3}\right) dx$

$= \int_{2}^{4} (x^{\frac{1}{2}} - x^{-3} + 4x^{3})\ dx$

$= \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^{-2}}{-2} + \frac{4x^{4}}{4}\right]_{2}^{4}$

$= \left[\frac{2}{3}(4)^{\frac{3}{2}} + \frac{1}{2(4)^{2}} + (4)^{4}\right] - \left[\frac{2}{3}(2)^{\frac{3}{2}} + \frac{1}{2(2)^{2}} + (2)^{4}\right]$

$= [261{,}365] - [18{,}011]$

$= 243{,}354$

3. $\int_{3}^{5} x^{-\frac{5}{3}}\ dx$

$= \left[\frac{x^{-\frac{2}{3}}}{-\frac{2}{3}}\right]_{3}^{5}$

$= \left[-\frac{3}{2}(5)^{-\frac{2}{3}}\right] - \left[-\frac{3}{2}(3)^{-\frac{2}{3}}\right]$

$= [-0{,}513] - [-0{,}721]$

$= 0{,}208$

4. $\int_{1}^{2} \left(e^{2x} + \frac{3}{x}\right) dx$

$= \left[\frac{e^{2x}}{2} + 3 \ln x\right]_{1}^{2}$

$= \left[\frac{1}{2}e^{2(2)} + 3 \ln 2\right] - \left[\frac{1}{2}e^{2(1)} + 3 \ln 1\right]$

$= [29{,}379] - [3{,}695]$

$= 25{,}684$

5. $$\int_2^3 \left(\frac{x+2}{x}\right)^2 dx$$
$$= \int_2^3 \frac{(x+2)(x+2)}{x^2}\ dx$$
$$= \int_2^3 \left(\frac{x^2+4x+4}{x^2}\right) dx$$
$$= \int_2^3 (1 + 4x^{-1} + 4x^{-2})\ dx$$
$$= \left[x + 4 \ln x + 4\frac{x^{-1}}{-1}\right]_2^3$$
$$= \left[x + 4 \ln x - \frac{4}{x}\right]_2^3$$
$$= \left[(3) + 4 \ln (3) - \frac{4}{(3)}\right] - \left[(2) + 4 \ln (2) - \frac{4}{(2)}\right]$$
$$= 6{,}061 - 2{,}773$$
$$= 3{,}288$$

6. $$\int_1^2 (3x^{-1} + 2e^{-x})\ dx$$
$$= [3 \ln x - 2e^{-x}]_1^2$$
$$= [3 \ln 2 - 2e^{-2}] - [3 \ln (1) - 2e^{-1}]$$
$$= 1{,}809 + 0{,}736$$
$$= 2{,}545$$

7. $$\int_{-2}^1 (4x-1)^2\ dx$$
$$= \int_{-2}^1 (4x-1)(4x-1)\ dx$$
$$= \int_{-2}^1 (16x^2 - 8x + 1)\ dx$$
$$= \left[\frac{16x^3}{3} - \frac{8x^2}{2} + x\right]_{-2}^1$$
$$= \left[\frac{16}{3}(1)^3 - 4(1)^2 + (1)\right] - \left[\frac{16}{3}(-2)^3 - 4(-2)^2 + (-2)\right]$$
$$= \left[\frac{16}{3} - 4 + 1\right] - \left[-\frac{128}{3} - 16 - 2\right]$$
$$= 2{,}333 - [-60{,}667]$$
$$= 62{,}9997$$
$$\approx 63$$

8. $$\int_{45°}^{90°} (\cos \theta - 3 \sin 2\theta)\ d\theta$$
$$= \left[\sin \theta - \frac{3 \cos 2\theta}{2}\right]_{45°}^{90°}$$
$$= \left[\sin 90° + \frac{3 \cos 2(90°)}{2}\right] - \left[\sin 45° + \frac{3 \cos 2(45°)}{2}\right]$$
$$= \left[1 - \frac{3(-1)}{2}\right] - \left[\frac{1}{\sqrt{2}} + \frac{3(0)}{2}\right]$$
$$= \left[\frac{5}{2}\right] + \left[\frac{1}{\sqrt{2}}\right]$$
$$= \frac{5-\sqrt{2}}{2} \text{ of } 1{,}7893$$

9. $$\int_0^a \left(a^4x - x^5 + \frac{1}{a^2}\right) dx$$
$$= \int_0^a (a^4x - x^5 + \frac{1}{a^2})\ dx$$
$$= \left[\frac{a^4x^2}{2} - \frac{x^6}{6} + \frac{x}{a^2}\right]_0^a$$
$$= \left[\frac{1}{2}a^4(a)^2 - \frac{1}{6}(a)^6 + \frac{(a)}{a^2}\right] - \left[\frac{1}{2}a^4(0)^2 - \frac{0^6}{6} + \frac{0}{a^2}\right]$$
$$= \left[\frac{1}{2}a^6 - \frac{1}{6}a^6 + \frac{1}{a}\right] - \left[0\right]$$
$$= \frac{1}{2}a^6 - \frac{1}{6}a^6 + \frac{1}{a}$$
$$= \frac{2}{6}a^6 + \frac{1}{a}$$
$$= \frac{1}{3}a^6 + \frac{1}{a}$$

10. $$\int_2^4 \left(y^{-0,2} + \frac{1}{\sqrt{y^3}} - \frac{3}{y^2}\right) dx$$
$$= \int_2^4 (y^{-0,2} + y^{-\frac{3}{2}} - 3y^{-2})\ dx$$
$$= \left[\frac{y^{0,8}}{0,8} + \frac{y^{-\frac{1}{2}}}{-\frac{1}{2}} - \frac{3y^{-1}}{-1}\right]_2^4$$
$$= \left[\frac{1}{0,8}(4)^{0,8} - \frac{2}{\sqrt{4}} + \frac{3}{(4)}\right] - \left[\frac{1}{0,8}\ (2)^{0,8} - \frac{2}{\sqrt{2}} + \frac{3}{(2)}\right]$$
$$= [3{,}539] - [2{,}262]$$
$$= 1{,}277$$

Assessment activity 2.26

1. $\Delta A = y.\Delta x$

$A = \int_{0°}^{180°} 2 \sin x \, dx$

$= [-2 \cos x]_{0°}^{180°}$

$= [-2 \cos 180°] - [-2 \cos 0°]$

$= [2] - [-2]$

$= 4 \text{ units}^2$

2. $\Delta A = y\Delta x$

$A = \int_1^2 x^3 \, dx$

$= \left[\frac{x^4}{4}\right]_1^2$

$= \left[\frac{(2)^4}{3} - \frac{(1)^4}{4}\right]$

$= 3{,}75 \text{ units}^2$

3. **a)** $y = 2x + 2$

x-intercept: $y = 0$

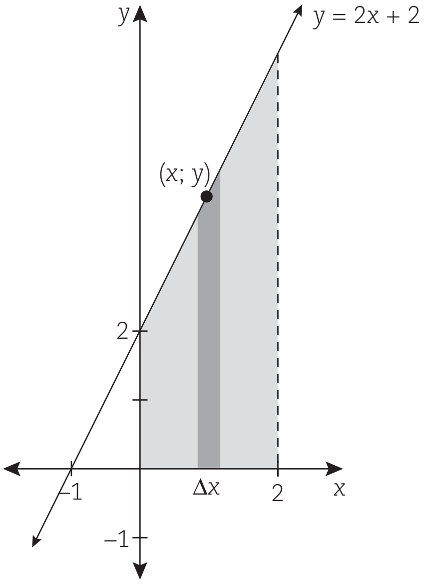
$0 = 2x + 2$

$2x = -2$

$\therefore x = -1$

y-intercept: $x = 0$

$\therefore y = 2$



b) $\Delta A = y\Delta x$

$A = \int_0^2 y \, dx$

$= \int_0^2 (2x + 2) \, dx$

$= \left[\frac{2x^2}{2} + 2x\right]_0^2$

$= [x^2 + 2x]_0^2$

$= [(2)^2 + 2(2)] - [0 + 2(0)]$

$= 8 \text{ units}^2$

4. $y = -x^2 + 2x + 3$

x-intercept: $y = 0$

$0 = -x^2 + 2x + 3$

$0 = x^2 - 2x - 3$

$0 = (x - 3)(x + 1)$

$\therefore x = 3; x = -1$

y-intercept: $x = 0$

$\therefore y = 3$

TP: $x = \frac{-b}{2a}$

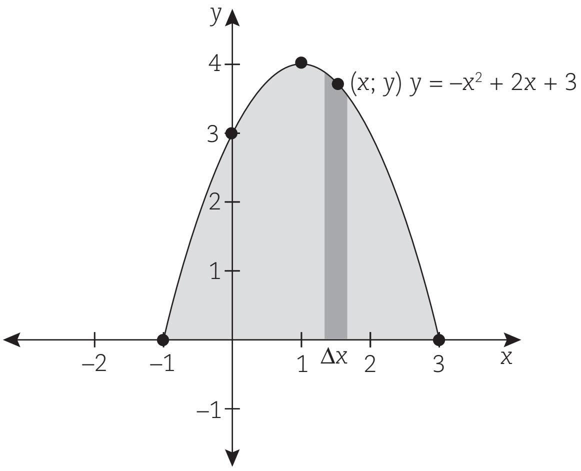
$= \frac{-2}{2(-1)}$

$= 1$

$y = -(1)^2 + 2(1) + 3$

$= 4$

$\therefore$ TP: (1; 4)



$$\Delta A = y\Delta x$$

$$A = \int_{-1}^{3} y\ dx$$

$$= \int_{-1}^{3} (-x^2 + 2x + 3)\ dx$$

$$= \left[-\frac{x^3}{3} + \frac{2x^2}{2} + 3x\right]_{-1}^{3}$$

$$= \left[-\tfrac{1}{3}(3)^3 + (3)^2 + 3(3)\right] - \left[-\tfrac{1}{3}(-1)^3 + (-1)^2 + 3(-1)\right]$$

$$= [9] - \left[-1\tfrac{2}{3}\right]$$

$$= 10\tfrac{2}{3} \text{ units}^2 \text{ or } 10{,}667 \text{ units}^2$$

5. **a)** $f(x) = (x + 2)^2 - 1$

x-intercept: $y = 0$

$0 = (x + 2)^2 - 1$

$0 = x^2 + 4x + 4 - 1$

$0 = x^2 + 4x + 3$

$0 = (x + 3)(x + 1)$

$\therefore\ x = -3;\ x = -1$

y-intercept: $x = 0$

$y = 3$

TP: $x = \frac{-b}{2a}$

$= \frac{-4}{2}$

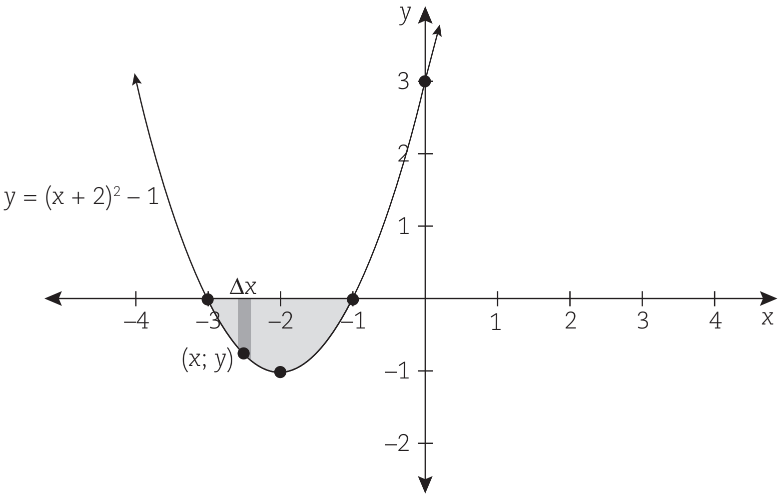
$= -2$

$y = (-2)^2 + 4(-2) + 3$

$= 4 - 8 + 3$

$= -1$

$\therefore$ TP: $(-2; -1)$



b) $\Delta A = y\Delta x$

$$A = -\int_{-3}^{-1} (x^2 + 4x + 3)\ dx$$

$$= -\left[\frac{x^3}{3} + 2x^2 + 3x\right]_{-3}^{-1}$$

$$= -\left\{\left[\tfrac{1}{3}(-1)^3 + 2(-1)^2 + 3(-1)\right] - \left[\tfrac{1}{3}(-3)^3 + 2(-3)^2 + 3(-3)\ \right]\right\}$$

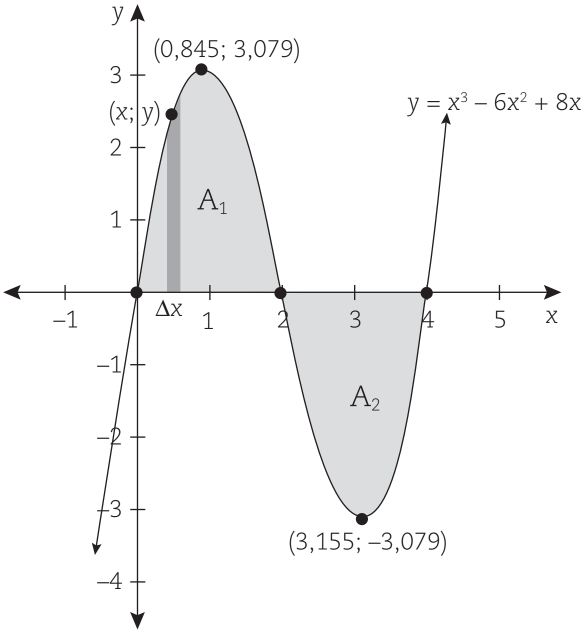
$$= -\{[-\tfrac{1}{3} + 2 - 3] - [-9 + 18 - 9]\}$$

$$= -\{[-1{,}333] - [0]\}$$

$$= -\{-1{,}333\}$$

$$= 1{,}333 \text{ units}^2$$

6. **a)** $f(x) = x^3 - 6x^2 + 8x$



y-intercept: $x = 0$

$\therefore y = 0$

x-intercept: $y = 0$

$x3 - 6x^2 + 8x = 0$

$x(x^2 - 6x + 8 = 0)$

$x(x - 4)(x - 2) = 0$

$x = 0$ or $x = 4$ or $x = 2$

Turning points:

$f\,'(x) = 3x^2 - 12x + 8 = 0$

$$x = \frac{-(-12) \pm \sqrt{(-12)^2 - (3)(8)}}{2(3)}$$

$$= \frac{12 \pm \sqrt{48}}{6}$$

$= 3{,}155$ or $0{,}845$

$f(3{,}155) = -3{,}079$

$f(0{,}8450 = 3{,}079$

b) $\Delta A = y\Delta x$

$$A_1 = \int_0^2 y\,dx$$
$$= \int_0^2 (x^3 - 6x^2 + 8x)\,dx$$
$$= \left[\frac{x^4}{4} - 2x^3 + 4x^2\right]_0^2$$
$$= \left[\frac{(2)^4}{4} - 2(2)^3 + 4(2)^2\right] - [0]$$
$$= 4 \text{ units}^2$$

$$A_2 = -\int_2^4 y\,dx$$
$$= -\int_2^4 (x^3 - 6x^2 + 8x)\,dx$$
$$= -\left[\frac{x^4}{4} - 2x^3 + 4x^2\right]_2^4$$
$$= -\left\{\left[\tfrac{1}{4}(4)^4 - 2(4)^3 + 4(4)^2\right] - \left[\tfrac{1}{4}(2)^4 - 2(2)^3 + 4(2)^2\right]\right\}$$
$$= -\{[0] - [4]\}$$
$$= 4 \text{ units}^2$$

$\therefore$ Total area $= A_1 + A_2$

$= (4 + 4)$ units^2

$= 8$ units^2

Solutions for summative assessments: Chapter 2

Summative assessment 1

Question 1

1.1 $x = \frac{b}{2a}$

$= -\frac{0}{2(4)}$

$= 0$ (1)

1.2 Yes: one-to-one function. (2)

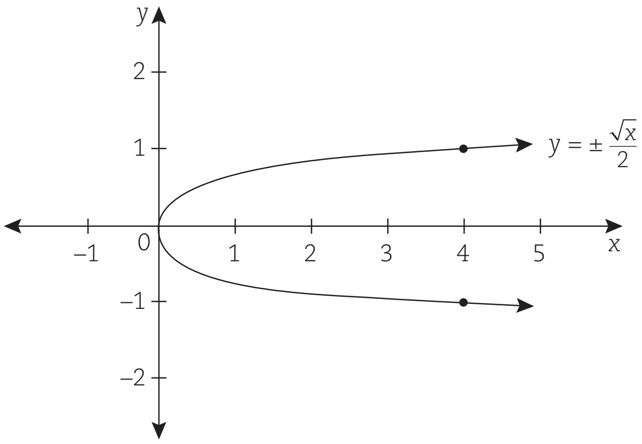
1.3 $f(x)\text{: } y = 4x^2$

$x = 4y^2$

$y^2 = \frac{x}{4}$

$y = \pm\ \frac{\sqrt{x}}{2}$ (1)

1.4



(4)

1.5 No. Every x-value has more than one y-value. (2)

1.6 $\{x\text{: } x \in \mathbb{R}\}$ (1)

1.7 $\{y\text{: } y \in \mathbb{R}\}$ (1)

1.8 Continuous (1)

[13]

Question 2

2.1 (0; 0) (1)

2.2 $y = 0$ (1)

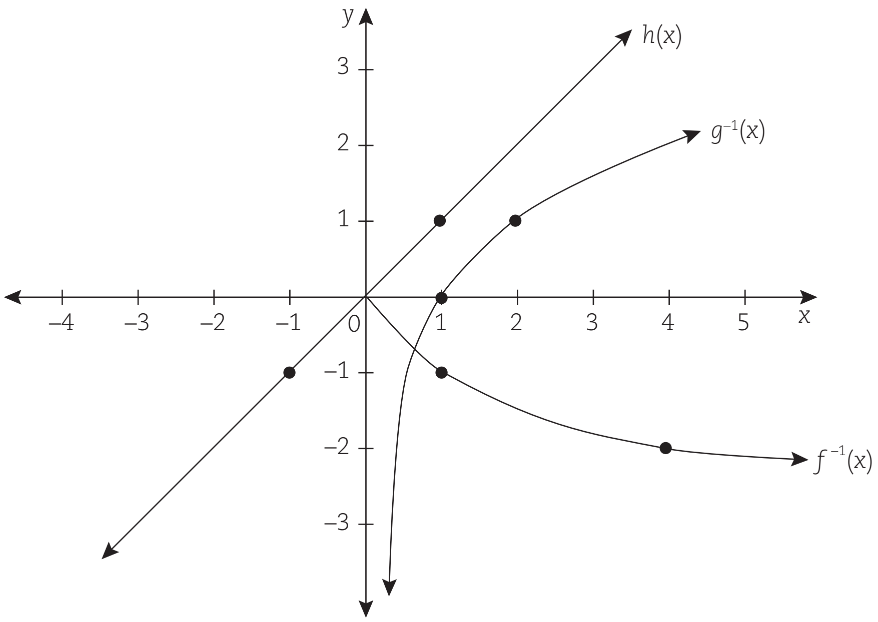
2.3 $x > 0$ (1)

2.4 $f(x) = x^2$

$x = y^2$

$\therefore\ y = \pm\ \sqrt{x}$ (2)

2.5



(2)

2.6 On the graph in question 2.5. (1)

2.7

$$f(x) = (x)$$
$$x^2 = x$$
$$x^2 - x = 0$$
$$x(x - 1) = 0$$
$$x = 0 \text{ or } x = 1$$
$$y = 0 \quad y = 1$$
$$(0; 0) \quad (1; 1)$$

(2)

2.8 $(1; 1), (–1; 1), (4; 2), (4; –2)$ (1)

2.9 $y \in \mathbb{R}$ or $\{y; y = 0; y \in \mathbb{R}\}$ (1)

2.10

$$g(x) = 2^x$$
$$g^{-1}(x)\text{: } x = 2^y$$

(1)

2.11 On the graph in question 2.5. (1)

2.12 $x > 0, x \in \mathbb{R}$ or $x \in (0; \infty)$ or $\{x\text{: } x > 0; x \in \mathbb{R}\}$ (1)

[15]

Question 3

3.1

$$2x^3 + 4x^2 - 6x - p = 0$$
$$(x - 3)(2x^2 + 10x + 24) = 0$$
$$\therefore\ 2x^3 - 6x^2 + 10x^2 - 30x + 24x - 72 = 0$$
$$2x^3 - 14x^2 - 6x - 72 = 0$$
$$\therefore\ p = -72$$

(3)

3.2

$$f(x) = 2x^3 - 3x^2 + px - 4$$
$$f(-2) = 8$$
$$\therefore\ 2(-2)^3 - 3(-2)^2 + p(-2) - 4 = 8$$
$$-16 - 12 - 2p - 4 = 8$$
$$-2p = 40$$
$$p = -20$$

(3)

3.3 **3.3.1** $x^3 - 5x^2 + 7x - 3$

$= (x - 1)(x^2 - 4x + 3)$
$= (x - 1)(x - 3)(x - 1)$ (3)

3.3.2 $2x^3 - 5x^2 - 4x + 3$
$= (x + 1)(2x^2 - 7x + 3)$
$= (x + 1)(2x - 1)(x - 3)$ (3)

3.3.3 $x^3 + 4x^2 + 5x + 20$
$= (x + 4)(x^2 + 5)$ (2)

3.4 $f(x) = 2x^3 + 3x^2 - 4x + 6$
$f(3) = 2(3)^3 + 3(3)^2 - 4(3) + 6$
$= 75$
$\therefore$ $x - 3$ is not a factor. (3)

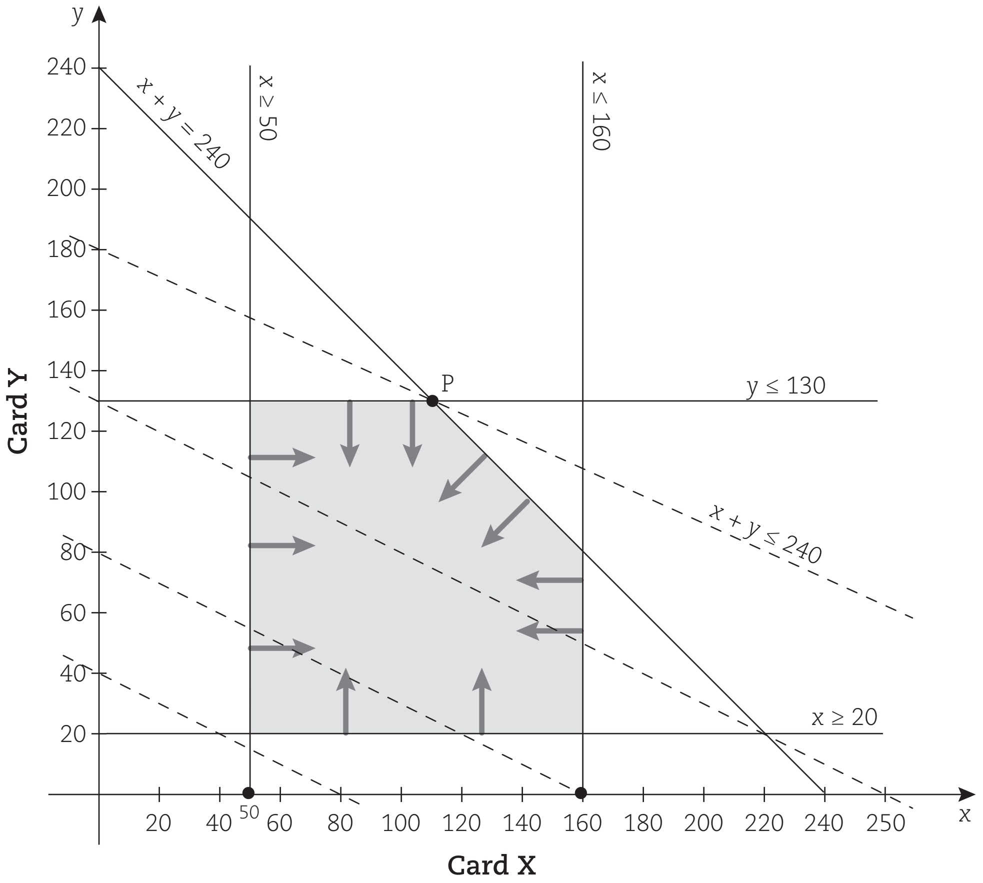
3.5 $f(x) = x^3 - 4x^2 + x + 6$
$= (x - 3)(x^2 - x + 2)$
$= (x - 3)(x - 2)(x + 1)$ (2)

[19]

Question 4

4.1
$x \leq 160$
$y \leq 130$
$x + y \leq 240$
$x \geq 20$
$x, y \in \mathbb{N}_0$ (5)

4.2 (4)



4.3 $P = 5x + 10y$ (1)

4.4 $y = -\frac{5x}{10} + \frac{p}{10}$

$= -\frac{1}{2}x + \frac{p}{10}$ (1)

4.5 At P: $x + y = 240$ and $y = 130$

$\therefore x = 240 - 130$

$= 110$

$P = 5(110) + 10(130)$

$= R1\ 850$ (2)

[13]

Total: [60]

Summative assessment 2

Question 1

1.1

$$f(x) = \frac{2}{x}$$

$$f(x+h) = \frac{2}{x+h}$$

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{\frac{2}{x+h} - \frac{2}{x}}{h} \\ &= \lim_{h\to 0} \frac{2x - 2(x+h)}{x(x+h)} \times \frac{1}{h} \\ &= \lim_{h\to 0} \frac{-2h}{x(x+h)h} \\ &= \lim_{h\to 0} \frac{-2}{x(x+h)} \\ &= \frac{-2}{x(x+0)} \\ &= \frac{-2}{x^2} \end{aligned}$$ (4)

1.2

$$f(x) = -x^2 + 3$$

$$\begin{aligned} f(x+h) &= -(x+h)^2 + 3 \\ &= -x^2 - 2xh - h^2 + 3 \\ &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{-x^2 - 2xh - h^2 + 3 - (-x^2 + 3)}{h} \\ &= \lim_{h\to 0} \frac{-2xh - h^2}{h} \\ &= \lim_{h\to 0} (-2x - h) \\ &= -2x \end{aligned}$$ (3)

1.3 **1.3.1**

$$y = \frac{3}{\sqrt{x^3}} - x^{-1} - 2x^3 + 2\cos 4x + \frac{3}{2x^2}$$

$$= 3x^{-\frac{3}{2}} - x^{-1} - 2x^3 + 2\cos 4x + \frac{3}{2}x^{-2}$$

$$\frac{dy}{dx} = -\frac{9}{2}x^{-\frac{5}{2}} + x^{-2} - 6x^2 - 8\sin 4x - 3x^{-3}$$

$$= -\frac{9}{2\sqrt{x^5}} + \frac{1}{x^2} - 6x^2 - 8\sin 4x - \frac{3}{x^3}$$

(5)

1.3.2

$$y = \frac{1}{2}e^{2x} - 4\ln x$$

$$\frac{dy}{dx} = e^{2x} - \frac{4}{x}$$

(2)

1.3.3

$$y = e^{-3x}(x^2 + 3)$$

$$= e^{-3x}(2x) + (-3e^{-3x})(x^2 + 3)$$

$$= e^{-3x}(2x) - 3e^{-3x}(x^2 + 2)$$

$$\text{or } = \frac{2x}{e^{3x}} - \frac{3x^2}{e^{3x}} - \frac{9}{e^{3x}}$$

$$\text{or } = \frac{2x - 3x^2 - 9}{e^{3x}}$$

(2)

1.4

$$\frac{d}{dx}\left(\frac{x^3}{3} - \frac{3}{x^3}\right)$$

$$= \frac{d}{dx}\left(\frac{x^3}{3} - 3x^{-3}\right)$$

$$= \frac{3x^2}{3} + 9x^{-4}$$

$$= x^2 + \frac{9}{x^4}$$

(2)

1.5

$$y = 5x^2.\tan x$$

$$\frac{dy}{dx} = 5x^2(\sec^2 x) + 10x.\tan x$$

$$= 5x^2 \,.\, \sec^2 x + 10x.\tan x$$

(3)

1.6

$$f(x) = \frac{2x^3 - 4}{3\ln 2x}$$

$$f'(x) = \frac{(3\ln 2x)(6x^2) - (2x^3 - 4)(\frac{3}{x})}{(3\ln 2x)^2}$$

$$= \frac{18x^2.\ln 2x - 6x^2 + \frac{12}{x}}{9\ln^2 2x}$$

(3)

1.7

$$y = e^{-3x + 4}$$

$$y = e^u \text{ where } u = -3x + 4$$

$$\frac{dy}{du} = e^u \qquad \frac{du}{dx} = -3$$

$$= e^{-3x + 4}$$

$$\therefore \frac{dy}{dx} = (e^{-3x + 4})(-3)$$

$$= -3e^{-3x + 4}$$

(3)

1.8 $y = (2x - 3)(3 - x)$ at $x = 2$

$= -2x^2 + 9x - 9$

$\frac{dy}{dx} = -4 + 9$

$m = -4(2) + 9$

$= 1$

$f(2) = -2(2)^2 + 9(2) - 9$

$= 1$

$(2; 1)$

$y = mx + c$

$1 = 1(2) + c$

$\therefore c = -1$

$\therefore y = x - 1$ (4)

[31]

Question 2

2.1 $s = t^3 + 10{,}5t^2 - 102t$

2.1.1 $u = \frac{ds}{dt} = 3t^2 + 21t - 102$

$u(3) = 3(3)^2 + 21(3) - 102$

$= -12$ m/s (3)

2.1.2 $a = \frac{du}{dt} = 6t + 21$

$a(3) = 6(3) + 21$

$= 39$ m/s^2 (3)

2.2 $s = 6t^2$

2.2.1 $s(4) = 6(4)^2$

$= 96$ m (3)

2.2.2 $u = \frac{ds}{dt}$

$= 12t$

$u(3) = 12(3)$

$= 36$ m/s (3)

2.2.3 $12t = 24$

$t = 2$ seconds (2)

2.2.4 $420 = 6t^2$

$t^2 = 70$

$t = \sqrt{70}$

$= 8{,}367$ seconds (3)

[17]

Question 3

3.1 $f(x) = x(x^2 - 12) + 3$

3.1.1 $f(x) = x^3 - 12x + 3$

$f'(x) = 3x^2 - 12$

$3x^2 - 12 = 0$

$(x - 2)(x + 2) = 0$

$x = 2$ or $x = -2$

$f(2) = -13$ $f(-2) = 19$

TP: (2; –13) and (–2; 19) (4)

3.1.2 $f''(x) = 6x$

$f''(2) = 6(2) = 12 > 0$ $\therefore$ minimum TP (2; –3)

$f''(-2) = 6(-2) = -12 < 0$ $\therefore$ maximum TP (–2; 19) (3)

3.1.3 $f''(x) = 0$

$\therefore 6x = 0$

$\therefore x = 0$

$\therefore f(0) = 3$

$\therefore (0; 3)$ (3)

3.2 $f(x) = x^3 - 8x^2 + 5x + 14$

y-intercept: $x = 0$

$\therefore y = 14$

x-intercept: $y = 0$

$\therefore x^3 - 8x^2 + 5x + 14 = 0$

$(x + 1)(x^2 - 9x + 14) = 0$

$(x + 1)(x - 7)(x - 2) = 0$

$\therefore x = -1$ or $x = 7$ or $x = 2$

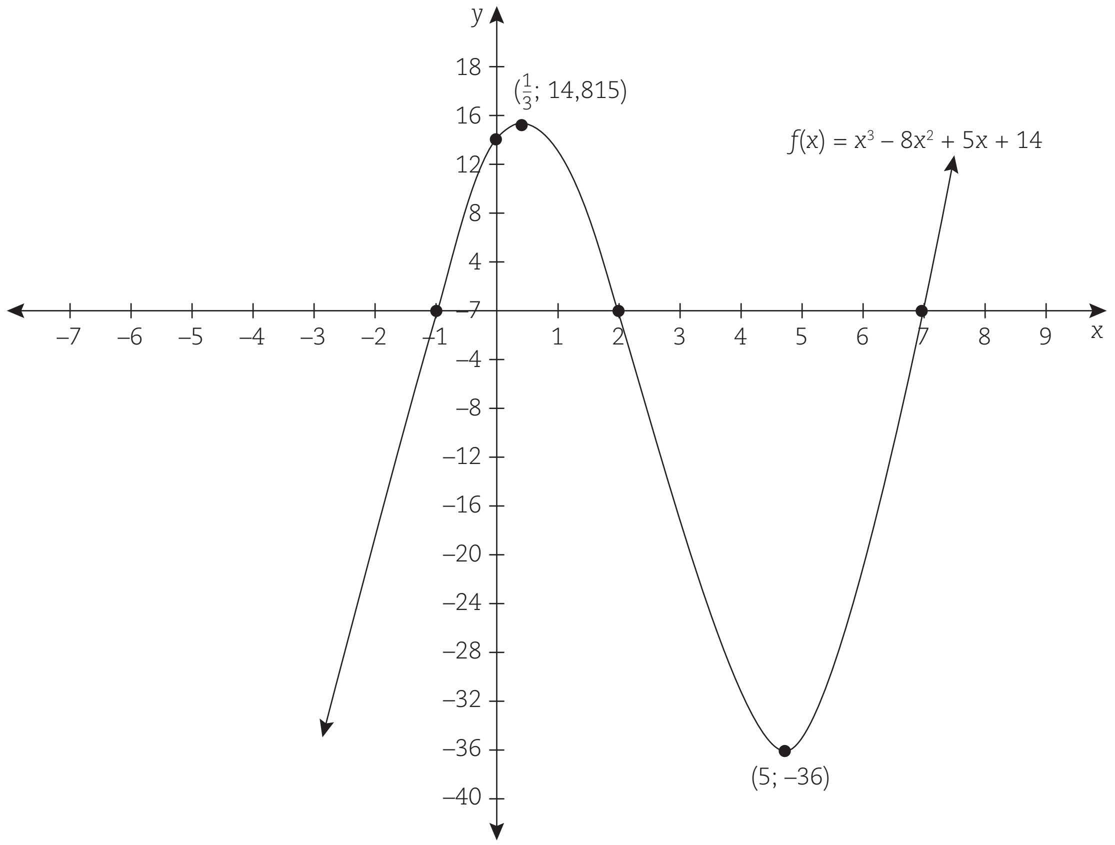
$f'(x) = 3x^2 - 16x + 5$

$(3x - 1)(x - 5) = 0$

$\therefore x = \frac{1}{3}$ or $x = 5$

$f\left(\frac{1}{3}\right) = 14{,}815$ $f(5) = -36$ (12)

The graph is shown on the next page.



[22]

Total: [70]

Summative assessment 3

Question 1

1.1 **1.1.1** $\int\left(x^3 + \frac{3}{x^2} - 4x^{-4} + 5p - 3\sqrt{x}\right) dx$

$= \int\left(x^3 + 3x^{-2} - 4x^{-4} + 5p - 3x^{\frac{1}{2}}\right) dx$

$= \frac{x^4}{4} + \frac{3x^{-1}}{-1} - \frac{4x^{-3}}{-3} + 5px - \frac{3x^{\frac{3}{2}}}{\frac{3}{2}}$

$= \frac{x^4}{4} - \frac{3}{x} + \frac{4}{3x^3} + 5px - 2\sqrt{x^3} + c$ (6)

1.1.2 $\int\left(e^{2x} + \frac{1}{3}\cos 3x - 2\sin 4x + \frac{3}{x}\right) dx$

$\int(e^{2x} + \frac{1}{3}\cos 3x - 2\sin 4x + 3x^{-1})\, dx$

$= \frac{e^{2x}}{2} + \frac{1}{3}\left(\frac{\sin 3x}{3}\right) - 2\left(\frac{-\cos 4x}{4}\right) + 3\ln x + c$

$= \frac{e^{2x}}{2} + \frac{\sin 3x}{9} + \frac{\cos 4x}{2} + 3\ln x + c$ (5)

1.2 **1.2.1** $\int_{-1}^{2} (-2x^2 + 4x)\, dx$

$= \left[\frac{-2x^3}{3} + \frac{4x^2}{2}\right]_{-1}^{2}$

$= \left[\frac{-2(2)^3}{3} + 2(2)^2\right] - \left[\frac{-2(-1)^3}{3} + 2(-1)^2\right]$

$= \left[\frac{8}{3}\right] - \left[\frac{8}{3}\right]$

$= 0$ (3)

1.2 **1.2.2** $\int_2^3 \left(\frac{x-1}{x}\right)^2 dx$

$= \int_2^3 \left(\frac{x^2-2x+1}{x^2}\right) dx$

$= \int_2^3 (1 - 2x^{-1} + x^{-2})\, dx$

$= \left[x - 2\ln x + \frac{x^{-1}}{-1}\right]_2^3$

$= \left[x - 2\ln x + \frac{1}{x}\right]_2^3$

$= \left[(3) - 2\ln(3) - \frac{1}{(3)}\right] - \left[(2) - 2\ln(2) - \frac{1}{(2)}\right]$

$= 0{,}469 - 1{,}189$

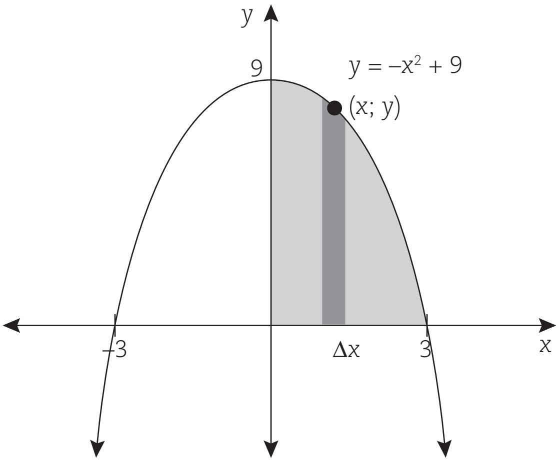
$= -0{,}72$ (4)

[18]

Question 2

2.1 **2.1.1** $y = -x^2 + 9$

(3)



2.1.2 $\Delta A = y\Delta x$

$= \int_0^3 (-x^2 + 9)\, dx$

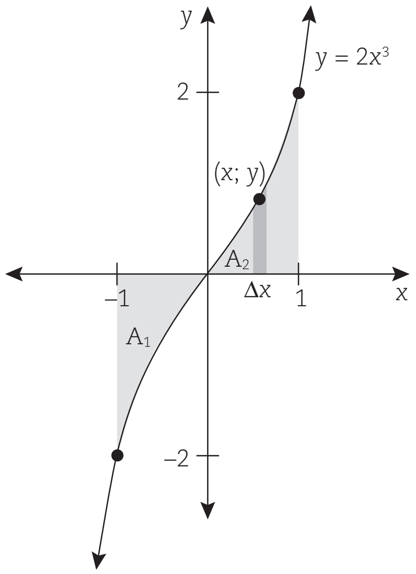
$= \left[-\frac{x^3}{3} + 9x\right]_0^3$

$= \left[-\frac{(3)^3}{3} + 9(3)\right] - \left[-\frac{(0)^3}{3} + 9(0)\right]$

$= [18] - [0]$

$= 18 \text{ units}^2$ (5)

2.2 **2.2.1** $y = 2x^3$



(3)

2.2.2 $\Delta A_1 = \int_{-1}^{0} 2x^3\,dx$

$$= -\left[\frac{2x^4}{4}\right]_{-1}^{0}$$

$$= -\left\{\left[\frac{2(0)^4}{4}\right] - \left[\frac{2(-1)^4}{4}\right]\right\}$$

$$= \frac{1}{2} \text{ units}^2$$

$\Delta A_2 = \int_{0}^{1} 2x^3\,dx$

$$= \left[\frac{2x^4}{4}\right]_{0}^{1}$$

$$= \left\{\frac{2(1)^4}{4} - \frac{2(0)^4}{4}\right\}$$

$$= \frac{1}{2} \text{ unit}^2$$

$\therefore A = \frac{1}{2} + \frac{1}{2} = 1 \text{ units}^2$ (6)

[17]

Total: [35]

Worked solutions • Chapter 3 Space, shape and measurement

Assessment activity 3.1

1. a)
$$\begin{aligned} x^2 + y^2 &= r^2 \\ (-3)^2 + (4)^2 &= r^2 \\ 25 &= r^2 \\ \therefore x^2 + y^2 &= 25 \end{aligned}$$

b)
$$\begin{aligned} x^2 + y^2 &= r^2 \\ (5)^2 + (-2)^2 &= r^2 \\ 29 &= r^2 \\ \therefore x^2 + y^2 &= 29 \end{aligned}$$

c)
$$\begin{aligned} x^2 + y^2 &= r^2 \\ (-4)^2 + (-2\sqrt{5})^2 &= r^2 \\ 36 &= r^2 \\ \therefore x^2 + y^2 &= 36 \end{aligned}$$

d)
$$\begin{aligned} x^2 + y^2 &= r^2 \\ x^2 + y^2 &= (4)^2 \\ \therefore x^2 + y^2 &= 16 \end{aligned}$$

e)
$$\begin{aligned} x^2 + y^2 &= r^2 \\ x^2 + y^2 &= (\sqrt{13})^2 \\ \therefore x^2 + y^2 &= 13 \end{aligned}$$

f)
$$\begin{aligned} x^2 + y^2 &= r^2 \\ x^2 + y^2 &= (2\sqrt{11})^2 \\ \therefore x^2 + y^2 &= 44 \end{aligned}$$

2. a)
$$\begin{aligned} x^2 + y^2 &= 16 \\ r^2 &= 16 \\ \therefore r &= 4 \text{ units} \end{aligned}$$

b)
$$\begin{aligned} x^2 + y^2 &= 24 \\ r^2 &= 24 \\ \therefore r &= 2\sqrt{6} \text{ units} \end{aligned}$$

c)
$$\begin{aligned} -3x^2 - 3y^2 &= -147 && : \div -3 \\ x^2 + y^2 &= 49 \\ r^2 &= 49 \\ \therefore r &= 7 \text{ units} \end{aligned}$$

d)
$$\begin{aligned} 25x^2 + 25y^2 &= 81 && : \div 25 \\ x^2 + y^2 &= \frac{81}{25} \\ r^2 &= \frac{81}{25} \\ \therefore r &= \frac{9}{5} \text{ units} \end{aligned}$$

e)
$$\begin{aligned} 16x^2 + 16y^2 - 144 &= 0 \\ 16x^2 + 16y^2 &= 144 && : \div 16 \\ x^2 + y^2 &= 9 \\ r^2 &= 9 \\ \therefore r &= 3 \text{ units} \end{aligned}$$

f)
$$\begin{aligned} \frac{x^2}{9} + \frac{y^2}{9} &= 25 && : \times 9 \\ x^2 + y^2 &= 225 \\ r^2 &= 225 \\ \therefore r &= 15 \text{ units} \end{aligned}$$

3.
$$\begin{aligned} x^2 + y^2 &= r^2 \\ (2\sqrt{3})^2 + (-5)^2 &= r^2 \\ 37 &= r^2 \\ \therefore r &= \sqrt{37} \text{ units} \end{aligned}$$

4.
$$\begin{aligned} x^2 + y^2 &= 97 \\ (-4)^2 + (k)^2 &= 97 \\ 16 + k^2 &= 97 \\ k^2 &= 97 - 16 \\ k^2 &= 81 \\ \therefore k &= \pm 9 \end{aligned}$$

5. **a)** centre (3; 2) and radius = 5

$(x - h)^2 + (y - k)^2 = r^2$

$(x - (3))^2 + (y - (2))^2 = (5)^2$

$\therefore (x - 3)^2 + (y - 2)^2 = 25$

b) centre (–1; 4) and radius = $3\sqrt{7}$

$(x - h)^2 + (y - k)^2 = r^2$

$(x - (-1))^2 + (y - (4))^2 = (3\sqrt{7})^2$

$\therefore (x + 1)^2 + (y - 4)^2 = 63$

c) centre (–5; –3) and point = (3; –2)

$(x - h)^2 + (y - k)^2 = r^2$

$((3) - (-5))^2 + ((-2) - (-3))^2 = r^2$

$64 + 1 = r^2$

$65 = r^2$

$(x - (-5))^2 + (y - (-3))^2 = 65$

$\therefore (x + 5)^2 + (y + 3)^2 = 65$

d) centre ($\sqrt{2}$; –4) and point = ($-3\sqrt{2}$; 5)

$(x - h)^2 + (y - k)^2 = r^2$

$((-3\sqrt{2}) - (\sqrt{2}))^2 + ((5) - (-4))^2 = r^2$

$32 + 81 = r^2$

$113 = r^2$

$(x - (\sqrt{2}))^2 + (y - (-4))^2 = 113$

$\therefore (x - \sqrt{2})^2 + (y + 4)^2 = 113$

6. **a)** $(x - 2)^2 + (y + 3)^2 = 169$

$r^2 = 169$

$\therefore r = 13$ units

$x - 2 = 0$ and $y + 3 = 0$

$x = 2$ $\quad y = -3$

$\therefore$ centre = $(h; k) = (2; -3)$

b) $(x + 2)^2 + y^2 = 27$

$(x + 2)^2 + (y - 0)^2 = 27$

$r^2 = 27$

$\therefore r = 3\sqrt{3}$ units

$x + 2 = 0$ and $y - 0 = 0$

$x = -2$ $\quad y = 0$

$\therefore$ centre = $(h; k) = (-2; 0)$

c) $x^2 + y^2 - 8y = 0$

$(x^2) + (y^2 - 8y) = 0$

$(x - 0)^2 + \left(y^2 - 8y + \left(\frac{1}{2} . -8\right)^2\right) = 0 + \left(\frac{1}{2} . -8\right)^2$

$(x - 0)^2 + (y^2 - 8y + 16) = 16$

$(x - 0)^2 + (y - 4)^2 = 16$

$r^2 = 16$

$\therefore r = 4$ units

$x - 0 = 0$ and $y - 4 = 0$

$x = 0$ $\quad y = 4$

$\therefore$ centre = $(h; k) = (0; 4)$

d)

$$x^2 + 4x + y^2 - 10y = 7$$
$$(x^2 + 4x) + (y^2 - 10y) = 7$$
$$(x^2 + 4x + \left(\frac{1}{2}.4\right)^2\Big) + (y^2 - 10y + \left(\frac{1}{2}.-10\right)^2\Big) = 7 + \left(\frac{1}{2}.4\right)^2 + \left(\frac{1}{2}.-10\right)^2$$
$$(x^2 + 4x + 4) + (y^2 - 10y + 25) = 7 + 4 + 25$$
$$(x + 2)^2 + (y - 5)^2 = 36$$

$r^2 = 36$

$\therefore r = 6$ units

$x + 2 = 0$ and $y - 5 = 0$

$x = -2$ $y = 5$

$\therefore$ centre $= (h; k) = (-2; 5)$

e)

$$x^2 - 3x + y^2 - 4y - 14 = 0$$
$$(x^2 - 3x) + (y^2 - 4y) = 14$$
$$\left(x^2 - 3x + \left(\frac{1}{2}.-3\right)^2\right) + (y^2 - 4y + \left(\frac{1}{2}.-4\right)^2\Big) = 14 + \left(\frac{1}{2}.-3\right)^2 + \left(\frac{1}{2}.-4\right)^2$$
$$\left(x^2 - 3x + \frac{9}{4}\right) + (y^2 - 4y + 4) = 14 + \frac{9}{4} + 4$$
$$\left(x - \frac{3}{2}\right)^2 + (y - 2)^2 = \frac{81}{4}$$

$r^2 = \frac{81}{4}$

$\therefore r = \frac{9}{2}$ units

$x - \frac{3}{2} = 0$ and $y - 2 = 0$

$x = \frac{3}{2}$ $y = 2$

$\therefore$ centre $= (h; k) = \left(\frac{3}{2}; 2\right)$

f)

$$-2x^2 - 2y^2 + 12x + 4y + 108 = 0 \qquad : \div -2$$
$$x^2 + y^2 - 6x - 2y - 54 = 0$$
$$(x^2 - 6x) + (y^2 - 2y) = 54$$
$$\left(x^2 - 6x + \left(\frac{1}{2}.-6\right)^2\right) + \left(y^2 - 2y + \left(\frac{1}{2}.-2\right)^2\right) = 54 + \left(\frac{1}{2}.-6\right)^2 + \left(\frac{1}{2}.-2\right)^2$$
$$(x^2 - 6x + 9) + (y^2 - 2y + 1) = 54 + 9 + 1$$
$$(x - 3)^2 + (y - 1)^2 = 64$$

$r^2 = 64$

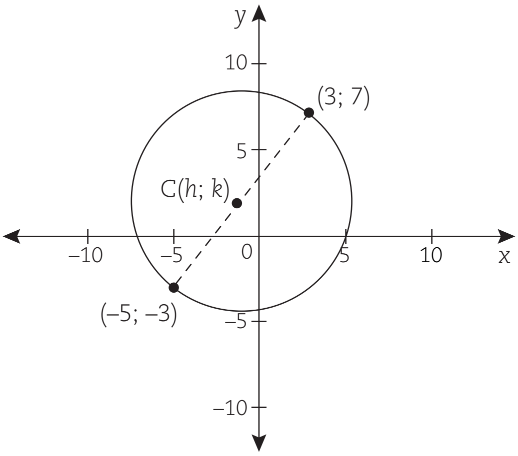
$\therefore r = 8$ units

$x - 3 = 0$ and $y - 1 = 0$

$x = 3$ $y = 1$

$\therefore$ centre $= (h; k) = (3; 1)$

7.



$$M = \left(\frac{x_1 + x_2}{2}; \frac{y_1 + y_2}{2}\right)$$

$$= \left(\frac{(-5) + (3)}{2}; \frac{(-3) + (7)}{2}\right)$$

$$M = (-1; 2)$$

$$(x - h)^2 + (y - k)^2 = r^2$$
$$((-5) - (-1))^2 + ((-3) - (2))^2 = r^2$$
$$(-4)^2 + (-5)^2 = r^2$$
$$41 = r^2$$
$$(x + 1)^2 + (y - 2)^2 = 41$$

$$((6) + 1)^2 + ((-4) - 2)^2 = r^2$$
$$85 = r^2$$
$$r = \sqrt{85} \text{ units}$$

Since $r = \sqrt{85}$ is greater than $r = \sqrt{41}$, therefore the point (6; –4) is outside the circle.

8.

$$(x + 3)^2 + (y - 20)^2 = 289$$
$$((m) + 3)^2 + ((5) - 20)^2 = 289$$
$$(m + 3)^2 + (-15)^2 = 289$$
$$(m + 3)^2 + 225 = 289$$
$$(m + 3)^2 = 289 - 225$$
$$(m + 3)^2 = 64$$
$$m + 3 = \pm 8$$

$\therefore m + 3 = 8$ and $m + 3 = -8$

$m = 8 - 3$ $\quad m = -8 - 3$

$m = 5$ $\quad m = -11$

Assessment activity 3.2

1. **a)** $P(x_1; y_1) = (9; -12)$

$$m_{OP} = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{(0) - (-12)}{(0) - (9)}$$
$$m_{OP} = -\frac{4}{3}$$

$$m_{OP} \times m_{tan} = -1$$
$$m_{tan} = \frac{-1}{m_{OP}}$$
$$= \frac{-1}{-\frac{4}{3}}$$
$$m_{tan} = \frac{3}{4}$$

$$y - y_1 = m(x - x_1)$$
$$y - (-12) = \frac{3}{4}(x - (9))$$
$$y + 12 = \frac{3}{4}x - \frac{27}{4}$$
$$y = \frac{3}{4}x - \frac{75}{4}$$

b) $P(x_1; y_1) = (-5; -12)$

$$m_{OP} = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{(0) - (-12)}{(0) - (-5)}$$
$$m_{OP} = \frac{12}{5}$$

$$m_{OP} \times m_{tan} = -1$$
$$m_{tan} = \frac{-1}{m_{OP}}$$
$$= \frac{-1}{\frac{12}{5}}$$
$$m_{tan} = -\frac{5}{12}$$

$$y - y_1 = m(x - x_1)$$
$$y - (-12) = -\frac{5}{12}(x - (-5))$$
$$y + 12 = -\frac{5}{12}x - \frac{25}{12}$$
$$y = -\frac{5}{12}x - \frac{169}{12}$$

c) $P(x_1; y_1) = (-1; 19)$

$(x - 6)^2 + (y + 5)^2 = 625$

$\therefore$ centre $= (h; k) = (6; -5)$

$$m_{OP} = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{(-5) - (19)}{(6) - (-1)}$$
$$m_{OP} = -\frac{24}{7}$$

$$m_{OP} \times m_{tan} = -1$$
$$m_{tan} = \frac{-1}{m_{OP}}$$
$$= \frac{-1}{-\frac{24}{7}}$$
$$m_{tan} = \frac{7}{24}$$

$$y - y_1 = m(x - x_1)$$
$$y - (19) = \frac{7}{24}(x - (-1))$$
$$y - 19 = \frac{7}{24}x + \frac{7}{24}$$
$$y = \frac{7}{24}x + \frac{463}{24}$$

d) $P(x_1; y_1) = (16; 38)$

$$x^2 + y^2 - 8x - 6y = 1344$$
$$(x^2 - 8x) + (y^2 - 6y) = 1344$$
$$\left(x^2 - 8x + \left(\tfrac{1}{2} . -8\right)^2\right) + \left(y^2 - 6y + \left(\tfrac{1}{2} . -6\right)^2\right) = 1\,344 + \left(\tfrac{1}{2} . -8\right)^2 + \left(\tfrac{1}{2} . -6\right)^2\Big)$$
$$(x^2 - 8x + 16) + (y^2 - 6y + 9) = 1\,344 + 16 + 9$$
$$(x - 4)^2 + (y - 3)^2 = 1\,369$$

$\therefore$ centre $= (h; k) = (4; 3)$

$$m_{OP} = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{(3) - (38)}{(4) - (16)}$$
$$m_{OP} = \frac{35}{12}$$

$$m_{OP} \times m_{tan} = -1$$
$$m_{tan} = \frac{-1}{m_{OP}}$$
$$= \frac{-1}{\frac{35}{12}}$$
$$m_{tan} = -\frac{12}{35}$$

$$y - y_1 = m(x - x_1)$$
$$y - (38) = -\frac{12}{35}\ (x - (16))$$
$$y - 38 = -\frac{12}{35} x + \frac{192}{35}$$
$$y = -\frac{12}{35} x + \frac{1522}{35}$$

2. a) $x^2 + y^2 = 289$

$\therefore r^2 = 289$

$(x_1; y_1) = (15; 8)$

$$x_1x + y_1y = r^2$$
$$(15)x + (8)y = 289$$
$$15x + 8y = 289$$
$$8y = -15x + 289$$
$$y = -\frac{15}{8} x + \frac{289}{8}$$

b) $x^2 + y^2 = 20$

$\therefore r^2 = 20$

$(x_1; y_1) = (-4; 2)$

$$x_1x + y_1y = r^2$$
$$(-4)x + (2)y = 20$$
$$-4x + 2y = 20$$
$$2y = 4x + 20$$
$$y = 2x + 10$$

c) $r = 5$

$(x_1; y_1) = (3; -4)$

$$x_1x + y_1y = r^2$$
$$(3)x + (-4)y = (5)^2$$
$$3x - 4y = 25$$
$$-4y = -3x + 25$$
$$y = \frac{3}{4} x - \frac{25}{4}$$

d) $r = \sqrt{13}$

$(x_1; y_1) = (-3; -2)$

$$x_1x + y_1y = r^2$$
$$(-3)x + (-2)y = (\sqrt{13})^2$$
$$-3x - 2y = 13$$
$$-2y = 3x + 13$$
$$y = -\frac{3}{2} x - \frac{13}{2}$$

3. **a)**

$$x^2 + y^2 = r^2$$
$$x^2 + y^2 = (\sqrt{19})^2$$
$$\therefore x^2 + y^2 = 19$$

b) $y = -2x + 4$ and $x^2 + y^2 = 19$

Substitute $y = -2x + 4$ into $x^2 + y^2 = 19$:

$$x^2 + (-2x + 4)^2 = 19$$
$$x^2 + 4x^2 - 16x + 16 = 19$$
$$5x^2 - 16x - 3 = 0$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-(-16) \pm \sqrt{(-16)^2 - 4(5)(-3)}}{2(5)}$$
$$x = \frac{16 \pm \sqrt{316}}{10}$$
$$x = \frac{16 + \sqrt{316}}{10} \text{ and } x = \frac{16 - \sqrt{316}}{10}$$
$$x = 3{,}378 \qquad x = -0{,}178$$

Substitute $x = 3{,}378$ and $x = -0{,}178$ into $y = -2x + 4$:

$y = -2(3{,}378) + 4$ and $y = -2(-0{,}178) + 4$
$y = -2{,}755$ $\qquad$ $y = 4{,}355$

$(3{,}378; -2{,}755)$ and $(-0{,}178; 4{,}355)$

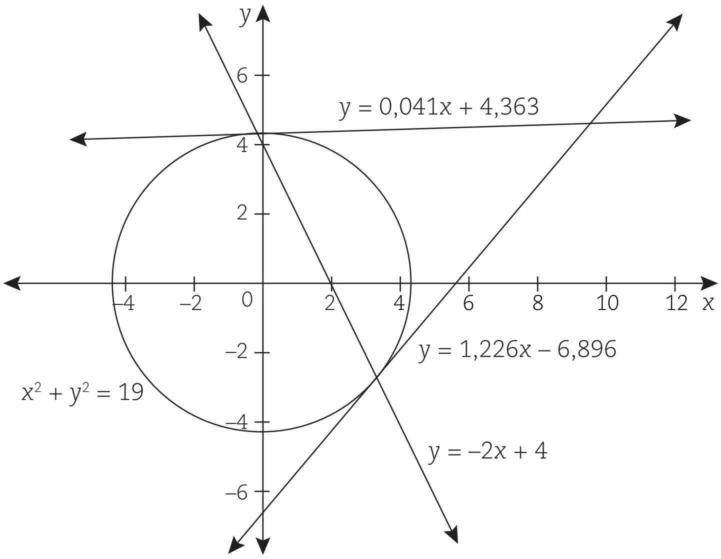
c) Tangent at $(3{,}378; -2{,}755)$, $r^2 = 19$ and $(x_1; y_1) = (3{,}378; -2{,}755)$

$$x_1x + y_1y = r^2$$
$$(3{,}378)x + (-2{,}755)y = 19$$
$$3{,}378x + -2{,}755y = 19$$
$$-2{,}755y = -3{,}378x + 19$$
$$y = 1{,}226x - 6{,}896$$

Tangent at $(-0{,}178; 4{,}355)$, $r^2 = 19$ and $(x_1; y_1) = (-0{,}178; 4{,}355)$

$$x_1x + y_1y = r^2$$
$$(-0{,}178)x + (4{,}355)y = 19$$
$$-0{,}178x + 4{,}355y = 19$$
$$4{,}355y = 0{,}178x + 19$$
$$y = 0{,}041x + 4{,}363$$

d)



4. **a)** $y = mx + c$ ① and $x^2 + y^2 = r^2$ ②

Substitute ① into ②:

$$x^2 + (mx + c)^2 = r^2$$
$$x^2 + m^2x^2 + 2mcx + c^2 = r^2$$
$$x^2\ (m^2 + 1) + 2mc.x + c^2 - r^2 = 0$$

Substitute into quadratic formula,

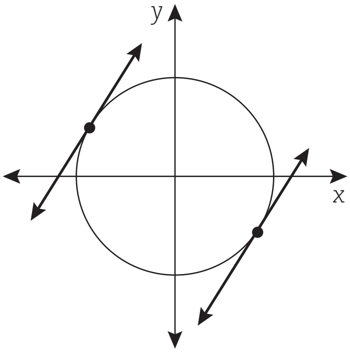
$$x = \frac{-(2mc) \pm \sqrt{(2mc)^2 - 4(m^2 + 1)(c^2 - r^2)}}{2(m^2 + 1)}$$

$$x = \frac{-mc \pm \sqrt{r^2(m^2 + 1) - c^2}}{m^2 + 1}$$

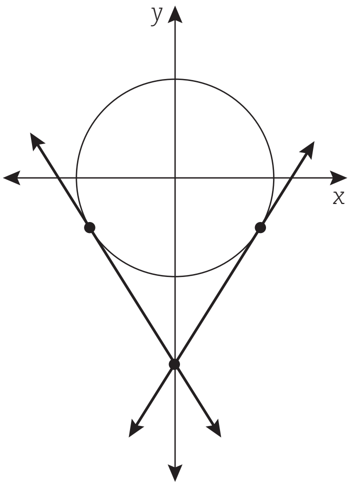
Since only one point of contact exists, therefore the discriminant will be equal to zero, that is,

$$r^2(m^2 + 1) - c^2 = 0$$
$$-c^2 = -r^2\ (m^2 + 1)$$
$$c^2 = r^2\ (m^2 + 1)$$

b) **i)** The tangents with the same gradients,



(ii) The tangents with the same y-intercepts,



c) **(i)** $x^2 + y^2 = 28$

$\therefore r^2 = 28$

$\sqrt{r^2} = \sqrt{28}$

$r = 2\sqrt{7}$ units

$c = 8$

$c^2 = r^2\,(m^2 + 1)$

$(8)^2 = (2\sqrt{7}\,)^2\,(m^2 + 1)$

$64 = 28\,(m^2 + 1)$

$\frac{64}{28} = m^2 + 1$

$\frac{64}{28} - 1 = m^2$

$\frac{9}{7} = m^2$

$\sqrt{\frac{9}{7}} = \sqrt{m^2}$

$\pm\,\frac{3}{\sqrt{7}} = m$

$y = \pm\,\frac{3}{\sqrt{7}}x + 8$

$y = +\,\frac{3}{\sqrt{7}}x + 8$ and $y = -\frac{3}{\sqrt{7}}x + 8$

(ii) $x^2 + y^2 - 36 = 0$

$x^2 + y^2 = 36$

$\therefore r^2 = 36$

$\sqrt{r^2} = \sqrt{36}$

$r = 6$ units

$m = 5$

$c^2 = r^2\,(m^2 + 1)$

$= (6)^2\,((5)^2 + 1)$

$c^2 = 936$

$\sqrt{c^2} = \sqrt{936}$

$c = \pm\,6\sqrt{26}$

$y = 5x \pm 6\sqrt{26}$

$y = 5x + 6\sqrt{26}$ and $y = 5x - 6\sqrt{26}$

(iii) $-\frac{1}{4}x^2 - \frac{1}{4}y^2 = -16$

$x^2 + y^2 = 64$

$\therefore r^2 = 64$

$\sqrt{r^2} = \sqrt{64}$

$r = 8$ units

$\theta = 45°$

$\therefore m = \tan\theta$

$= \tan 45°$

$m = 1$

$c^2 = r^2\,(m^2 + 1)$

$= (8)^2\,((1)^2 + 1)$

$c^2 = 128$

$\sqrt{c^2} = \sqrt{128}$

$c = \pm\,8\sqrt{2}$

$y = x \pm 8\sqrt{2}$

$y = x + 8\sqrt{2}$ and $y = x - 8\sqrt{2}$

5. a) $4x^2 + 4y^2 = 100$

$x^2 + y^2 = 25$

$\therefore r^2 = 25$

$\sqrt{r^2} = \sqrt{25}$

$r = 5$ units

Since (4; –5) is a point on the tangents,

$y = mx + c$

$-5 = m(4) + c$

$-5 = 4m + c$

$\therefore c = -4m - 5$

Substitute $c = -4m - 5$ and $r^2 = 25$ into $c^2 = r^2(m^2 + 1)$,

$c^2 = r^2(m^2 + 1)$

$(-4m - 5)^2 = 25(m^2 + 1)$

$16m^2 + 40m + 25 = 25m^2 + 25$

$-9m^2 + 40m = 0$

$9m^2 - 40m = 0$

$m(9m - 40) = 0$

$m = 0$ and $9m - 40 = 0$

$9m = 40$

$m = \frac{40}{9}$

Substitute $m = 0$ and $m = \frac{40}{9}$ into $c = -4m - 5$,

$c = -4(0) - 5$ and $c = -4\left(\frac{40}{9}\right) - 5$

$c = -5$ $\qquad c = -\frac{205}{9}$

Therefore, the equation of the tangents are,

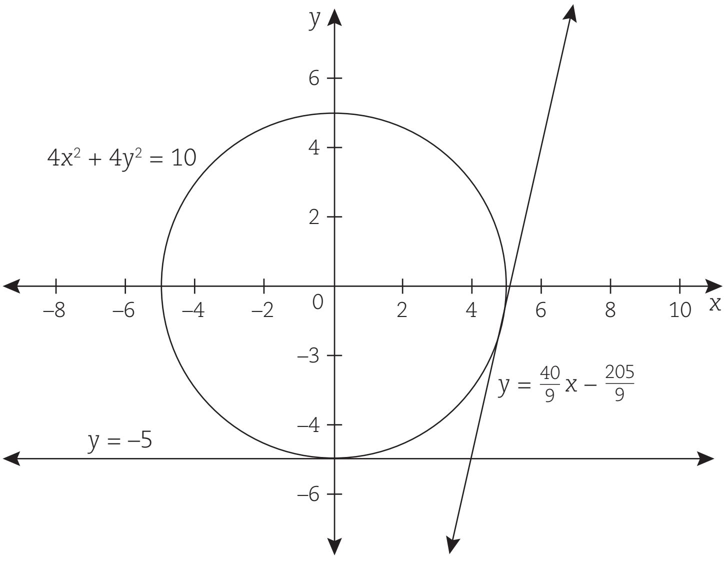
$y = (0)x + (-5)$

$y = -5$

$y = \left(\frac{40}{9}\right)x + \left(-\frac{205}{9}\right)$

$y = \frac{40}{9}x - \frac{205}{9}$

b)



6. $(x_1; y_1) = (10; -2)$

$(x + 2)^2 + (y - 3)^2 = 13^2$

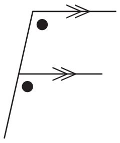
$\therefore r^2 = 13^2$

$\therefore$ centre $= (h; k) = (-2; 3)$

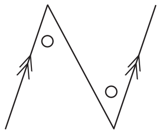
$$(x_1 - h)(x - h) + (y_1 - k)(y - k) = r^2$$
$$((10) - (-2))(x - (-2)) + ((-2) - (3))(y - (3)) = 13^2$$
$$12x + 24 - 5y + 15 = 169$$
$$-5y = -12x + 130$$
$$y = \frac{12}{5}x - 26$$

Assessment activity 3.3

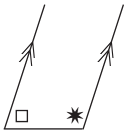
1.



Corresponding angles



Consecutive interior angles



Alternate interior angles

2. $\hat{s} = 117°$

- Vertically opposite angle.

$\hat{t} + \hat{s} = 180°$

$\hat{t} + 117° = 180°$

$\therefore \hat{t} = 180° - 117°$

$\hat{t} = 63°$

- Supplementary angles.

$\hat{u} = \hat{t}$

$\therefore \hat{u} = 63°$

- Vertically opposite angles.

$\hat{v} + \hat{u} = 180°$

$\hat{v} + 63° = 180°$

$\therefore \hat{v} = 180° - 63°$

$\hat{v} = 117°$

- Consecutive interior angles.

$\hat{x} + \hat{v} = 180°$

$\hat{x} + 117° = 180°$

$\therefore \hat{x} = 180° - 117°$

$\hat{x} = 63°$

- Supplementary angles.

$\hat{y} = \hat{x}$

$\therefore \hat{y} = 63°$

- Vertically opposite angles.

$\hat{z} + \hat{y} = 180°$

$\hat{z} + 63° = 180°$

$\therefore \hat{z} = 180° - 63°$

$\hat{z} = 117°$

- Supplementary angles.

3.

$$\begin{aligned} 2x - 10^\circ &= 110^\circ \\ 2x &= 110^\circ + 10^\circ \\ 2x &= 120^\circ \\ \therefore x &= 60^\circ \end{aligned}$$

- Alternate exterior angles.

4.

$$\begin{aligned} N\hat{O}M + M\hat{O}Q + Q\hat{O}P &= 180^\circ \\ 4q + 90^\circ + 2q &= 180^\circ \\ 6q + 90^\circ &= 180^\circ \\ 6q &= 180^\circ - 90^\circ \\ 6q &= 90^\circ \\ \therefore q &= 15^\circ \end{aligned}$$

- Straight angle.

$$\begin{aligned} \therefore N\hat{O}M &= 4q \\ &= 4(15^\circ) \\ N\hat{O}M &= 60^\circ \end{aligned}$$

$$\begin{aligned} \therefore P\hat{O}Q &= 2q \\ &= 2(15^\circ) \\ P\hat{O}Q &= 30^\circ \end{aligned}$$

5.

$$\begin{aligned} R\hat{O}S &= 25^\circ \\ \hat{a} &= 25^\circ \end{aligned}$$

- Vertically opposite angles.

$$\begin{aligned} X\hat{O}U + U\hat{O}T &= 90^\circ \\ 25^\circ + \hat{b} &= 90^\circ \\ \therefore \hat{b} &= 90^\circ - 25^\circ \\ \hat{b} &= 65^\circ \end{aligned}$$

- Complementary angles.

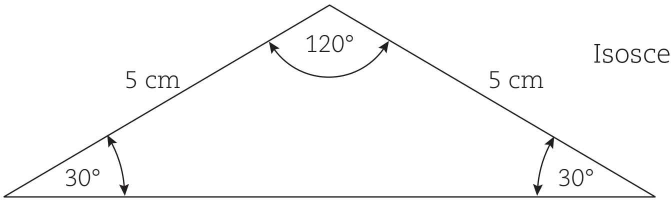
$$\begin{aligned} X\hat{O}R + R\hat{O}S &= 180^\circ \\ \hat{c} + \hat{a} &= 180^\circ \\ \hat{c} + 25^\circ &= 180^\circ \\ \therefore \hat{c} &= 180^\circ - 25^\circ \\ \hat{c} &= 155^\circ \end{aligned}$$

- Straight angles.

Assessment activity 3.4

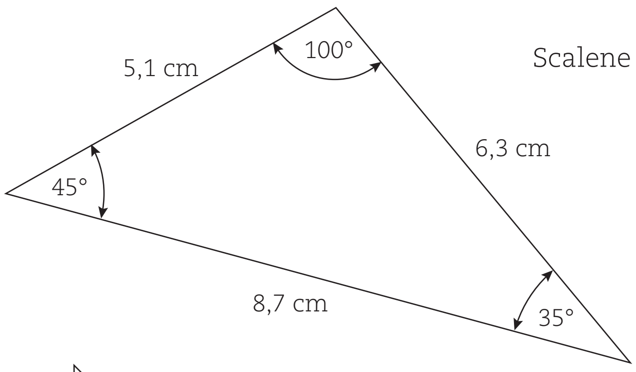
1. **a)** Scalene triangle: sides are all different lengths and all three angles different.
   b) Isosceles triangle: two equal sides and the angles opposite the sides are also equal.
   c) Equilateral triangle: all three sides are equal and each angle measures 60°.
   d) Acute-angled triangle: all interior angles are less than 90°.
   e) Obtuse-angled triangle: one interior angle is more than 90°.
   f) Right-angled triangle: one interior angle is a right angle, that is 90°.

2. **a)**



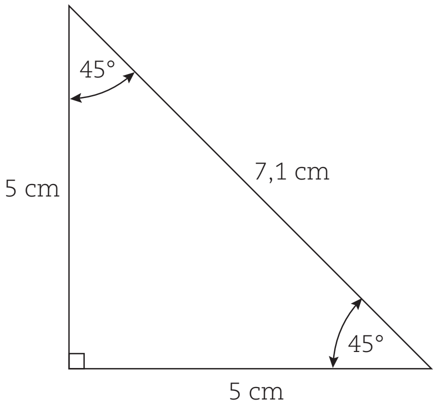
Isosceles obtuse triangle.

b)



Scalene obtuse triangle.

c)



Isosceles right-angled triangle.

d) Scalene isosceles triangle.
Not possible to have a scalene (all sides different lengths) and isosceles (two equal sides) triangle at the same time.

3. $\hat{a} + 69° + 74° = 180°$
$\hat{a} + 143° = 180°$
$\therefore \hat{a} = 180° - 143°$
$\hat{a} = 37°$

- Sum of the interior angles of a triangle is equal to 180°.

$\hat{b} + 102° = 180°$
$\therefore \hat{b} = 180° - 102°$
$\hat{b} = 78°$

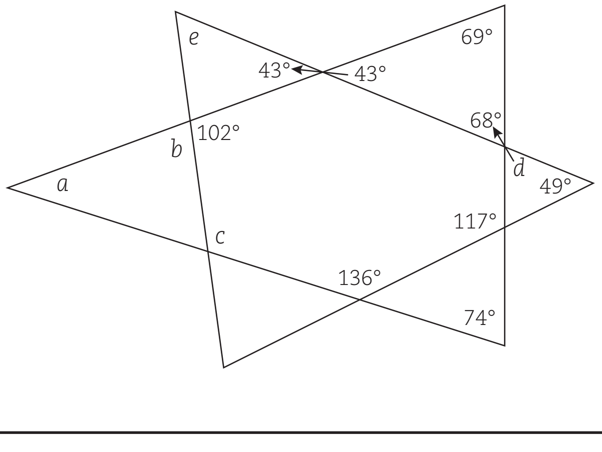
- Straight angle.

$\hat{c} = \hat{a} + \hat{b}$
$= 37° + 78°$
$\hat{c} = 115°$

- Exterior angle of a triangle is equal to the sum of the two opposite interior angles.

$\hat{d} + 49° = 117°$
$\therefore \hat{d} = 117° - 49°$
$\hat{d} = 68°$

- Exterior angle of a triangle is equal to the sum of the two opposite interior angles.



$$\hat{e} + 43° = 102°$$
$$\therefore \hat{e} = 102° - 43°$$
$$\hat{e} = 59°$$

- Exterior angle of triangle is equal to the sum of the two opposite interior angles.

4.

$$\hat{a} = 90°$$

- Corresponding angles.

$$\hat{b} + 37° + 90° = 180°$$
$$\hat{b} + 127° = 180°$$
$$\therefore \hat{b} = 180° - 127°$$
$$\hat{b} = 53°$$

- Sum of the interior angles of a triangle is equal to 180°.

$$\hat{c} + \hat{b} = 180°$$
$$\hat{c} + 53° = 180°$$
$$\therefore \hat{c} = 180° - 53°$$
$$\hat{c} = 127°$$

- Straight angles.

$$\hat{d} = \hat{c}$$
$$\therefore \hat{d} = 127°$$

- Alternate interior angles.

$$\hat{e} = \hat{b}$$
$$\therefore \hat{e} = 53°$$

- Alternate exterior angles.

$$\hat{f} + 2\hat{e} = 180°$$
$$\hat{f} + 2(53°) = 180°$$
$$\hat{f} + 106° = 180°$$
$$\therefore \hat{f} = 180° - 106°$$
$$\hat{f} = 74°$$

- Sum of the interior angles.

$$\hat{g} = \hat{e} + \hat{f}$$
$$= 53° + 74°$$
$$\therefore \hat{g} = 127°$$

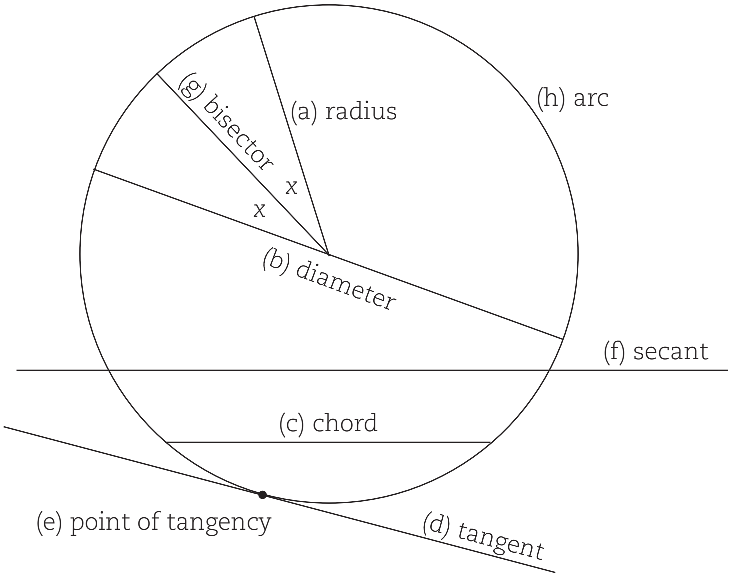
- Exterior angle of a triangle is equal to the sum of the two opposite interior angles.

$$\hat{h} = \hat{g}$$
$$\therefore \hat{h} = 127°$$

- Alternate interior angles.

Assessment activity 3.5

1.



2. **a)** OR, OP, OG and OE **b)** RG and PE
c) EG, PR and LI **d)** MS and FJ
e) N and H **f)** AD
g) KO and OQ **h)** ΔOPR and ΔOGE

3. **a)** Midpoint is a point exactly half way along a line segment.
b) A perpendicular line is a line that is at a right angle to another line.
c) A bisector is a line that cuts an angle or line segment exactly in half.
d) A right angle implies at 90°.
e) Collinear points are points that lie on the same straight line.
f) A set of points is said to be concyclic (or cocyclic) if they lie on a common circle.
g) A quadrilateral that has each of its vertices on the circumference of a circle.
h) Supplementary angles are two angles that add up to 180°.
i) Exterior angle is defined as an angle of a polygon contained between one side extended and the adjacent side.
j) Interior opposite angle is defined as an angle of a polygon contained between two adjacent sides opposite to the supplementary angle of the exterior angle.
k) Point of contact is also referred to as point of tangency, and defined as the point where the tangent touches the circle.
l) The alternate segment is the segment on the opposite side of the chord to the angle formed between the tangent and the chord at the point of contact.

4. A chord is defined as a straight line joining any two points on a circle, therefore a diameter is a chord as it satisfies the definition of a chord.

Assessment activity 3.6

1. **a)** Midpoint; perpendicular; chord; bisects.

b) **(i)** **(a)** $OC = OB$ • Radii of the circle.

$r = x$

(b) $DC = \frac{1}{2} AC$ • A line is drawn from the centre of the circle perpendicular to the chord, then it bisects the chord.

$= \frac{1}{2}(150)$

$DC = 75$ units

$OD = OB - BD$

$OD = x - 25$ units

$$OD^2 + DC^2 = OC^2$$ • Pythagoras

$$(x - 25)^2 + (75)^2 = (x)^2$$

$$x^2 - 50x + 625 + 5\,625 = x^2$$

$$-50x + 6\,250 = 0$$

$$-50x = -6\,250$$

$$x = 125 \text{ units}$$

Since $r = x$, therefore $r = 125$ units.

(ii) $OG = \frac{1}{2}GH$ • Radius = $\frac{1}{2}$(Diameter)

$= \frac{1}{2}(90)$

$OG = 45$ mm

$EF = FG = 15\sqrt{5}$ mm • A line is drawn from the centre of a circle to the midpoint of a chord, that line is perpendicular to the chord.

$OF^2 + FG^2 = OG^2$ • Pythagoras

$OF^2 + (15\sqrt{5})^2 = (45)^2$

$OF^2 + 1\ 125 = 2\ 025$

$\therefore OF^2 = 2\ 025 - 1\ 125$

$OF^2 = 900$

$OF = 30$ mm

2. **a)** Circumference; twice.

b) **(i)** $\hat{x} + K\hat{O}I = 360°$ • Angle at a point.

$\hat{x} + 90° = 360°$

$\therefore \hat{x} = 360° - 90°$

$\hat{x} = 270°$

$\hat{y} = \frac{1}{2}\hat{x}$ • Angle subtended at the circumference is half the angle at the centre subtended by the same arc.

$= \frac{1}{2}(270°)$

$\therefore \hat{y} = 135°$

$\hat{z} = \frac{1}{2}(90°)$ • Angle subtended at the circumference is half the angle at the centre subtended by the same arc.

$\hat{z} = 45°$

(ii) $N\hat{P}Q = \frac{1}{2}(250°)$ • Angle subtended at the circumference is half the angle at the centre subtended by the same arc.

$N\hat{P}Q = 125°$

$N\hat{O}Q = 360° - 250°$ • Angle at a point.

$N\hat{O}Q = 110°$

$N\hat{P}Q + N\hat{O}Q \neq 180°$

Therefore, NPQO is not a cyclic quadrilateral.

3. **a)** Diameter; right; chord; diameter

b) **(i)** **(a)** $RT = 2OR$ • Diameter = 2(Radius)

$= 2(6{,}5)$

$RT = 13$ units

$ST^2 + RS^2 = RT^2$ • Angle subtended by a diameter at the circumference is equal to a right angle.

$ST^2 + (5)^2 = (13)^2$

$ST^2 + 25 = 169$

$\therefore ST^2 = 169 - 25$

$ST^2 = 144$

$ST = 12$ units

(b) $R\hat{T}U = \frac{1}{2}(90^\circ)$
$R\hat{T}U = 45^\circ$

- ΔRUT is an isosceles triangle therefore the base angles are equal.

(ii) $V\hat{O}Z + Z\hat{O}Y = 180^\circ$

- Straight angle.

$V\hat{O}Z + 70^\circ = 180^\circ$
$\therefore V\hat{O}Z = 180^\circ - 70^\circ$
$V\hat{O}Z = 110^\circ$

$O\hat{V}Z = \frac{1}{2}(180^\circ - 110^\circ)$
$O\hat{V}Z = 35^\circ$

- ΔVOZ is an isosceles triangle with base angles equal.

$X\hat{V}O + O\hat{V}Z = Z\hat{V}X$
$X\hat{V}O + 35^\circ = 60^\circ$
$X\hat{V}O = 60^\circ - 35^\circ$
$X\hat{V}O = 25^\circ$

$V\hat{Y}X + X\hat{V}O = 90^\circ$

- Complementary angles.

$V\hat{Y}X + 25^\circ = 90^\circ$
$\therefore V\hat{Y}X = 90^\circ - 25^\circ$
$V\hat{Y}X = 65^\circ$

4. **a)** Equal; concyclic

b) **(i)** $B\hat{A}C = B\hat{D}A$
$B\hat{A}C = 54^\circ$

- Angles in the same segment of a circle are equal.

$A\hat{B}C = \frac{1}{2}(180^\circ - B\hat{A}C)$
$= \frac{1}{2}(180^\circ - 54^\circ)$
$A\hat{B}C = 63^\circ$
$\therefore \hat{x} = 63^\circ$

- ΔABC is an isosceles triangle.

(ii) $H\hat{G}I + G\hat{H}I + H\hat{I}G = 180^\circ$

- Sum of the interior angles of a triangle is equal to 180°.

$H\hat{G}I + 35^\circ + 120^\circ = 180^\circ$
$H\hat{G}I + 155^\circ = 180^\circ$
$\therefore H\hat{G}I = 180^\circ - 155^\circ$
$H\hat{G}I = 25^\circ$

$E\hat{F}H = H\hat{G}I$
$E\hat{F}H = 25^\circ$

- Angles in the same segment of a circle are equal.

5. **a)** Supplementary; cyclic

b) **i)** $J\hat{M}L = \frac{1}{2}(140^\circ)$
$J\hat{M}L = 70^\circ$

- Angle subtended at the circumference is half the angle at the centre subtended by the same arc.

$J\hat{O}L + 140^\circ = 360^\circ$
$\therefore J\hat{O}L = 360^\circ - 140^\circ$
$J\hat{O}L = 220^\circ$

- Angle at a point.

$J\hat{K}L = \frac{1}{2}(J\hat{O}L)$
$= \frac{1}{2}(220°)$
$J\hat{K}L = 110°$

- Angle subtended at the circumference is half the angle at the centre subtended by the same arc.

$J\hat{M}L + J\hat{K}L = 180°$

- Opposite angles of a cyclic quadrilateral are supplementary.

(ii) $Q\hat{P}R + R\hat{P}N + N\hat{R}Q = 180°$
$\hat{x} + 90° + 75° = 180°$
$\hat{x} + 165° = 180°$
$\therefore \hat{x} = 180° - 165°$
$\hat{x} = 15°$

- Opposite angles of a cyclic quadrilateral are supplementary.

6. **a)** Interior; exterior

b) **(i)** **(a)** $V\hat{T}U = T\hat{V}U$
$V\hat{T}U = 50°$

- An isosceles triangle, base angles are equal.

$T\hat{U}V = 180° - (V\hat{T}U + T\hat{V}U)$
$= 180° - (50° + 50°)$
$T\hat{U}V = 80°$

- Sum of the interior angles of a triangle is equal to 180°.

(b) $S\hat{X}T = T\hat{U}V$
$S\hat{X}T = 80°$

- Exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.

(ii) **(a)** $D\hat{A}C = A\hat{D}C = 45°$

- ΔADC is an isosceles triangle with base angles equal.

$Z\hat{D}C = C\hat{A}B$
$\therefore Z\hat{D}C = 88°$

- Exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.

$Z\hat{D}A + A\hat{D}C = Z\hat{D}C$
$Z\hat{D}A + 45° = 88°$
$\therefore Z\hat{D}A = 88° - 45°$
$Z\hat{D}A = 43°$

$Y\hat{Z}D = Z\hat{D}A$
$2\hat{x} = 43°$
$\hat{x} = 21{,}5°$

- Alternate interior angles, AD // ZY.

(b) $D\hat{A}Z + D\hat{A}C + C\hat{A}B = 180°$
$D\hat{A}Z + 45° + 88° = 180°$
$D\hat{A}Z + 133° = 180°$
$\therefore D\hat{A}Z = 180° - 133°$
$D\hat{A}Z = 47°$

- Sum of the interior angles of a triangle is equal to 180°.

7. **a)** Perpendicular; tangent

b) **(i)** $G\hat{O}I + 290^\circ = 360^\circ$

- Angle at a point.

$\therefore G\hat{O}I = 360^\circ - 290^\circ$

$G\hat{O}I = 70^\circ$

Since ΔGOI is an isosceles triangle, $O\hat{G}I = O\hat{I}G = 55^\circ$.

$G\hat{E}I = \frac{1}{2}(G\hat{O}I)$

$= \frac{1}{2}(70^\circ)$

$G\hat{E}I = 35^\circ$

- Angle subtended at the circumference is half the angle at the centre subtended by the same arc.

$G\hat{E}I + E\hat{I}O + O\hat{I}G + I\hat{G}O + E\hat{G}O = 180^\circ$

$35^\circ + \hat{x} + 55^\circ + 55^\circ + \hat{x} = 180^\circ$

$2x + 145^\circ = 180^\circ$

$\therefore 2\hat{x} = 180^\circ - 145^\circ$

$2\hat{x} = 35^\circ$

$\hat{x} = 17{,}5^\circ$

- Sum of interior angles of a triangle is equal to 180°.

$E\hat{G}F + E\hat{G}O = 90^\circ$

$E\hat{G}F + \hat{x} = 90^\circ$

$E\hat{G}F + 17{,}5^\circ = 90^\circ$

$\therefore E\hat{G}F = 90^\circ - 17{,}5^\circ$

$E\hat{G}F = 72{,}5^\circ$

- A tangent to a circle is perpendicular to the radius drawn from the point of contact.

(ii) $O\hat{J}L = O\hat{L}J = 64^\circ$

- ΔOJL is an isosceles triangle, base angles are equal.

$J\hat{L}K + O\hat{L}J = 90^\circ$

$\hat{x} + 64^\circ = 90^\circ$

$\therefore \hat{x} = 90^\circ - 64^\circ$

$\hat{x} = 26^\circ$

- A tangent to a circle is perpendicular to the radius drawn from the point of contact.

$O\hat{K}L + K\hat{L}O + K\hat{O}L = 180^\circ$

$\hat{y} + 90^\circ + 52^\circ = 180^\circ$

$\hat{y} + 142^\circ = 180^\circ$

$\therefore \hat{y} = 180^\circ - 142^\circ$

$\hat{y} = 38^\circ$

- Sum of interior angles of a triangle is equal to 180°.

8. **a)** Equal

b) **(i)** **(a)** OT bisects $U\hat{T}S$

$\therefore U\hat{T}O = S\hat{T}O = 28^\circ$

$O\hat{U}T = 90^\circ$

- A tangent to a circle is perpendicular to the radius drawn from the point of contact.

$T\hat{O}U + O\hat{U}T + U\hat{T}O = 180^\circ$

$T\hat{O}U + 90^\circ + 28^\circ = 180^\circ$

$T\hat{O}U + 118^\circ = 180^\circ$

$T\hat{O}U = 180^\circ - 118^\circ$

$T\hat{O}U = 62^\circ$

- Sum of interior angles of a triangle is equal to 180°.

Since ΔOUT and ΔOST is congruent, therefore $T\hat{O}U = T\hat{O}S = 62°$.

Therefore, $S\hat{O}U = 2(62°)$
$S\hat{O}U = 124°$

$U\hat{Q}S = \frac{1}{2}(124°)$
$U\hat{Q}S = 62°$

- Angle subtended at the circumference is half the angle at the centre subtended by the same arc.

(b) $U\hat{P}S = U\hat{Q}S$
$U\hat{P}S = 62°$

- Angles in the same segment of a circle are equal.

$P\hat{S}R = U\hat{P}S$
$P\hat{S}R = 62°$

- Alternate interior angles, PU // RT.

(ii) $Y\hat{X}Z + Z\hat{X}O = 90°$
$\hat{x} + 27° = 90°$
$\therefore \hat{x} = 90° - 27°$
$\hat{x} = 63°$

- A tangent to a circle is perpendicular to the radius drawn from the point of contact.

ΔXYZ is an isosceles triangle with base angles equal, that is, $Y\hat{X}Z = Y\hat{Z}X = 63°$.

$X\hat{Y}Z + Y\hat{X}Z + Y\hat{Z}X = 180°$
$\hat{y} + 63° + 63° = 180°$
$\hat{y} + 126° = 180°$
$\hat{y} = 180° - 126°$
$\hat{y} = 54°$

- Sum of interior angles of a triangle is equal to 180°.

9. **a)** Alternate; tangent

b) **(i)** $D\hat{F}B = B\hat{A}D$
$\hat{x} = 50°$

- Angles in the same segment of a circle are equal.

$D\hat{B}F = E\hat{D}F$
$\hat{y} = 20°$

- Angle between a tangent and a chord drawn from the point of contact is equal to an angle in the alternate segment.

$B\hat{D}C = B\hat{F}D$
$\hat{z} = 50°$

- Angle between a tangent and a chord drawn from the point of contact is equal to an angle in the alternate segment.

(ii) ΔHOJ is an isosceles triangle, therefore $J\hat{H}O = O\hat{J}H = 35°$.

$I\hat{H}J + J\hat{H}O = 90°$
$\hat{x} + 35° = 90°$
$\therefore \hat{x} = 90° - 35°$
$\hat{x} = 55°$

- A tangent to a circle is perpendicular to the radius drawn from the point of contact.

$J\hat{K}H = I\hat{H}J$
$\hat{y} = \hat{x}$
$\hat{y} = 55°$

- Angle between a tangent and a chord drawn from the point of contact is equal to an angle in the alternate segment.

Assessment activity 3.7

1. **a)** $\sin(\alpha + 35^\circ)$
$= \sin\alpha.\cos 35^\circ + \sin 35^\circ.\cos\alpha$

b) $\cos(4x - 20^\circ)$
$= \cos 4x.\cos 20^\circ + \sin 4x.\sin 20^\circ$

c) $\sin(3y - z)$
$= \sin 3y.\cos z - \sin z.\cos 3y$

d) $\cos(\alpha + 2\beta)$
$= \cos\alpha.\cos 2\beta - \sin\alpha.\ \sin 2\beta$

e) $\sin(50^\circ + 3z)$
$= \sin 50^\circ.\cos 3z + \sin 3z.\cos 50^\circ$

f) $\cos(65^\circ - \theta)$
$= \cos 65^\circ.\cos\theta + \sin 65^\circ.\sin\theta$

2. **a)** $\sin 3x.\cos 45^\circ + \sin 45^\circ.\cos 3x$
$= \sin(3x + 45^\circ)$

b) $\cos 7x.\cos 3x + \sin 7x.\ \sin 3x$
$= \cos(7x - 3x)$
$= \cos 4x$

c) $\sin 2\alpha.\cos\beta - \cos 2\alpha.\sin\beta$
$= \sin 2\alpha.\cos\beta - \sin\beta.\cos 2\alpha$
$= \sin(2\alpha - \beta)$

d) $\cos 30^\circ.\cos 2z - \sin 30^\circ.\sin 2z$
$= \cos(30^\circ + 2z)$

e) $\sin 27^\circ.\cos 33^\circ + \cos 27^\circ.\sin 33^\circ$
$= \sin 27^\circ.\cos 33^\circ + \sin 33^\circ.\cos 27^\circ$
$= \sin(27^\circ + 33^\circ)$
$= \sin 60^\circ$
$= \frac{\sqrt{3}}{2}$

f) $\cos 71^\circ.\cos 26^\circ + \sin 26^\circ.\sin 71^\circ$
$= \cos 71^\circ.\cos 26^\circ + \sin 71^\circ.\sin 26^\circ$
$= \cos(71^\circ - 26^\circ)$
$= \cos 45^\circ$
$= \frac{1}{\sqrt{2}}$

3. **a)** $\sin 105^\circ$

$= \sin(45^\circ + 60^\circ)$

$= \sin 45^\circ.\cos 60^\circ + \sin 60^\circ.\cos 45^\circ$

$= \left(\frac{1}{\sqrt{2}}\right).\left(\frac{1}{2}\right) + \left(\frac{\sqrt{3}}{2}\right).\left(\frac{1}{\sqrt{2}}\right)$
$= \frac{1}{2\sqrt{2}} + \frac{\sqrt{3}}{2\sqrt{2}}$
$= \frac{1 + \sqrt{3}}{2\sqrt{2}}$
$= \frac{1 + \sqrt{3}}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$
$= \frac{\sqrt{2} + \sqrt{6}}{4}$

b) $\sin 37^\circ.\cos 38^\circ + \cos 37^\circ.\ \sin 38^\circ$

$= \sin 37^\circ.\cos 38^\circ + \sin 38^\circ.\cos 37^\circ$

$= \sin(37^\circ + 38^\circ)$

$= \sin 75^\circ$

$= \sin(30^\circ + 45^\circ)$

$= \sin 30^\circ.\cos 45^\circ + \sin 45^\circ.\cos 30^\circ$

$= \left(\frac{1}{2}\right).\left(\frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}}\right).\left(\frac{\sqrt{3}}{2}\right)$
$= \frac{1}{2\sqrt{2}} + \frac{\sqrt{3}}{2\sqrt{2}}$
$= \frac{1 + \sqrt{3}}{2\sqrt{2}}$
$= \frac{1 + \sqrt{3}}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$
$= \frac{\sqrt{2} + \sqrt{6}}{4}$

c) $\cos 75°$

$= \cos(30° + 45°)$

$= \cos 30°.\cos 45° - \sin 30°.\sin 45°$

$= \left(\frac{\sqrt{3}}{2}\right).\left(\frac{1}{\sqrt{2}}\right) - \left(\frac{1}{2}\right).\left(\frac{1}{\sqrt{2}}\right)$

$= \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}$

$= \frac{\sqrt{3} - 1}{2\sqrt{2}}$

$= \frac{\sqrt{3} - 1}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$

$= \frac{\sqrt{6} - \sqrt{2}}{4}$

d) $\cos 30°.\cos 45° + \sin 30°.\sin 45°$

$= \left(\frac{\sqrt{3}}{2}\right).\left(\frac{1}{\sqrt{2}}\right) + \left(\frac{1}{2}\right).\left(\frac{1}{\sqrt{2}}\right)$

$= \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}}$

$= \frac{\sqrt{3} + 1}{2\sqrt{2}}$

$= \frac{\sqrt{3} + 1}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$

$= \frac{\sqrt{6} + \sqrt{2}}{4}$

e) $\tan 15°$

$= \tan(45° - 30°)$

$= \frac{\sin(45° - 30°)}{\cos(45° - 30°)}$

$= \frac{\sin 45°.\cos 30° - \sin 30°.\cos 45°}{\cos 45°.\cos 30° + \sin 45°.\sin 30°}$

$= \frac{(\frac{1}{\sqrt{2}}).(\frac{\sqrt{3}}{2}) - (\frac{1}{2}).(\frac{1}{\sqrt{2}})}{(\frac{1}{\sqrt{2}}).(\frac{\sqrt{3}}{2}) + (\frac{1}{\sqrt{2}}).(\frac{1}{2})}$

$= \frac{(\frac{\sqrt{3}}{2\sqrt{2}}) - (\frac{1}{2\sqrt{2}})}{(\frac{\sqrt{3}}{2\sqrt{2}}) + (\frac{1}{2\sqrt{2}})}$

$= \frac{\frac{\sqrt{3}-1}{2\sqrt{2}}}{\frac{\sqrt{3}+1}{2\sqrt{2}}}$

$= \frac{\sqrt{3} - 1}{2\sqrt{2}} \times \frac{2\sqrt{2}}{\sqrt{3} + 1}$

$= \frac{\sqrt{3} - 1}{\sqrt{3} + 1}$

$= \frac{\sqrt{3} - 1}{\sqrt{3} + 1} \times \frac{\sqrt{3} - 1}{\sqrt{3} - 1}$

$= \frac{3 - 2\sqrt{3} + 1}{3 - 1}$

$= \frac{4 - 2\sqrt{3}}{2}$

$= \frac{2(2 - \sqrt{3})}{2}$

$= 2 - \sqrt{3}$

f) $\cos 47°.\cos 58° - \sin 47°.\sin 58°$

$= \cos(47° + 58°)$

$= \cos 105°$

$= \cos(45° + 60°)$

$= \cos 45°.\cos 60° - \sin 45°.\sin 60°$

$= \left(\frac{1}{\sqrt{2}}\right).\left(\frac{1}{2}\right) - \left(\frac{1}{\sqrt{2}}\right).\left(\frac{\sqrt{3}}{2}\right)$

$= \frac{1}{2\sqrt{2}} - \frac{\sqrt{3}}{2\sqrt{2}}$

$= \frac{1 - \sqrt{3}}{2\sqrt{2}}$

$= \frac{1 - \sqrt{3}}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$

$= \frac{\sqrt{2} - \sqrt{6}}{4}$

4. **a)** sin 165°

$$= \sin(180° - 15°)$$
$$= \sin 15°$$
$$= \sin(45° - 30°)$$
$$= \sin 45°.\cos 30° - \sin 30°.\cos 45°$$
$$= \left(\frac{1}{\sqrt{2}}\right).\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{1}{2}\right).\left(\frac{1}{\sqrt{2}}\right)$$
$$= \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}$$
$$= \frac{\sqrt{3}-1}{2\sqrt{2}}$$
$$= \frac{\sqrt{3}-1}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$
$$= \frac{\sqrt{6}-\sqrt{2}}{4}$$

b) cos 195°

$$= \cos(180° + 15°)$$
$$= -\cos 15°$$
$$= -\cos(45° - 30°)$$
$$= -[\cos 45°.\cos 30° + \sin 45°.\sin 30°]$$
$$= -\left[\left(\frac{1}{\sqrt{2}}\right).\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{1}{\sqrt{2}}\right).\left(\frac{1}{2}\right)\right]$$
$$= -\left[\frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}}\right]$$
$$= -\left[\frac{\sqrt{3}+1}{2\sqrt{2}}\right]$$
$$= \frac{-\sqrt{3}-1}{2\sqrt{2}}$$
$$= \frac{-\sqrt{3}-1}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$
$$= \frac{-\sqrt{6}-\sqrt{2}}{4}$$

c) tan 285°

$$= \tan(360° - 75°)$$
$$= -\tan 75°$$
$$= -\tan(30° + 45°)$$
$$= -\left[\frac{\sin(30° + 45°)}{\cos(30° + 45°)}\right]$$
$$= -\left[\frac{\sin 30°.\cos 45° + \sin 45°.\cos 30°}{\cos 30°.\cos 45° - \sin 30°.\sin 45°}\right]$$
$$= -\left[\frac{\left(\frac{1}{2}\right).\left(\frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}}\right).\left(\frac{\sqrt{3}}{2}\right)}{\left(\frac{\sqrt{3}}{2}\right).\left(\frac{1}{\sqrt{2}}\right) - \left(\frac{1}{2}\right).\left(\frac{1}{\sqrt{2}}\right)}\right]$$
$$= -\left[\frac{\left(\frac{1}{2\sqrt{2}}\right) + \left(\frac{\sqrt{3}}{2\sqrt{2}}\right)}{\left(\frac{\sqrt{3}}{2\sqrt{2}}\right) - \left(\frac{1}{2\sqrt{2}}\right)}\right]$$
$$= -\left[\frac{\frac{1+\sqrt{3}}{2\sqrt{2}}}{\frac{\sqrt{3}-1}{2\sqrt{2}}}\right]$$
$$= -\left[\frac{1+\sqrt{3}}{2\sqrt{2}} \times \frac{2\sqrt{2}}{\sqrt{3}-1}\right]$$
$$= -\left[\frac{1+\sqrt{3}}{\sqrt{3}-1}\right]$$
$$= \frac{1+\sqrt{3}}{1-\sqrt{3}}$$
$$= \frac{1+\sqrt{3}}{1-\sqrt{3}} \times \frac{1+\sqrt{3}}{1+\sqrt{3}}$$
$$= \frac{1+2\sqrt{3}+3}{1-3}$$
$$= \frac{4+2\sqrt{3}}{-2}$$
$$= \frac{2(2+\sqrt{3})}{-2}$$
$$= -2 - \sqrt{3}$$

d) sin 105°

$$= \sin(180° - 75°)$$
$$= \sin 75°$$
$$= \sin(30° + 45°)$$
$$= \sin 30°.\cos 45° + \sin 45°.\cos 30°$$
$$= \left(\frac{1}{2}\right).\left(\frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}}\right).\left(\frac{\sqrt{3}}{2}\right)$$
$$= \frac{1}{2\sqrt{2}} + \frac{\sqrt{3}}{2\sqrt{2}}$$
$$= \frac{1+\sqrt{3}}{2\sqrt{2}}$$
$$= \frac{1+\sqrt{3}}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$
$$= \frac{\sqrt{2}+\sqrt{6}}{4}$$
$$= \frac{\sqrt{6}+\sqrt{2}}{4}$$

e) $\cos 255°$

$= \cos(180° + 75°)$

$= -\cos 75°$

$= -\cos(30° + 45°)$

$= -[\cos 30°.\cos 45° - \sin 30°.\sin 45°]$

$$= -\left[\left(\frac{\sqrt{3}}{2}\right).\left(\frac{1}{\sqrt{2}}\right) - \left(\frac{1}{2}\right).\left(\frac{1}{\sqrt{2}}\right)\right]$$

$$= -\left[\frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}\right]$$

$$= -\left[\frac{\sqrt{3} - 1}{2\sqrt{2}}\right]$$

$$= \frac{-\sqrt{3} + 1}{2\sqrt{2}}$$

$$= \frac{-\sqrt{3} + 1}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$

$$= \frac{-\sqrt{6} + \sqrt{2}}{4}$$

f) $\tan 345°$

$= \tan(360° - 15°)$

$= -\tan 15°$

$= -\tan(45° - 30°)$

$$= -\left[\frac{\sin(45° - 30°)}{\cos(45° - 30°)}\right]$$

$$= -\left[\frac{\sin 45°.\cos 30° - \sin 30°.\cos 45°}{\cos 45°.\cos 30° + \sin 45°.\sin 30°}\right]$$

$$= -\left[\frac{(\frac{1}{\sqrt{2}}).(\frac{\sqrt{3}}{2}) - (\frac{1}{2}).(\frac{1}{\sqrt{2}}).}{(\frac{1}{\sqrt{2}}).(\frac{\sqrt{3}}{2}) + (\frac{1}{\sqrt{2}}).(\frac{1}{2})}\right]$$

$$= -\left[\frac{(\frac{\sqrt{3}}{2\sqrt{2}}) - (\frac{1}{2\sqrt{2}})}{(\frac{\sqrt{3}}{2\sqrt{2}}) + (\frac{1}{2\sqrt{2}})}\right]$$

$$= -\left[\frac{\frac{\sqrt{3}-1}{2\sqrt{2}}}{\frac{\sqrt{3}+1}{2\sqrt{2}}}\right]$$

$$= -\left[\frac{\sqrt{3} - 1}{2\sqrt{2}} \times \frac{2\sqrt{2}}{\sqrt{3} + 1}\right]$$

$$= -\left[\frac{\sqrt{3} - 1}{\sqrt{3} + 1}\right]$$

$$= \frac{-\sqrt{3} + 1}{\sqrt{3} + 1}$$

$$= \frac{-\sqrt{3} + 1}{\sqrt{3} + 1} \times \frac{\sqrt{3} - 1}{\sqrt{3} - 1}$$

$$= \frac{-3 + 2\sqrt{3} - 1}{3 - 1}$$

$$= \frac{-4 + 2\sqrt{3}}{2}$$

$$= \frac{2(-2 + \sqrt{3})}{2}$$

$= -2 + \sqrt{3}$

5. $\sin 2\alpha = \sin(\alpha + \alpha)$

$= \sin\alpha.\cos\alpha + \sin\alpha.\cos\alpha$

$\sin 2\alpha = 2\sin\alpha.\cos\alpha$

Therefore,

$\sin 120° = \sin[2(60°)]$

$= 2\sin 60°.\cos 60°$

$$= 2\left(\frac{\sqrt{3}}{2}\right).\left(\frac{1}{2}\right)$$

$$\sin 120° = \frac{\sqrt{3}}{2}$$

6. **a)** $\cos 2\alpha = \cos(\alpha + \alpha)$

$= \cos\alpha.\cos\alpha - \sin\alpha.\sin\alpha$

$= \cos^2\alpha - \sin^2\alpha$

$= (1 - \sin^2\alpha) - \sin^2\alpha$

$\cos 2\alpha = 1 - 2\sin^2\alpha$

b) $\cos 150° = \cos[2(75°)]$

$= 1 - 2\sin^2 75°$

$= 1 - 2(\sin 75°)^2$

$= 1 - 2(\sin(30° + 45°))^2$

$= 1 - 2(\sin 30°.\cos 45° + \sin 45°.\cos 30°)^2$

$= 1 - 2\left[\left(\frac{1}{2}\right).\left(\frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}}\right).\left(\frac{\sqrt{3}}{2}\right)\right]^2$

$= 1 - 2\left[\frac{1}{2\sqrt{2}} + \frac{\sqrt{3}}{2\sqrt{2}}\right]^2$

$= 1 - 2\left[\frac{1 + \sqrt{3}}{2\sqrt{2}}\right]^2$

$= 1 - 2\left[\frac{1 + 2\sqrt{3} + 3}{8}\right]$

$= 1 - 2\left[\frac{4 + 2\sqrt{3}}{8}\right]$

$= 1 + \frac{-8 - 4\sqrt{3}}{8}$

$= \frac{8 - 8 - 4\sqrt{3}}{8}$

$= \frac{-4\sqrt{3}}{8}$

$\cos 150° = -\frac{\sqrt{3}}{2}$

c) $\cos(90° + \alpha) = \cos 90°.\cos\alpha - \sin 90°.\sin\alpha$

$= 0.\cos\alpha - 1.\sin\alpha$

$\cos(90° + \alpha) = -\sin\alpha$

d) $\cos 150° = \cos(90° + 60°)$

$= -\sin 60°$

$\cos 150° = -\frac{\sqrt{3}}{2}$

Assessment activity 3.8

1. **a)** $\sin(x + 30°) - \sin(x - 30°) = \cos x$

L.H.S. $\sin(x + 30°) - \sin(x - 30°)$

$= (\sin x.\cos 30° + \sin 30°.\cos x) - (\sin x.\cos 30° - \sin 30°.\cos x)$

$= \sin x.\cos 30° + \sin 30°.\cos x - \sin x.\cos 30° + \sin 30°.\cos x$

$= 2\sin 30°.\cos x$

$= 2\left(\frac{1}{2}\right).\cos x$

$= \cos x$

$\therefore$ L.H.S. = R.H.S.

b) $\sin 58° + \sin 32° = \sqrt{2}\cos 13°$

L.H.S. $\sin 58° + \sin 32°$

$= \sin(45° + 13°) + \sin(45° - 13°)$

$= \sin 45°.\cos 13° + \sin 13°.\cos 45° + \sin 45°.\cos 13° - \sin 13°.\cos 45°$

$= 2\sin 45°.\cos 13°$

$= 2\left(\frac{1}{\sqrt{2}}\right).\cos 13°$

$= \sqrt{2}\cos 13°$

$\therefore$ L.H.S. = R.H.S.

c) $\cos 53° - \cos 7° = -\sin 23°$

L.H.S. $\cos 53° - \cos 7°$

$= \cos(23° + 30°) - \cos(30° - 23°)$

$= (\cos 23°.\cos 30° - \sin 23°.\sin 30°) - (\cos 30°.\cos 23° + \sin 30°.\sin 23°)$

$= \cos 23°.\cos 30° - \sin 23°.\sin 30° - \cos 30°.\cos 23° - \sin 30°.\sin 23°$

$= -2 \sin 30°.\sin 23°$

$= -2\left(\frac{1}{2}\right).\sin 23°$

$= -\sin 23°$

$\therefore$ L.H.S. = R.H.S.

d) $\cos 3x = 4\cos^3 x - 3\cos x$

L.H.S. $\cos 3x$

$= \cos(2x + x)$

$= \cos 2x.\cos x - \sin 2x.\sin x$

$= (2\cos^2 x - 1).\cos x - (2\sin x.\cos x)\sin x$

$= (2\cos^2 x - 1).\cos x - 2\sin^2 x.\cos x$

$= (2\cos^2 x - 1).\cos x - 2(1 - \cos^2 x).\cos x$

$= 2\cos^3 x - \cos x - 2\cos x + 2\cos^3 x$

$= 4\cos^3 x - 3\cos x$

$\therefore$ L.H.S. = R.H.S.

2. **a)** $\dfrac{1 - \cos 2\theta}{\tan(360° - \theta).\sin 2\theta}$

$= \dfrac{1 - (1 - 2\sin^2\theta)}{-\tan\theta.2\sin\theta.\cos\theta}$

$= \dfrac{1 - 1 + 2\sin^2\theta}{-\frac{\sin\theta}{\cos\theta}.2\sin\theta.\cos\theta}$

$= \dfrac{2\sin^2\theta}{-2\sin^2\theta}$

$= -1$

b) $(\sin\theta - \cos\theta)^2$

$= (\sin\theta - \cos\theta)(\sin\theta - \cos\theta)$

$= \sin^2\theta - 2\sin\theta.\cos\theta + \cos^2\theta$

$= (\sin^2\theta + \cos^2\theta) - 2\sin\theta.\cos\theta$

$= 1 - \sin 2\theta$

c) $\dfrac{2\tan\theta}{1 + \tan^2\theta}$

$= \dfrac{2\frac{\sin\theta}{\cos\theta}}{1 + \frac{\sin^2\theta}{\cos^2\theta}}$

$= \dfrac{2\frac{\sin\theta}{\cos\theta}}{\frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta}}$

$= \dfrac{2\sin\theta}{\cos\theta} \times \dfrac{\cos^2\theta}{\cos^2\theta + \sin^2\theta}$

$= \dfrac{2\sin\theta}{\cos\theta} \times \dfrac{\cos^2\theta}{1}$

$= 2\sin\theta.\cos\theta$

$= \sin 2\theta$

d) $\dfrac{1-\tan^2 x}{1+\tan^2 x}$

$$= \frac{1-\frac{\sin^2 x}{\cos^2 x}}{1+\frac{\sin^2 x}{\cos^2 x}}$$

$$= \frac{\frac{\cos^2 x-\sin^2 x}{\cos^2 x}}{\frac{\cos^2 x+\sin^2 x}{\cos^2 x}}$$

$$= \frac{\cos^2 x-\sin^2 x}{\cos^2 x} \times \frac{\cos^2 x}{\cos^2 x+\sin^2 x}$$

$$= \frac{\cos^2 x-\sin^2 x}{\cos^2 x+\sin^2 x}$$

$$= \frac{\cos 2x}{1}$$

$$= \cos 2x$$

e) $\dfrac{\cos 2x}{\sin x.\cos x}$

$$= \frac{\cos^2 x-\sin^2 x}{\sin x.\cos x}$$

$$= \frac{\cos^2 x}{\sin x.\cos x} - \frac{\sin^2 x}{\sin x.\cos x}$$

$$= \frac{\cos x}{\sin x} - \frac{\sin x}{\cos x}$$

$$= \frac{1}{\tan x} - \tan x$$

f) $\dfrac{\sin 2x}{1+\cos 2x}$

$$= \frac{2\sin x.\cos x}{1+2\cos^2 x-1}$$

$$= \frac{2\sin x.\cos x}{2\cos^2 x}$$

$$= \frac{\sin x}{\cos x}$$

$$= \tan x$$

3. **a)** $\dfrac{\cos 2\theta}{\sin 2\theta-\tan\theta} = \dfrac{1}{\tan\theta}$

L.H.S. $\dfrac{\cos 2\theta}{\sin 2\theta-\tan\theta}$

$$= \frac{2\cos^2\theta-1}{2\sin\theta.\cos\theta-\frac{\sin\theta}{\cos\theta}}$$

$$= \frac{2\cos^2\theta-1}{\frac{2\sin\theta.\cos^2\theta-\sin\theta}{\cos\theta}}$$

$$= \frac{2\cos^2\theta-1}{\frac{\sin\theta(2\cos^2\theta-1)}{\cos\theta}}$$

$$= \frac{2\cos^2\theta-1}{1} \times \frac{\cos\theta}{\sin\theta(2\cos^2\theta-1)}$$

$$= \frac{\cos\theta}{\sin\theta}$$

$$= \frac{1}{\tan\theta}$$

$\therefore$ L.H.S. = R.H.S.

b) $\dfrac{2\sin\theta-1}{\sin 2\theta-\cos\theta} = \dfrac{1}{\cos\theta}$

L.H.S. $\dfrac{2\sin\theta-1}{\sin 2\theta-\cos\theta}$

$$= \frac{2\sin\theta-1}{2\sin\theta.\cos\theta-\cos\theta}$$

$$= \frac{2\sin\theta-1}{\cos\theta(2\sin\theta-1)}$$

$$= \frac{1}{\cos\theta}$$

$\therefore$ L.H.S. = R.H.S.

c) $\dfrac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)-\cos(90^\circ-\alpha).\sin(180^\circ+\beta)} = \tan\alpha+\tan\beta$

L.H.S. $\dfrac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)-\cos(90^\circ-\alpha).\sin(180^\circ+\beta)}$

$$= \frac{\sin\alpha.\cos\beta+\sin\beta.\cos\alpha}{\cos\alpha.\cos\beta-\sin\alpha.\cos\beta-\sin\alpha.-\sin\beta}$$

$$= \frac{\sin\alpha.\cos\beta+\sin\beta.\cos\alpha}{\cos\alpha.\cos\beta}$$

$$= \frac{\sin\alpha.\cos\beta}{\cos\alpha.\cos\beta} + \frac{\sin\beta.\cos\alpha}{\cos\alpha.\cos\beta}$$

$$= \frac{\sin\alpha}{\cos\alpha} + \frac{\sin\beta}{\cos\beta}$$

$$= \tan\alpha+\tan\beta$$

$\therefore$ L.H.S. = R.H.S.

d) $\frac{1-\cos 2x}{\tan x} = \sin 2x$

$$\text{L.H.S. } \frac{1-\cos 2x}{\tan x}$$

$$= \frac{1-(1-2\sin^2 x)}{\frac{\sin x}{\cos x}}$$

$$= \frac{1-1+2\sin^2 x}{\frac{\sin x}{\cos x}}$$

$$= \frac{2\sin^2 x}{\frac{\sin x}{\cos x}}$$

$$= \frac{2\sin^2 x}{1} \times \frac{\cos x}{\sin x}$$

$$= 2\sin x.\cos x$$

$$= \sin 2x$$

$\therefore$ L.H.S. = R.H.S.

e) $\frac{1-\tan x}{1+\tan x} = \frac{\cos 2x}{1+\sin 2x}$

$$\text{L.H.S. } \frac{1-\tan x}{1+\tan x}$$

$$= \frac{1-\frac{\sin x}{\cos x}}{1+\frac{\sin x}{\cos x}}$$

$$= \frac{\frac{\cos x - \sin x}{\cos x}}{\frac{\cos x + \sin x}{\cos x}}$$

$$= \frac{\cos x - \sin x}{\cos x} \times \frac{\cos x}{\cos x + \sin x}$$

$$= \frac{\cos x - \sin x}{\cos x + \sin x}$$

$$= \frac{\cos x - \sin x}{\cos x + \sin x} \times \frac{\cos x + \sin x}{\cos x + \sin x}$$

$$= \frac{\cos^2 x - \sin^2 x}{\cos^2 x + 2\sin x.\cos x + \sin^2 x}$$

$$= \frac{\cos^2 x - \sin^2 x}{\sin^2 x + \cos^2 x + 2\sin x.\cos x}$$

$$= \frac{\cos 2x}{1+\sin 2x}$$

$\therefore$ L.H.S. = R.H.S.

f) $\frac{\sin x + \sin 2x}{1+\cos 2x + \cos x} = \tan x$

$$\text{L.H.S. } \frac{\sin x + \sin 2x}{1+\cos 2x + \cos x}$$

$$= \frac{\sin x + 2\sin x.\cos x}{1+2\cos^2 x - 1 + \cos x}$$

$$= \frac{\sin x + 2\sin x.\cos x}{2\cos^2 x + \cos x}$$

$$= \frac{\sin x(2\cos x + 1)}{\cos x(2\cos x + 1)}$$

$$= \frac{\sin x}{\cos x}$$

$$= \tan x$$

$\therefore$ L.H.S. = R.H.S.

4. a) $\sin 137°$

$= \sin (90° + 47°)$

$= + \cos 47°$

$= + k$

b) $\cos 133°$

$= \cos (180° - 47°)$

$= -\cos 47°$

$= -k$

c) $\sin 47°$

$= \sqrt{1-\cos^2 47°}$

$= \sqrt{1-(k)^2}$

$= \sqrt{1-k^2}$

d) $\cos 94°$

$= \cos [2(47°)]$

$= 2\cos^2 47° - 1$

$= 2(k)^2 - 1$

$= 2k^2 - 1$

e) $\sin 13°$

$= \sin (60° - 47°)$

$= \sin 60°.\cos 47° - \sin 47°.\cos 60°$

$= \left(\frac{\sqrt{3}}{2}\right).\cos 47° - \sin 47°.\left(\frac{1}{2}\right)$

$= \frac{\sqrt{3}}{2}.\cos 47° - \frac{1}{2}\sin 47°$

$= \frac{\sqrt{3}}{2}.\cos 47° - \frac{1}{2}\sqrt{1 - \cos^2 47°}$

$= \frac{\sqrt{3}}{2}.k - \frac{1}{2}.\sqrt{1 - k^2}$

f) $\cos 92°$

$= \cos (45° + 47°)$

$= \cos 45°.\cos 47° - \sin 45°.\sin 47°$

$= \left(\frac{1}{\sqrt{2}}\right).\cos 47° - \left(\frac{1}{\sqrt{2}}\right).\sin 47°$

$= \frac{1}{\sqrt{2}}.\cos 47° - \frac{1}{\sqrt{2}}.\sqrt{1 - \cos^2 47°}$

$= \frac{1}{\sqrt{2}}.k - \frac{1}{\sqrt{2}}.\sqrt{1 - k^2}$

5. $\cos^2 x = \cos (x + x)$

$= \cos x.\cos x - \sin x.\sin x$

$= \cos^2 x - \sin^2 x$

$= \cos^2 x - (1 - \cos^2 x)$

$= \cos^2 x - 1 + \cos^2 x$

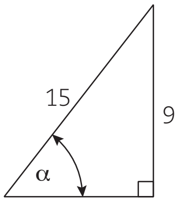
$\cos 2x = 2\cos^2 x - 1$ if $\cos x = \frac{7}{9}$

$= 2\left(\frac{7}{9}\right)^2 - 1$

$= \frac{98}{81} - 1$

$\cos 2x = \frac{17}{81}$

6. $\sin \alpha = \frac{9}{15}$



$x^2 + y^2 = r^2$

$x^2 + (9)^2 = (15)^2$

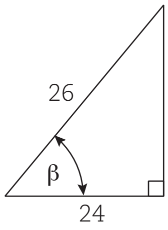
$x^2 + 81 = 225$

$x^2 = 225 - 81$

$x^2 = 144$

$x = 12$

$\cos \beta = \frac{24}{26}$



$x^2 + y^2 = r^2$

$(24)^2 + y^2 = (26)^2$

$576 + y^2 = 676$

$y^2 = 676 - 576$

$y^2 = 100$

$y = 10$

a) $\cos (\alpha + \beta)$

$= \cos \alpha.\cos \beta - \sin \alpha.\sin \beta$

$= \left(\frac{12}{15}\right).\left(\frac{24}{26}\right) - \left(\frac{9}{15}\right).\left(\frac{10}{26}\right)$

$= \frac{48}{65} - \frac{3}{13}$

$= \frac{33}{65}$

b) $\sin (180° - \beta)$

$= \sin \beta$

$= \frac{10}{26}$

$= \frac{5}{13}$

c) $\cos(90° + \alpha)$
$= -\sin\alpha$
$= -\frac{9}{15}$
$= -\frac{3}{5}$

d) $\sin(\alpha - \beta)$
$= \sin\alpha.\cos\beta - \sin\beta.\cos\alpha$
$= \left(\frac{9}{15}\right).\left(\frac{24}{26}\right) - \left(\frac{10}{26}\right).\left(\frac{12}{15}\right)$
$= \frac{36}{65} - \frac{4}{13}$
$= \frac{16}{65}$

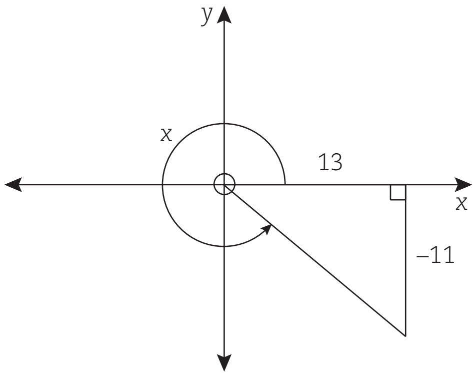
e) $\cos(360° + \alpha)$
$= \cos\alpha$
$= \frac{12}{15}$
$= \frac{4}{5}$

f) $\sin(90° - \beta)$
$= \cos\beta$
$= \frac{24}{26}$
$= \frac{12}{13}$

7. $\tan x = -\frac{11}{13}, x \in [180°; 360°]$

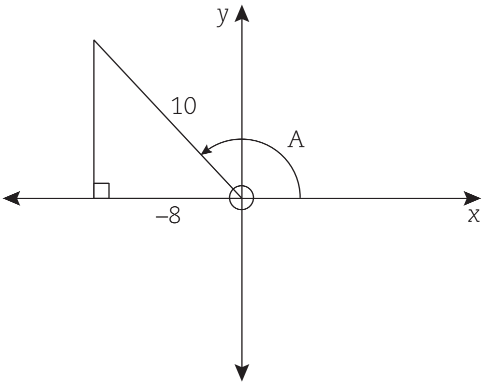
$x^2 + y^2 = r^2$
$(13)^2 + (-11)^2 = r^2$
$169 + 121 = r^2$
$290 = r^2$
$\therefore r = \sqrt{290}$

$\sin 2x = 2\sin x.\cos x$
$= 2\left(\frac{-11}{\sqrt{290}}\right)\left(\frac{13}{\sqrt{290}}\right)$
$\sin 2x = -\frac{143}{145}$



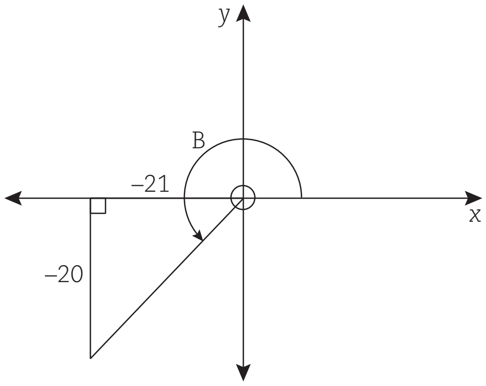
8. $\cos A = -\frac{8}{10}, A \in [0°; 180°]$

$x^2 + y^2 = r^2$
$(-8)^2 + y^2 = (10)^2$
$64 + y^2 = 100$
$y^2 = 100 - 64$
$y^2 = 36$
$y = 6$



$\tan B = \frac{20}{21}, B \in [90°; 360°]$

$x^2 + y^2 = r^2$
$(-21)^2 + (20)^2 = r^2$
$441 + 400 = r^2$
$841 = r^2$
$r = 29$



a) $\sin(A + B)$

$= \sin A.\cos B + \sin B.\cos A$

$= \left(\frac{6}{10}\right).\left(\frac{-21}{29}\right) + \left(\frac{-20}{29}\right).\left(\frac{-8}{10}\right)$

$= -\frac{63}{145} + \frac{16}{29}$

$= \frac{17}{145}$

b) $\cos 2A$

$= 2\cos^2 A - 1$

$= 2\left(-\frac{8}{10}\right)^2 - 1$

$= \frac{7}{25}$

c) $\sin(180° + A)$

$= -\sin A$

$= -\left(\frac{6}{10}\right)$

$= -\frac{3}{5}$

d) $\cos(A - B)$

$= \cos A.\cos B + \sin A.\sin B$

$= \left(\frac{-8}{10}\right).\left(\frac{-21}{29}\right) + \left(\frac{6}{10}\right).\left(\frac{-20}{29}\right)$

$= \frac{84}{145} - \frac{12}{29}$

$= \frac{24}{145}$

e) $\sin 2B$

$= 2\sin B.\cos B$

$= 2\left(\frac{-20}{29}\right)\left(\frac{-21}{29}\right)$

$= \frac{840}{841}$

f) $\tan(90° - B)$

$= \frac{\sin(90° - B)}{\cos(90° - B)}$

$= \frac{\cos B}{\sin B}$

$= \frac{-\frac{21}{29}}{-\frac{20}{29}}$

$= -\frac{21}{29} \times -\frac{29}{20}$

$= \frac{21}{20}$

$= 1\frac{1}{20}$

Assessment activity 3.9

1. a) $\sin 52° = \sin x.\cos 17° + \sin 17°.\cos x$

$\sin 52° = \sin(x + 17°)$

$\therefore 52° = x + 17°$

$x = 35°$

OR

$\therefore 52° = 180° - (x + 17°)$

$x = 111°$

But $x \in [0°; 90°]$,

$x = 35°$ and $x \neq 111°$

b) $\sin 3x = \cos x.\cos 30° - \sin x.\sin 30°$

$\sin 3x = \cos(x + 30°)$

$\sin 3x = \sin[90° - (x + 30°)]$

$\therefore 3x = 90° - x - 30°$

$4x = 60°$

$x = 15°$

OR

$\therefore\ 3x = 180° - (90° - x - 30°)$

$2x = 120°$

$x = 60°$

$\sin 3x = \cos x.\cos 30° - \sin x.\sin 30°$

$\sin 3x = \cos (x + 30°)$

$\sin 3x = \sin [90° + (x + 30°)]$

$\therefore\ 3x = 90° + x + 30°$

$2x = 120°$

$x = 60°$

OR

$\therefore\ 3x = 180° - (90° + x + 30°)$

$4x = 60°$

$x = 15°$

But $x \in [0°; 90°]$,

$x = 15°$ and $x = 60°$

c) $\sin x.\cos 25° - \sin 25°.\cos x = \cos 80°$

$\sin (x - 25°) = \cos 80°$

$\sin (x - 25°) = \sin 10°$

$\therefore\ x - 25° = 10°$

$x = 35°$

OR

$\therefore\ x - 25° = 180° - 10°$

$x = 195°$

$\sin x.\cos 25° - \sin 25°.\cos x = \cos 80°$

$\sin (x - 25°) = \cos 80°$

$\sin (x - 25°) = \sin 170°$

$\therefore\ x - 25° = 170°$

$x = 195°$

OR

$\therefore\ x - 25° = 180° - 170°$

$x = 35°$

But $x \in [0°; 90°]$,

$x = 35°$ and $x \neq 195°$

d) $\cos 3x.\cos 15° + \sin 3x.\sin 15° = -\cos 60°$

$\cos (3x - 15°) = -\cos 60°$

$\therefore\ 3x - 15° = 180° - 60°$

$3x = 135°$

$x = 45°$

OR

$\therefore\ 3x - 15° = 180° + 60°$

$3x = 255°$

$x = 85°$

But $x \in [0°; 90°]$,

$x = 45°$ and $x = 85°$

e) $\cos(90° - x) = \sin 77° - \sin 43°$

$$\sin x = \sin(60° + 17°) - \sin(60° - 17°)$$
$$\sin x = \sin 60°.\cos 17° + \sin 17°.\cos 60° - \sin 60°.\cos 17° + \sin 17°.\cos 60°$$
$$\sin x = 2\sin 17°.\cos 60°$$
$$= 2\sin 17°.\ \frac{1}{2}$$
$$\sin x = \sin 17°$$

$$\therefore\ x = 17°$$

OR

$$\therefore\ x = 180° - 17°$$
$$x = 163°$$

But $x \in [0°; 90°]$,

$x = 17°$ and $x \neq 163°$

2. **a)**

$$3\cos 2\theta - \cos\theta + 2 = 0$$
$$3(2\cos^2\theta - 1) - \cos\theta + 2 = 0$$
$$6\cos^2\theta - 3 - \cos\theta + 2 = 0$$
$$6\cos^2\theta - \cos\theta - 1 = 0$$
$$(3\cos\theta + 1)(2\cos\theta - 1) = 0$$

$3\cos\theta + 1 = 0$	$2\cos\theta - 1 = 0$
$3\cos\theta = -1$	$2\cos\theta = 1$
$\cos\theta = \ominus\ \frac{1}{3}$	$\cos\theta = \oplus\ \frac{1}{2}$

cos –; 2nd	cos –; 3rd	cos +; 1st	cos +; 4th
$\theta = 180° - \theta_{ref}$	$\theta = 180° + \theta_{ref}$	$\theta = \theta_{ref}$	$\theta = 360° - \theta_{ref}$
$\theta = 180° - \cos^{-1}\left(\frac{1}{3}\right)$	$\theta = 180° + \cos^{-1}\left(\frac{1}{3}\right)$	$\theta = \cos^{-1}\left(\frac{1}{2}\right)$	$\theta = 360° - \cos^{-1}\left(\frac{1}{2}\right)$
$\theta = 109{,}471°$	$\theta = 250{,}529°$	$\theta = 60°$	$\theta = 300°$

Since all the angles are within the restriction $x \in [0°; 360°]$, the solutions will be $\theta = 60°$, $\theta = 109{,}471°$, $\theta = 250{,}529°$ and $\theta = 300°$.

b)

$$3\sin\theta + 1 = 2\cos 2\theta$$
$$3\sin\theta + 1 = 2(1 - 2\sin^2\theta)$$
$$3\sin\theta + 1 = 2 - 4\sin^2\theta$$
$$4\sin^2\theta + 3\sin\theta - 1 = 0$$
$$(4\sin\theta - 1)(\sin\theta + 1) = 0$$

$4\sin\theta - 1 = 0$	$\sin\theta + 1 = 0$
$4\sin\theta = 1$	$\sin\theta = \ominus\ 1$
$\sin\theta = \oplus\ \frac{1}{4}$	

sin +; 1st	sin +; 2nd	sin –; 3rd	sin –; 4th
$\theta = \theta_{ref}$	$\theta = 180° - \theta_{ref}$	$\theta = 180° + \theta_{ref}$	$\theta = 360° - \theta_{ref}$
$\theta = \sin^{-1}\left(\frac{1}{4}\right)$	$\theta = 180° - \sin^{-1}\left(\frac{1}{4}\right)$	$\theta = 180° + \sin^{-1}(1)$	$\theta = 360° - \sin^{-1}(1)$
$\theta = 14{,}478°$	$\theta = 165{,}523°$	$\theta = 270°$	$\theta = 270°$

Since all the angles are within the restriction $x \in [0°; 360°]$, the solutions will be $\theta = 14{,}478°$, $\theta = 165{,}523°$ and $\theta = 270°$.

c)

$$10 \cos 2\theta = \frac{7}{2} \sin 2\theta - 2 [\cos (180° + \theta)]^2$$

$$10 (\cos^2\theta - \sin^2\theta) = \frac{7}{2} (2 \sin \theta.\cos \theta) - 2 [-\cos \theta)]^2$$

$$10 \cos^2\theta - 10 \sin^2\theta = 7 \sin \theta.\cos \theta - 2 \cos^2\theta$$

$$-10 \sin^2\theta - 7 \sin \theta.\cos \theta + 12 \cos^2\theta = 0$$

$$10 \sin^2\theta + 7 \sin \theta.\cos \theta - 12 \cos^2\theta = 0$$

$$10 \tan^2\theta + 7 \tan \theta - 12 = 0$$

$$(5 \tan \theta - 4)(2 \tan \theta + 3) = 0$$

$$5 \tan \theta - 4 = 0$$

$$5 \tan \theta = 4$$

$$\tan \theta = \oplus \frac{4}{5}$$

tan +; 1st

$$\theta = \theta_{ref}$$

$$\theta = \tan^{-1}\left(\frac{4}{5}\right)$$

$$\theta = 38{,}660°$$

tan +; 3rd

$$\theta = 180° + \theta_{ref}$$

$$\theta = 180° + \tan^{-1}\left(\frac{4}{5}\right)$$

$$\theta = 218{,}660°$$

$$2 \tan \theta + 3 = 0$$

$$2 \tan \theta = -3$$

$$\tan \theta = \ominus \frac{3}{2}$$

tan -; 2nd

$$\theta = 180° - \theta_{ref}$$

$$\theta = 180° - \tan^{-1}\left(\frac{3}{2}\right)$$

$$\theta = 123{,}690°$$

tan –; 4th

$$\theta = 360° - \theta_{ref}$$

$$\theta = 360° - \tan^{-1}\left(\frac{3}{2}\right)$$

$$\theta = 303{,}690°$$

Since all the angles are within the restriction $x \in [0°; 360°]$, the solutions will be $\theta = 38{,}660°$, $\theta = 123{,}690°$, $\theta = 218{,}660°$ and $\theta = 303{,}690°$.

d)

$$\frac{\cos 2\theta}{\cos^2 \theta} + 2 = 0$$

$$\cos 2\theta + 2 \cos^2\theta = 0$$

$$2 \cos^2\theta - 1 + 2 \cos^2\theta = 0$$

$$4 \cos^2\theta - 1 = 0$$

$$4 \cos^2\theta = 1$$

$$\cos^2\theta = \frac{1}{4}$$

$$\cos \theta = \pm \frac{1}{2}$$

$$\cos \theta = \oplus \frac{1}{2}$$

cos +; 1st

$$\theta = \theta_{ref}$$

$$\theta = \cos^{-1}\left(\frac{1}{2}\right)$$

$$\theta = 60°$$

cos +; 4th

$$\theta = 360° - \theta_{ref}$$

$$\theta = 360° - \cos^{-1}\left(\frac{1}{2}\right)$$

$$\theta = 300°$$

$$\cos \theta = \ominus \frac{1}{2}$$

cos -; 2nd

$$\theta = 180° - \theta_{ref}$$

$$\theta = 180° - \cos^{-1}\left(\frac{1}{2}\right)$$

$$\theta = 120°$$

cos –; 3rd

$$\theta = 180° + \theta_{ref}$$

$$\theta = 180° + \cos^{-1}\left(\frac{1}{2}\right)$$

$$\theta = 240°$$

Since all the angles are within the restriction $x \in [0°; 360°]$, the solutions will be $\theta = 60°$, $\theta = 120°$, $\theta = 240°$ and $\theta = 300°$.

e)

$$(\cos 60°.\cos\theta + \sin 60°.\sin\theta) - (\cos 60°.\cos\theta - \sin 60°.\sin\theta) = \cos 2\theta$$

$$\cos 60°.\cos\theta + \sin 60°.\sin\theta - \cos 60°.\cos\theta + \sin 60°.\sin\theta = 1 - 2\sin^2\theta$$

$$2\sin 60°.\sin\theta = 1 - 2\sin^2\theta$$

$$2\left(\frac{\sqrt{3}}{2}\right).\sin\theta = 1 - 2\sin^2\theta$$

$$\sqrt{3}\ \sin\theta = 1 - 2\sin^2\theta$$

$$2\sin^2\theta + \sqrt{3}\sin\theta - 1 = 0$$

$$\sin\theta = \frac{-(\sqrt{3}) \pm \sqrt{(\sqrt{3})^2 - 4(2)(-1)}}{2(2)}$$

$$\sin\theta = \frac{-\sqrt{3} \pm \sqrt{11}}{4}$$

$$\sin\theta = \frac{-\sqrt{3} + \sqrt{11}}{4} \qquad \sin\theta \neq \frac{-\sqrt{3} - \sqrt{11}}{4}$$

sin +; 1st

$$\theta = \theta_{ref}$$

$$\theta = \sin^{-1}\left(\frac{-\sqrt{3} + \sqrt{11}}{4}\right)$$

$$\theta = 23{,}337°$$

sin +; 2nd

$$\theta = 180° - \theta_{ref}$$

$$\theta = 180° - \sin^{-1}\left(\frac{-\sqrt{3} + \sqrt{11}}{4}\right)$$

$$\theta = 156{,}663°$$

Since all the angles are within the restriction $x \in [0°; 360°]$, the solutions will be $\theta = 23{,}337°$ and $\theta = 156{,}663°$.

Assessment activity 3.10

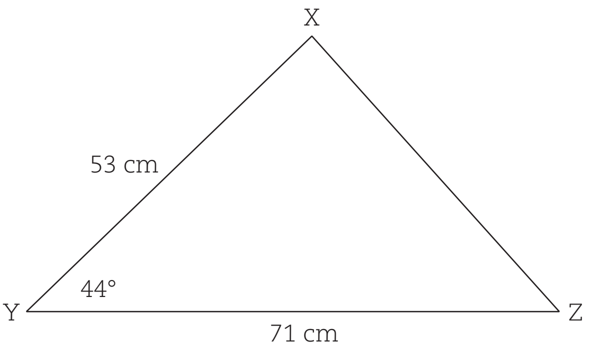
1.

$$XZ^2 = XY^2 + YZ^2 - 2(XY)(YZ)\cos X\hat{Y}Z$$

$$= (53)^2 + (71)^2 - 2(53)\,(71)\cos 44°$$

$$XZ^2 = 2\ 436{,}249$$

$$\therefore XZ = 49{,}358 \text{ cm}$$



$$\frac{\sin\hat{X}}{YZ} = \frac{\sin\hat{Y}}{XZ}$$

$$\sin\hat{X} = \frac{YZ\sin\hat{Y}}{XZ}$$

$$\hat{X} = \sin^{-1}\left[\frac{YZ\sin\hat{Y}}{XZ}\right]$$

$$= \sin^{-1}\left[\frac{71\sin 44°}{49{,}358}\right]$$

$$\hat{X} = 87{,}763°$$

$$\hat{X} + \hat{Y} + \hat{Z} = 180°$$

$$\therefore \hat{Z} = 180° - (\hat{X} + \hat{Y})$$

$$= 180° - (87{,}763° + 44°)$$

$$\hat{Z} = 48{,}237°$$

2. In ΔPQR, $\sin 45° = \frac{PR}{QR}$

$$\sin 45° = \frac{PR}{\sqrt{2}}$$

$$\therefore PR = \sqrt{2}\ \sin 45°$$

$$= \sqrt{2}\left(\frac{1}{\sqrt{2}}\right)$$

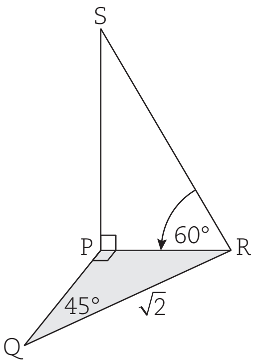
$$PR = 1 \text{ unit}$$

In ΔPRS, $\tan 60° = \frac{SP}{PR}$

$$\therefore SP = PR \tan 60°$$

$$= (1)\left(\frac{\sqrt{3}}{1}\right)$$

$$SP = \sqrt{3} \text{ units}$$

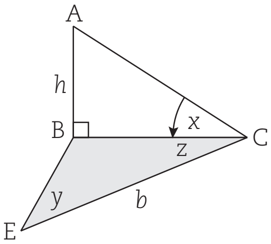


3. **a)** In ΔBCE, $E\hat{B}C = 180° - (y + z)$

$$\frac{BC}{\sin B\hat{E}C} = \frac{EC}{\sin E\hat{B}C}$$

$$\frac{BC}{\sin y} = \frac{b}{\sin[180° - (y + z)]}$$

$$BC = \frac{b \sin y}{\sin(y + z)}$$



In ΔABC, $\tan x = \frac{AB}{BC}$

$$\therefore AB = BC \tan x$$

$$h = \frac{b \sin y}{\sin(y + z)}.\tan x$$

$$h = \frac{b \sin y.\tan x}{\sin(y + z)}$$

b) $h = \frac{b \sin y.\tan x}{\sin(y + z)}$

$$= \frac{650 \sin 41,8°.\tan 14,9°}{\sin(41,8° + 66,7°)}$$

$$h = 121,560 \text{ m}$$

4. **a)** In ΔABC, $\frac{AB}{\sin A\hat{C}B} = \frac{BC}{\sin B\hat{A}C}$

$$\frac{AB}{\sin \alpha} = \frac{h}{\sin \theta}$$

$$AB = \frac{h \sin \alpha}{\sin \theta}$$

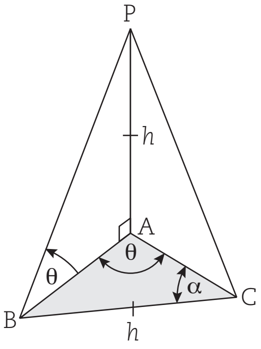
b) In ΔBAP, $\tan \theta = \frac{AP}{AB}$

$$\frac{\sin \theta}{\cos \theta} = \frac{h}{\frac{h \sin \alpha}{\sin \theta}}$$

$$\frac{\sin \theta}{\cos \theta} = \frac{\sin \theta}{\sin \alpha}$$

$$\sin \theta.\sin \alpha = \sin \theta.\cos \theta$$

$$\therefore \sin \alpha = \cos \theta$$

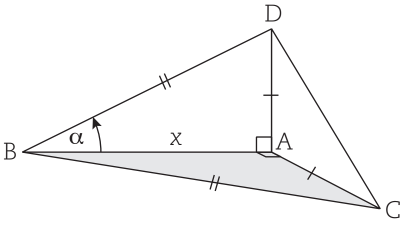


5. **a)** In ΔABD, $\cos A\hat{B}D = \frac{AB}{BD}$

$$\cos \alpha = \frac{x}{BD}$$

$$BD \cos \alpha = x$$

$$BD = \frac{x}{\cos \alpha}$$



But, BC = BD

$$\therefore\ BC = \frac{x}{\cos\alpha}$$

$$\tan A\hat{B}D = \frac{AD}{AB}$$

$$\tan\alpha = \frac{AD}{x}$$

$$x\tan\alpha = AD$$

But, AC = AD

$$\therefore\ AC = x\tan\alpha$$

b) In ΔACD, $CD^2 = AC^2 + AD^2$

$$= (x\tan\alpha)^2 + (x\tan\alpha)^2$$

$$CD^2 = 2x^2\tan^2\alpha$$

c) In ΔBCD, $CD^2 = BC^2 + BD^2 - 2(BC)(BD)\cos C\hat{B}D$

$$CD^2 = \left(\frac{x}{\cos\alpha}\right)^2 + \left(\frac{x}{\cos\alpha}\right)^2 - 2\left(\frac{x}{\cos\alpha}\right)\left(\frac{x}{\cos\alpha}\right)\cos C\hat{B}D$$

$$2x^2\tan^2\alpha = \frac{2x^2}{\cos^2\alpha} - \frac{2x^2}{\cos^2\alpha}\cos C\hat{B}D$$

$$2x^2.\frac{\sin^2\alpha}{\cos^2\alpha} = \frac{2x^2}{\cos^2\alpha} - \frac{2x^2}{\cos^2\alpha}\cos C\hat{B}D$$

$$\frac{2x^2\sin^2\alpha}{\cos^2\alpha} = \frac{2x^2}{\cos^2\alpha}(1 - \cos C\hat{B}D)$$

$$\sin^2\alpha = 1 - \cos C\hat{B}D$$

$$\sin^2\alpha - 1 = -\cos C\hat{B}D$$

$$1 - \sin^2\alpha = \cos C\hat{B}D$$

$$\cos^2\alpha = \cos C\hat{B}D$$

Solutions for summative assessment: Chapter 3

Section A

1.

$$x^2 + y^2 = 69 + 12y - 16x$$

$$(x^2 + 16x) + (y^2 - 12y) = 69$$

$$\left(x^2 + 16x + \left(\frac{1}{2}.16\right)^2\right) + \left(y^2 - 12y + \left(\frac{1}{2}.-12\right)^2\right) = 69 + \left(\frac{1}{2}.16\right)^2 + \left(\frac{1}{2}.-12\right)^2$$

$$(x^2 + 16x + 64) + (y^2 - 12y + 36) = 69 + 64 + 36$$

$$(x + 8)^2 + (y - 6)^2 = 169$$

$$r^2 = 169$$

$$\sqrt{r^2} = \sqrt{169}$$

$$\therefore r = 13 \text{ units}$$

$x + 8 = 0$ and $y - 6 = 0$

$x = -8$ $y = 6$

$\therefore$ centre $= (h; k) = (-8; 6)$ (4)

2. $M = \left(\frac{x_1 + x_2}{2}; \frac{y_1 + y_2}{2}\right)$

$= \left(\frac{(2) + (4)}{2}; \frac{(6) + (6)}{2}\right)$

$M = (3; 6)$

$$(x - h)^2 + (y - k)^2 = r^2$$

$$((2) - (3))^2 + ((6) - (6))^2 = r^2$$

$$(-1)^2 + (0)^2 = r^2$$

$$1 = r^2$$

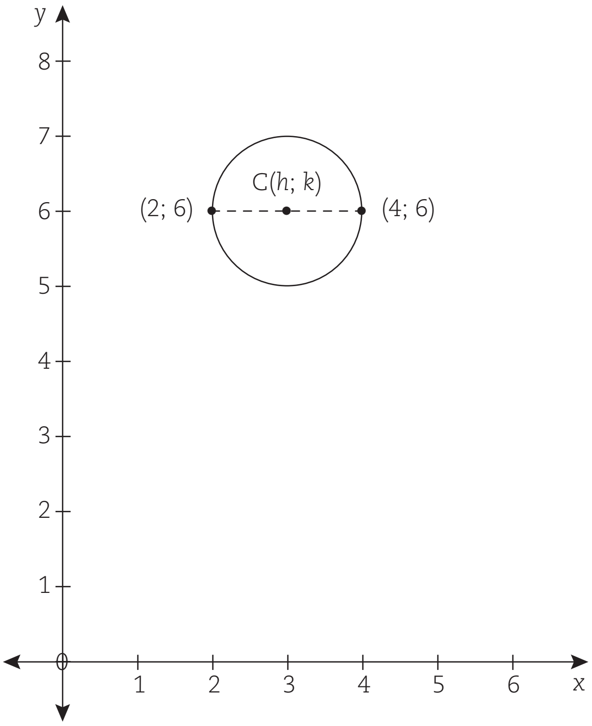
$$(x - 3)^2 + (y - 6)^2 = 1$$

$$((4) - 3)^2 + ((3) - 6)^2 = r^2$$

$$10 = r^2$$

$$r = \sqrt{10} \text{ units}$$

Since $r = \sqrt{10}$ is greater than $r = 1$, therefore the point (4; 3) is outside the circle.



(4)

3. a)

$$x^2 + y^2 - 112 = 14x - 16y$$

$$(x^2 - 14x) + (y^2 + 16y) = 112$$

$$\left(x^2 - 14x + \left(\frac{1}{2}.-14\right)^2\right) + \left(y^2 + 16y + \left(\frac{1}{2}.16\right)^2\right) = 112 + \left(\frac{1}{2}.-14\right)^2 + \left(\frac{1}{2}.16\right)^2$$

$$(x^2 - 14x + 49) + (y^2 + 16y + 64) = 112 + 49 + 64$$

$$(x - 7)^2 + (y - 8)^2 = 225$$

$$r^2 = 225$$

$$\sqrt{r^2} = \sqrt{225}$$

$$\therefore r = 15 \text{ units}$$

$x - 7 = 0$ and $y + 8 = 0$

$x = 7$ $y = -8$

$\therefore$ centre $= (h; k) = (7; -8)$ (4)

b) $P(x_1; y_1) = (-2; 4)$ and $P(x_2; y_2) = (7; -8)$

$$m_{OP} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{(-8) - (4)}{(7) - (-2)}$$

$$m_{OP} = -\frac{4}{3}$$

(2)

c) $m_{OP} \times m_{tan} = -1$

$$m_{tan} = \frac{-1}{m_{OP}}$$

$$= \frac{-1}{-\frac{4}{3}}$$

$$m_{tan} = \frac{3}{4}$$

$$y - y_1 = m(x - x_1)$$

$$y - (4) = \frac{3}{4}(x - (-2))$$

$$y - 4 = \frac{3}{4}x + \frac{3}{2}$$

$$y = \frac{3}{4}x + \frac{11}{2}$$

(2)

4. **a)** The condition of tangency refers to the relationship between the y–intercept (c), the radius (r) and the gradient (m), that is, $c^2 = r^2(m^2 + 1)$ (2)

b) **(i)** $x^2 + y^2 - 49 = 0$

$$x^2 + y^2 = 49$$

$$\therefore r^2 = 49$$

$$\sqrt{r^2} = \sqrt{49}$$

$$r = 7 \text{ units}$$

$$\theta = 150°$$

$$\therefore m = \tan\theta$$

$$= \tan 150°$$

$$m = -\frac{\sqrt{3}}{3}$$

$$c^2 = r^2(m^2 + 1)$$

$$= (7)^2\left(\left(-\frac{\sqrt{3}}{3}\right)^2 + 1\right)$$

$$c^2 = \frac{196}{3}$$

$$\sqrt{c^2} = \sqrt{\frac{196}{3}}$$

$$c = \pm\frac{14\sqrt{3}}{3}$$

$$y = -\frac{\sqrt{3}}{3}x \pm \frac{14\sqrt{3}}{3}$$

(4)

(ii) $x^2 + y^2 = 18$

$$\therefore r^2 = 18$$

$$\sqrt{r^2} = \sqrt{18}$$

$$r = 3\sqrt{2} \text{ units}$$

$$c = -5$$

$$c^2 = r^2(m^2 + 1)$$

$$(-5)^2 = (3\sqrt{2})^2(m^2 + 1)$$

$$25 = 18(m^2 + 1)$$

$$\frac{25}{18} = m^2 + 1$$

$$\frac{25}{18} - 1 = m^2$$

$$\frac{7}{18} = m^2$$

$$\sqrt{\frac{7}{18}} = \sqrt{m^2}$$

$$\pm\frac{\sqrt{14}}{6} = m$$

$$y = \pm\frac{\sqrt{14}}{6}x - 5$$

(5)

5.

$$P(x_1; y_1) = (-20; 7)$$

$$(x + 4)^2 + (y + 5)^2 = 400$$

$$\therefore r^2 = 400$$

$$\therefore \text{centre} = (h; k) = (-4; -5)$$

$$(x_1 - h)(x - h) + (y_1 - k)(y - k) = r^2$$

$$((-20) - (-4))(x - (-4)) + ((7) - (-5))(y - (-5)) = 400$$

$$-16x - 64 + 12y + 60 = 400$$

$$12y = 16x + 404$$

$$y = \frac{4}{3}x + \frac{101}{3}$$

(3)

Total [30]

Section B

1. **a)** Secant: a straight line cutting a circle or other curve. (1)

b) Bisector: a straight line that divides another line or angle into two equal parts. (1)

c) Tangent: a straight line touching the curve at a point. (1)

2.

a)	$\hat{a} = 3\hat{x}$	• Vertically opposite angles.	
	$6\hat{x} + \hat{a} = 180°$	• Consecutive interior angles.	
	$6\hat{x} + 3\hat{x} = 180°$		
	$9\hat{x} = 180°$		
	$\hat{x} = 20°$		(3)
b)	$\hat{a} = 3\hat{x}$		
	$= 3(20°)$		
	$\hat{a} = 60°$		
	$\hat{b} + \hat{a} = 180°$	• Straight angle.	
	$\hat{b} = 180° - \hat{a}$		
	$= 180° - 60°$		
	$\hat{b} = 120°$		
	$\hat{c} = \hat{b}$	• Vertically opposite angles.	
	$\hat{c} = 120°$		
	$\hat{d} = \hat{a}$	• Alternate exterior angles.	
	$\hat{d} = 60°$		
	$\hat{e} = \hat{d}$	• Vertically opposite angles.	
	$\hat{e} = 60°$		
	$\hat{f} = \hat{b}$	• Alternate exterior angles.	
	$\hat{f} = 120°$		
	$\hat{g} = \hat{f}$	• Vertically opposite angles.	
	$\hat{g} = 120°$		
	$\hat{h} = \hat{i}$	• Isosceles triangle, base angles are equal.	
	$\hat{h} + \hat{i} = \hat{d}$	• Exterior angle of a triangle is equal to the sum of the two opposite interior angles.	
	$2\hat{h} = \hat{d}$		
	$2\hat{h} = 60°$		
	$\hat{h} = 30°$		
	$\hat{i} = \hat{h}$	• Isosceles triangle, base angles are equal.	
	$\hat{i} = 30°$		(9)

3. **a)** $\hat{a} = 62°$

- Two tangents drawn from an external point are the same length, therefore an isosceles triangle.

$\hat{b} = 180° - 2\hat{a}$
$= 180° - 2(62°)$
$\hat{b} = 56°$

- Sum of the interior angles of a triangle is equal to 180°.

$\hat{c} = 62°$

- Alternate interior angles.

$\hat{d} = \hat{c}$
$\hat{d} = 62°$

- Angle between a tangent and a chord drawn from the point of contact is equal to an angle in the alternate segment. (8)

b) $\hat{e} = \frac{1}{2}(110°)$
$\hat{e} = 55°$

- Angle subtended at the circumference is half the angle at the centre subtended by the same arc.

$\hat{f} = \hat{e}$
$\hat{f} = 55°$

- Exterior angle of a cyclic quadrilateral is equal to the interior opposite angle. (4)

c) $OL = OK$
$\therefore O\hat{L}K = O\hat{K}L$
$\therefore O\hat{L}K = \frac{1}{2}(180° - 110°)$
$O\hat{L}K = 35°$

- Radii of circle.
- ΔLOK is an isosceles triangle with base angles equal.
- Sum of the interior angles of a triangle is equal to 180°.

$O\hat{L}K + \hat{g} = 90°$
$\hat{g} = 90° - O\hat{L}K$
$= 90° - 35°$
$\hat{g} = 55°$

- A tangent to a circle is perpendicular to the radius drawn from the point of contact.

(3)

Total [30]

Section C

1. **a)** $\sin(x - 23°)$
$= \sin x.\cos 23° - \sin 23°.\cos x$

b) $\cos(30° - y)$
$= \cos 30°.\cos y + \sin 30°.\sin y$

c) $\sin(4x + \pi)$
$= \sin 4x.\cos \pi + \sin \pi.\cos 4x$

d) $\cos(\mu + \varphi)$
$= \cos \mu.\cos \varphi - \sin \mu.\sin \varphi$ (4)

2. **a)** $\cos 5x.\cos 2x - \sin 2x.\sin 5x$
$= \cos(5x + 2x)$
$= \cos 7x$

b) $\sin 77°.\cos 17° - \sin 17°.\cos 77°$
$= \sin(77° - 17°)$
$= \sin 60°$
$= \frac{\sqrt{3}}{2}$

c) $\cos 45°.\cos 30° + \sin 45°.\sin 30°$
$= \cos(45° - 30°)$
$= \cos 15°$

d) $\sin \lambda.\cos 2\psi + \sin 2\psi.\cos \lambda$
$= \sin(\lambda + 2\psi)$ (4)

3. a) $\cos 105°$

$= \cos (45° + 60°)$

$= \cos 45°.\ \cos 60° - \sin 45°.\ \sin 60°$

$= \left(\frac{1}{\sqrt{2}}\right).\left(\frac{1}{2}\right) - \left(\frac{1}{\sqrt{2}}\right).\left(\frac{\sqrt{3}}{2}\right)$

$= \frac{1}{2\sqrt{2}} - \frac{\sqrt{3}}{2\sqrt{2}}$

$= \frac{1-\sqrt{3}}{2\sqrt{2}}$

$= \frac{1-\sqrt{3}}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$

$= \frac{\sqrt{2}-\sqrt{6}}{4}$ (5)

b) $\sin 75°$

$= \sin(30° + 45°)$

$= \sin 30°.\cos 45° + \sin 45°.\cos 30°$

$= \left(\frac{1}{2}\right).\left(\frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}}\right).\left(\frac{\sqrt{3}}{2}\right)$

$= \frac{1}{2\sqrt{2}} + \frac{\sqrt{3}}{2\sqrt{2}}$

$= \frac{1+\sqrt{3}}{2\sqrt{2}}$

$= \frac{1+\sqrt{3}}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$

$= \frac{\sqrt{6}+\sqrt{2}}{4}$ (7)

c) (i) $\cos 2\alpha = \cos (\alpha + \alpha)$

$= \cos \alpha\ .\cos \alpha - \sin \alpha.\ \sin \alpha$

$= \cos^2 \alpha - \sin^2 \alpha$

$= \cos^2 \alpha - (1 - \cos^2 \alpha)$

$\cos^2 \alpha = 2\cos^2 \alpha - 1$ (3)

(ii) $\cos 120° = \cos [2(60°)]$

$= 2\cos^2 60° - 1$

$= 2\left(\frac{1}{2}\right)^2 - 1$

$= \frac{1}{2} - 1$

$\cos 120° = -\frac{1}{2}$ (3)

(iii) $\cos 120° = \cos (90° + 30°)$

$= -\sin 30°$

$= -\left(\frac{1}{2}\right)$

$\cos 120° = -\frac{1}{2}$ (2)

4. a) $\tan x + \frac{1}{\tan x}$

$= \frac{\sin x}{\cos x} + \frac{1}{\frac{\sin x}{\cos x}}$

$= \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}$

$= \frac{\sin x(\sin x) + \cos x(\cos x)}{\sin x.\cos x}$

$= \frac{\sin^2 x + \cos^2 x}{\sin x.\cos x}$

$= \frac{1}{\sin x.\cos x}$

$= \frac{1}{\frac{1}{2}\sin 2x}$

$= \frac{2}{\sin 2x}$ (4)

b) $\sin 4x = 4 \sin x.\cos x.\cos 2x$

L.H.S. $\sin 4x$

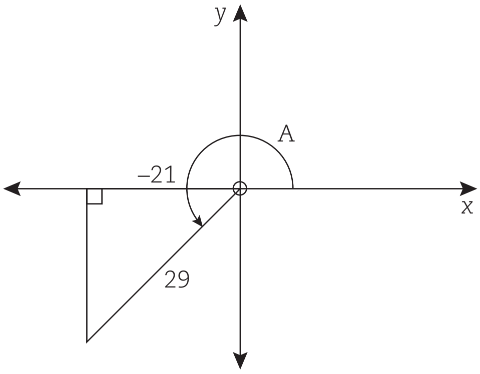
$= 2 \sin 2x.\cos 2x$

$= 2(2 \sin x.\cos x).\cos 2x$

$= 4 \sin x.\cos x.\cos 2x$

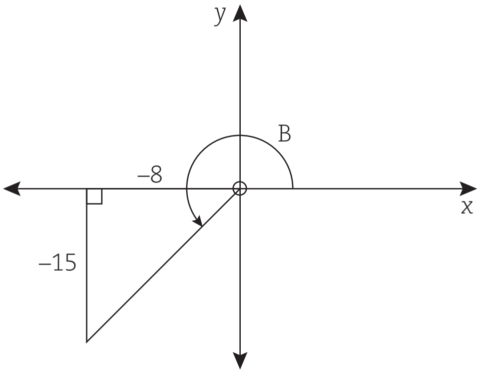
$\therefore$ L.H.S. = R.H.S. (3)

c) $\cos A = -\frac{21}{29}$, $A \in [180°; 360°]$



$$
\begin{aligned}
x^2 + y^2 &= r^2\\
(-21)^2 + y^2 &= (29)^2\\
441 + y^2 &= 841\\
y^2 &= 841 - 441\\
y^2 &= 400\\
y &= -20
\end{aligned}
$$

$\tan B = \frac{15}{8}$, $B \in [180°; 360°]$



$$
\begin{aligned}
x^2 + y^2 &= r^2\\
(-8)^2 + (-15)^2 &= r^2\\
64 + 225 &= r^2\\
289 &= r^2\\
r^2 &= 289\\
r &= 17
\end{aligned}
$$

(i) $\sin(180° - A)$

$= \sin A$

$= -\frac{20}{29}$

(ii) $\cos(A + B)$

$= \cos A.\cos B - \sin A.\sin B$

$= \left(-\frac{21}{29}\right).\left(-\frac{8}{17}\right) - \left(-\frac{20}{29}\right).\left(-\frac{15}{17}\right)$

$= \frac{168}{493} - \frac{300}{493}$

$= -\frac{132}{493}$

(iii) $\sin 2A$

$= 2\sin A.\cos A$

$= 2\left(-\frac{20}{29}\right)\left(-\frac{21}{29}\right)$

$= \frac{840}{841}$

(iv) $\sin(A - B)$

$= \sin A.\cos B - \sin B.\cos A$

$= \left(-\frac{20}{29}\right).\left(-\frac{8}{17}\right) - \left(-\frac{15}{17}\right)\left(-\frac{21}{29}\right)$

$= \frac{160}{493} - \frac{315}{493}$

$= -\frac{155}{493}$

(v) $\cos 2B$

$= \cos(B + B)$

$= \cos B.\cos B - \sin B.\sin B$

$= \cos^2 B - \sin^2 B$

$= \left(-\frac{8}{17}\right)^2 - \left(-\frac{15}{17}\right)^2$

$= \frac{64}{289} - \frac{225}{289}$

$= -\frac{161}{289}$

(vi) $\tan(90° + B)$

$= \frac{\sin(90° + B)}{\cos(90° + B)}$

$= \frac{\cos B}{-\sin B}$

$= \frac{-\frac{8}{17}}{-(-\frac{15}{17})}$

$= -\frac{8}{15}$

(8)

5. **a)** $\sin(x - 40°) = \cos 40°.\cos x - \sin 40°.\sin x$

$\sin(x - 40°) = \cos(40° + x)$

$\sin(x - 40°) = \sin[90° - (40° + x)]$

$\therefore x - 40° = 90° - 40° - x$

$2x = 90°$

$x = 45°$

OR

$x - 40° = 180° - (90° - 40° - x)$

$0 = 170°$

$\sin(x - 40°) = \cos 40°.\cos x - \sin 40°.\sin x$

$\sin(x - 40°) = \cos(40° + x)$

$\sin(x - 40°) = \sin[90° + (40° + x)]$

$\therefore x - 40° = 90° + 40° + x$

$0 = 170°$

OR

$\therefore x - 40° = 180° - (90° + 40° + x)$

$2x = 90°$

$x = 45°$

But $x \in [0°; 90°]$, $x = 45°$. (4)

b)

$\cos 2\theta = \cos\theta$

$2\cos^2\theta - 1 = \cos\theta$

$2\cos^2\theta - \cos\theta - 1 = 0$

$(2\cos\theta + 1)(\cos\theta - 1) = 0$

$2\cos\theta + 1 = 0$

$2\cos\theta = -1$

$\cos\theta = \textcircled{$-$}\ \frac{1}{2}$

cos –; 2nd

$\theta = 180° - \theta_{ref}$

$= 180° - \cos^{-1}\left(\frac{1}{2}\right)$

$\theta = 120°$

cos–; 3rd

$\theta = 180° + \theta_{ref}$

$= 180° + \cos^{-1}\left(\frac{1}{2}\right)$

$\theta = 240°$

$\cos\theta - 1 = 0$

$\cos\theta = \textcircled{$+$}\ 1$

cos +; 1st

$\theta = \theta_{ref}$

$= \cos^{-1}(1)$

$\theta = 0°$

cos +; 4th

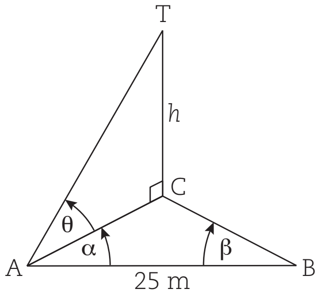
$\theta = 360° - \theta_{ref}$

$= 360° - \cos^{-1}(1)$

$\theta = 360°$

Since all the angles are within the restriction $\theta \in [0°; 360°]$, the solutions will be $\theta = 0°$, $\theta = 120°$, $\theta = 240°$ and $\theta = 360°$. (6)

6.



a)

$$\frac{AC}{\sin\beta} = \frac{AB}{\sin[180° - (\alpha + \beta)]}$$

$$AC = \frac{AB \sin\beta}{\sin[180° - (\alpha + \beta)]}$$

$$AC = \frac{25 \sin\beta}{\sin(\alpha + \beta)}$$

$$\tan\theta = \frac{h}{AC}$$

$$\therefore h = AC \tan\theta$$

$$= \frac{25 \sin\beta}{\sin(\alpha + \beta)} . \tan\theta$$

$$h = \frac{25 \sin\beta . \tan\theta}{\sin(\alpha + \beta)}$$ (5)

b)

$$h = \frac{25 \sin\beta . \tan\theta}{\sin(\alpha + \beta)}$$

$$= \frac{25 \sin 53° . \tan 26°}{\sin(44° + 53°)}$$

$h = 9{,}811$ m (2)

Total [60]

Worked solutions • Chapter 4

Data handling and probability models

Assessment activity 4.1

Group projects

Assessment activity 4.2

1.

Ordered data
1
2
2
3
4
5
7
8
12
$x_{10} = 19$
$x_{11} = 20$
21
22
27
34
35
44
45
46

$$P_{Q_2} = \frac{1}{2}(x + 1) = \frac{1}{2}(20 + 1) = 10{,}5$$

$$Q_2 = \frac{x_{10} + x_{11}}{2} = \frac{19 + 20}{2} = 19{,}5$$

$$\bar{x} = \frac{\sum x_i}{n} = \frac{377}{20} = 18{,}850$$

Traditional method

x_i	$x_i - \bar{x}$	$(x_i - \bar{x})^2$
21	2,150	4,622
34	15,150	229,523
2	–16,850	283,923
7	–11,850	140,423
27	8,150	66,423
19	0,150	0,022
8	–10,850	117,723
44	25,150	632,523
20	1,150	1,323
12	–6,850	46,923
5	–13,850	191,823
22	3,150	9,922
1	–17,850	318,623
3	–15,850	251,223
46	27,150	737,123
45	26,150	683,823
2	–16,850	283,923
20	1,150	1,323
35	16,150	260,823
4	–14,850	220,523
$\sum x_i = 377$	$\sum(x_i - \bar{x}) = 0{,}000$	$\sum(x_i - \bar{x})^2 = 4\,482{,}550$

$$s^2 = \frac{\sum(x_i - \bar{x})^2}{n - 1}$$

$$= \frac{4\,482{,}550}{20 - 1}$$

$$s^2 = 235{,}924$$

$$\therefore s = \sqrt{235{,}924}$$

$$s = 15{,}360$$

Shorthand method

x_i	x_i^2
21	441
34	1156
2	4
7	49
27	729
19	361
8	64
44	1936
20	400
12	144
5	25
22	484
1	1
3	9
46	2116
45	2025
2	4
20	400
35	1225
4	16
$\sum x_i = 377$	$\sum x_i^2 = 11\,589$

$$s^2 = \frac{1}{n-1}\,[\sum x_i^2 - n\bar{x}^2]$$

$$= \frac{1}{20-1}\,[11\,589 - 20(18{,}850)^2]$$

$$s^2 = 235{,}924$$

$$\therefore\ s = \sqrt{235{,}924}$$

$$s = 15{,}360$$

Since $\bar{x} - Q_2 < 0$, therefore the data are skewed to the left, that is, negatively skewed.
Since the data does not represent a normal distribution, therefore a 68% confidence interval cannot be computed.

2.

	Ordered data
	7
	12
	14
	18
	19
	20
	21
	23
	26
	31
	35
	46
x_{13} =	㊻
	48
	50
	55
	62
	69
	69
	69
	76
	80
	82
	91
	97

$$P_{Q_2} = \frac{1}{2}(n+1) = \frac{1}{2}(25+1) = 13$$

$$Q_2 = x_{13} = 46{,}000$$

$$\bar{x} = \frac{\sum x_i}{n} = \frac{1\,166}{25} = 46{,}640$$

Traditional method

x_i	$x_i - \overline{x}$	$(x_i - \overline{x})^2$
76	29,360	862,010
48	1,360	1,850
7	–39,640	1571,330
50	3,360	11,290
62	15,360	235,930
97	50,360	2536,130
82	35,360	1250,330
80	33,360	1112,890
31	–15,640	244,610
69	22,360	499,970
91	44,360	1967,810
23	–23,640	558,850
46	–0,640	0,410
14	–32,640	1065,370
20	–26,640	709,690
69	22,360	499,970
12	–34,640	1199,930
26	–20,640	426,010
19	–27,640	763,970
21	–25,640	657,410
35	–11,640	135,490
46	–0,640	0,410
18	–28,640	820,250
69	22,360	499,970
55	8,360	69,890
$\sum x_i = 1\ 166$	$\sum(x_i - \overline{x}) = 0{,}000$	$\sum(x_i - \overline{x})^2 = 17\ 701{,}760$

$$s^2 = \frac{\sum(x_i - \overline{x})^2}{n-1}$$

$$= \frac{17\ 701{,}760}{25-1}$$

$$s^2 = 737{,}573$$

$$\therefore\ s = \sqrt{737{,}573}$$

$$s = 27{,}158$$

Shorthand method

x_i	x_i^2
76	5776
48	2304
7	49
50	2500
62	3844
97	9409
82	6724
80	6400
31	961
69	4761
91	8281
23	529
46	2116
14	196
20	400
69	4761
12	144
26	676
19	361
21	441
35	1225
46	2116
18	324
69	4761
55	3025
$\sum x_i = 1\ 166$	$\sum x_i^2 = 72\ 084$

$$s^2 = \frac{1}{n-1}[\sum x_i^2 - n\overline{x}^2]$$

$$= \frac{1}{25-1}[72\ 084 - 25(46{,}640)^2]$$

$$s^2 = 737{,}573$$

$$\therefore s = \sqrt{737{,}573}$$

$$s = 27{,}158$$

Since $\overline{x} - Q_2 \approx 0$, therefore the data has a normal distribution. Since the data has a normal distribution, 68% confidence interval will be,

$$\overline{x} - s \leq x \leq \overline{x} + s$$

$$46{,}640 - 27{,}158 \leq x \leq 46{,}640 + 27{,}158$$

$$19{,}482 \leq x \leq 73{,}798$$

3.

Ordered data
18
18
19
23
23
27
31
$x_8 = \textcircled{31}$
36
36
40
42
46
54
59

$$P_{Q_2} = \frac{1}{2}(n+1) = \frac{1}{2}(15+1) = 8$$

$$\longleftarrow Q_2 = x_8 = 31{,}000$$

$$\overline{x} = \frac{\sum x_i}{n} = \frac{503}{15} = 33{,}533$$

Traditional method

x_i	$x_i - \overline{x}$	$(x_i - \overline{x})^2$
19	–14,533	211,218
31	–2,533	6,418
59	25,467	648,551
27	–6,533	42,684
42	8,467	71,684
36	2,467	6,084
40	6,467	41,818
31	–2,533	6,418
46	12,467	155,418
18	–15,533	241,284
36	2,467	6,084
18	–15,533	241,284
54	20,467	418,884
23	–10,533	110,951
23	–10,533	110,951
$\sum x_i = 503$	$\sum(x_i - \overline{x}) = 0{,}000$	$\sum(x_i - \overline{x})^2 = 2\ 319{,}733$

$$s^2 = \frac{\sum(x_i - \overline{x})^2}{n-1}$$

$$= \frac{2\ 319{,}733}{15-1}$$

$$s^2 = 165{,}695$$

$$\therefore s = \sqrt{165{,}695}$$

$$s = 12{,}872$$

Shorthand method

x_i	x_i^2
19	361
31	961
59	3481
27	729
42	1764
36	1296
40	1600
31	961
46	2116
18	324
36	1296
18	324
54	2916
23	529
23	529
$\sum x_i = 503$	$\sum x_i^2 = 19\ 187$

$$s^2 = \frac{1}{n-1}[\sum x_i^2 - n\overline{x}^2]$$

$$= \frac{1}{15-1}[19\ 187 - 15(33{,}533)^2]$$

$$s^2 = 165{,}695$$

$$\therefore s = \sqrt{165{,}695}$$

$$s = 12{,}872$$

Since $\overline{x} - Q_2 > 0$, therefore the data are skewed to the right, that is, positively skewed.
Since the data does not represent a normal distribution, therefore a 68% confidence interval cannot be computed.

4.

Ordered data
25
25
25
37
41
42
44
$x_8 =$ 48
$x_9 =$ 50
51
53
60
60
63
69
69

$$P_{Q_2} = \frac{1}{2}(n + 1) = \frac{1}{2}(16 + 1) = 8{,}5$$

$$Q_2 = \frac{x_8 + x_9}{2} = \frac{48 + 50}{2} = 49{,}000$$

$$\overline{x} = \frac{\sum x_i}{n} = \frac{762}{16} = 47{,}625$$

Traditional method

x_i	$x_i - \overline{x}$	$(x_i - \overline{x})^2$
53	5,375	28,891
44	–3,625	13,141
60	12,375	153,141
25	–22,625	511,891
60	12,375	153,141
63	15,375	236,391
69	21,375	456,891
69	21,375	456,891
50	2,375	5,641
48	0,375	0,141
41	–6,625	43,891
25	–22,625	511,891
42	–5,625	31,641
51	3,375	11,391
37	–10,625	112,891
25	–22,625	511,891
$\sum x_i = 762$	$\sum(x_i - \overline{x}) = 0{,}000$	$\sum(x_i - \overline{x})^2 = 3\ 239{,}750$

$$s^2 = \frac{\sum(x_i - \overline{x})^2}{n - 1}$$

$$= \frac{3\ 239{,}750}{16 - 1}$$

$$s^2 = 215{,}983$$

$$\therefore s = \sqrt{215{,}983}$$

$$s = 14{,}696$$

Shorthand method

x_i	x_i^2
53	2809
44	1936
60	3600
25	625
60	3600
63	3969
69	4761
69	4761
50	2500
48	2304
41	1681
25	625
42	1764
51	2601
37	1369
25	625
$\sum x_i = 762$	$\sum x_i^2 = 39\ 530$

$$s^2 = \frac{1}{n-1}[\sum x_i^2 - n\bar{x}^2]$$

$$= \frac{1}{16-1}[39\ 530 - 16(47{,}625)^2]$$

$$s^2 = 215{,}983$$

$$\therefore s = \sqrt{215{,}983}$$

$$s = 14{,}696$$

Since $\bar{x} - Q_2 < 0$, therefore the data are skewed to the left, that is, negatively skewed.
Since the data does not represent a normal distribution, therefore a 68% confidence interval cannot be computed.

5.

Ordered data
32
35
36
43
44
46
47
47
$x_9 = \textcircled{48}$
$x_{10} = \textcircled{51}$
52
54
54
56
58
62
62
65

$$P_{Q_2} = \frac{1}{2}(n+1) = \frac{1}{2}(18+1) = 9{,}5$$

$$Q_2 = \frac{x_9 + x_{10}}{2} = \frac{48+51}{2} = 49{,}500$$

$$\bar{x} = \frac{\sum x_i}{n} = \frac{892}{18} = 49{,}556$$

Traditional method

x_i	$x_i - \overline{x}$	$(x_i - \overline{x})^2$
47	–2,556	6,531
58	8,444	71,309
35	–14,556	211,864
46	–3,556	12,642
52	2,444	5,975
44	–5,556	30,864
56	6,444	41,531
65	15,444	238,531
62	12,444	154,864
43	–6,556	42,975
51	1,444	2,086
54	4,444	19,753
36	–13,556	183,753
32	–17,556	308,198
48	–1,556	2,420
47	–2,556	6,531
54	4,444	19,753
62	12,444	154,864
$\sum x_i = 892$	$\sum(x_i - \overline{x}) = 0{,}000$	$\sum(x_i - \overline{x})^2 = 1\ 514{,}444$

$$s^2 = \frac{\sum(x_i - \overline{x})^2}{n - 1}$$

$$= \frac{1\ 514{,}444}{18 - 1}$$

$$s^2 = 89{,}085$$

$$\therefore s = \sqrt{89{,}085}$$

$$s = 9{,}438$$

Shorthand method

x_i	x_i^2
32	1024
35	1225
36	1296
43	1849
44	1936
46	2116
47	2209
47	2209
48	2304
51	2601
52	2704
54	2916
54	2916
56	3136
58	3364
62	3844
62	3844
65	4225
$\sum x_i = 892$	$\sum x_i^2 = 45\ 718$

$$s^2 = \frac{1}{n - 1}[\sum x_i^2 - n\overline{x}^2]$$

$$= \frac{1}{18 - 1}[45\ 718 - 18(49{,}556)^2]$$

$$s^2 = 89{,}085$$

$$\therefore s = \sqrt{89{,}085}$$

$$s = 9{,}438$$

Since $\overline{x} - Q_2 \approx 0$, therefore the data has a normal distribution. Since the data has a normal distribution, 68% confidence interval will be,

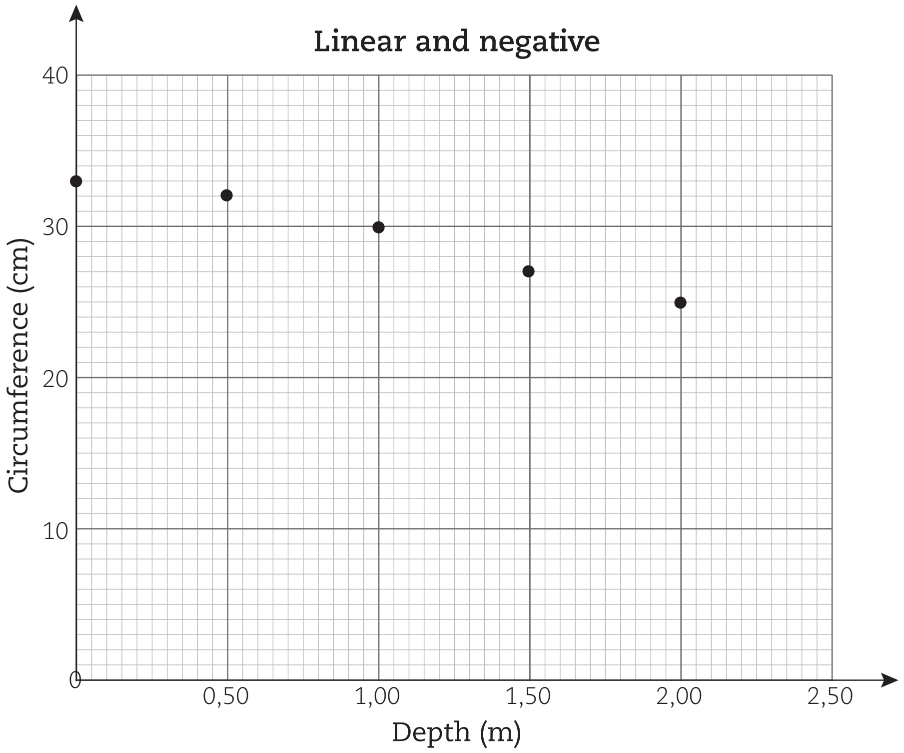
$$\overline{x} - s \leq x \leq \overline{x} + s$$

$$49{,}556 - 9{,}438 \leq x \leq 49{,}556 + 9{,}438$$

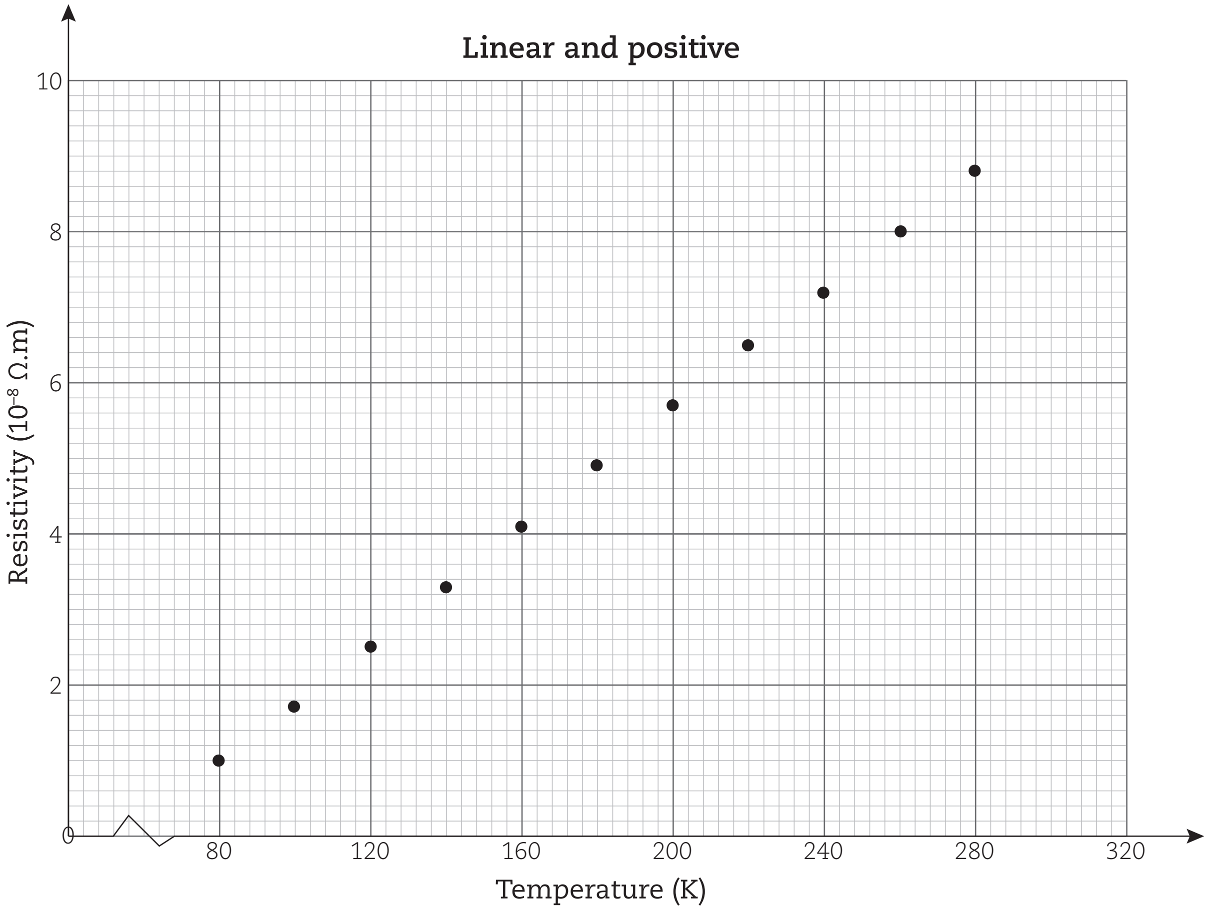
$$40{,}117 \leq x \leq 58{,}994$$

Assessment activity 4.3

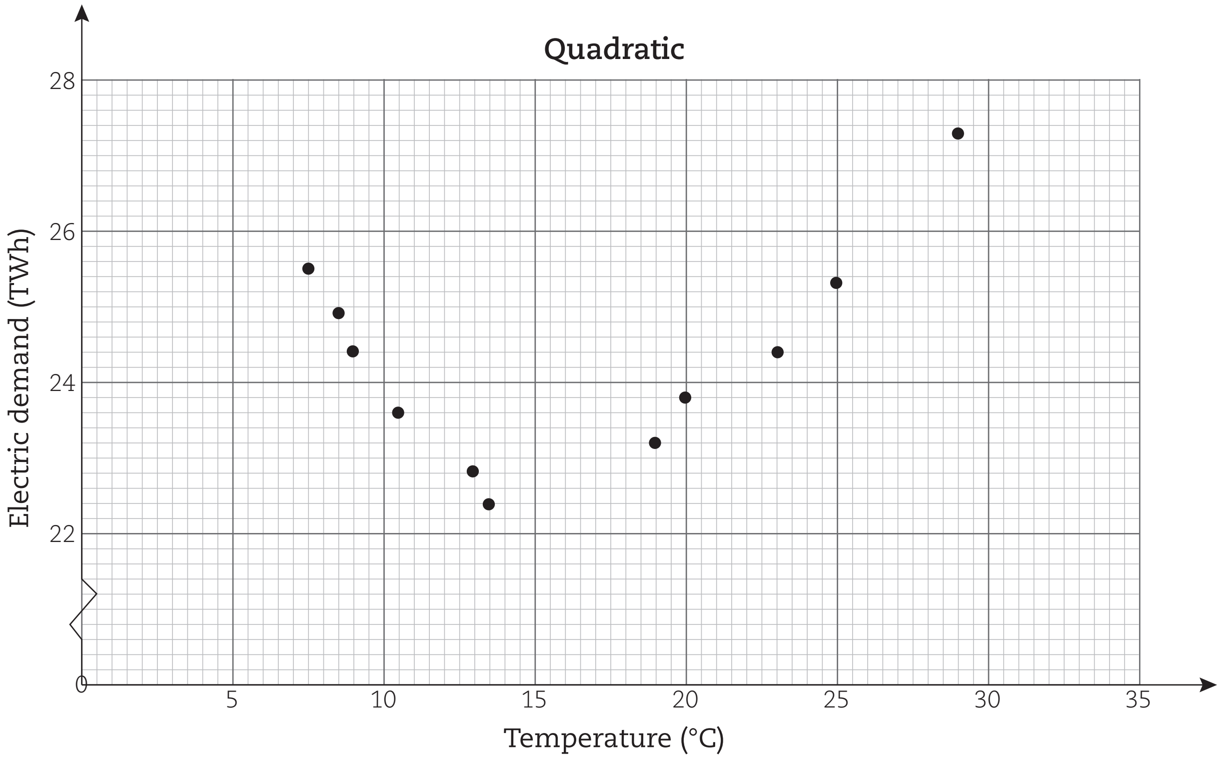
1.



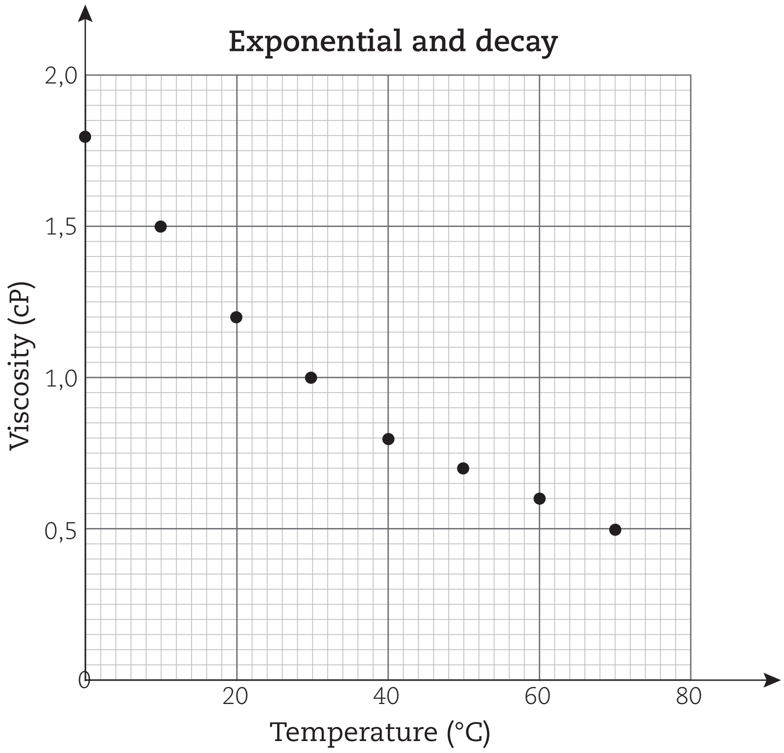
2.



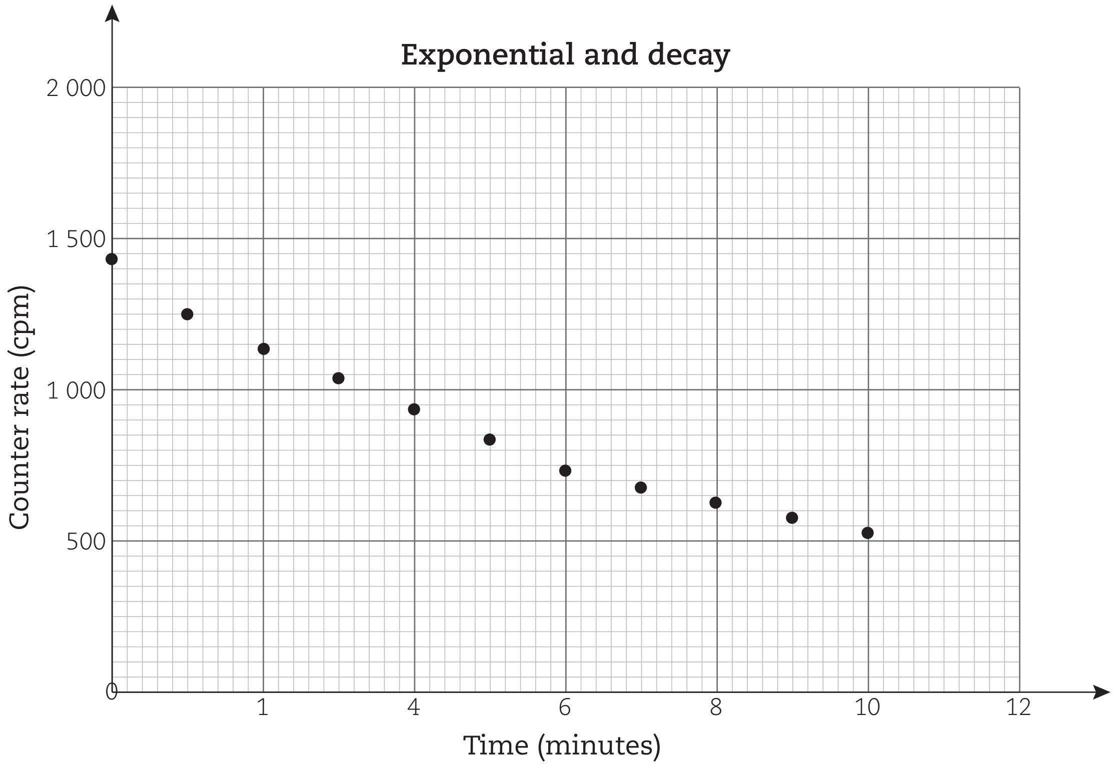
3.


Quadratic
Electric demand (TWh)
28
26
24
22
0
5
10
15
20
25
30
35
Temperature (°C)

4.

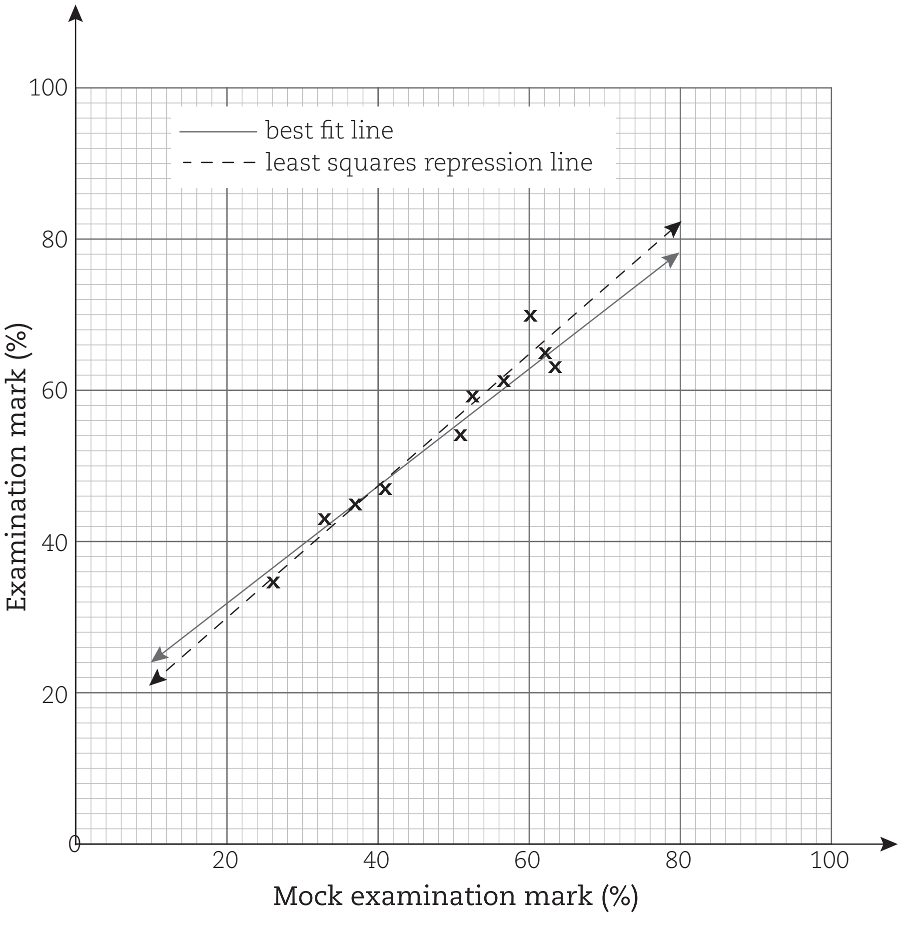

Exponential and decay
Viscosity (cP)
2,0
1,5
1,0
0,5
0
20
40
60
80
Temperature (°C)

5.



Assessment activity 4.4

1. a)



b) The line of best fit has a positive strong association.

c)

x	x^2	y	xy
28	784	34	952
33	1089	43	1419
37	1369	45	1665
41	1681	47	1927
51	2601	54	2754
53	2809	59	3127
57	3249	62	3534
60	3600	70	4200
62	3844	65	4030
64	4096	63	4032
$\sum x = 486$	$\sum x^2 = 25\ 122$	$\sum y = 542$	$\sum xy = 27\ 640$

$\overline{x} = \frac{\sum x}{n} = \frac{486}{10} = 48{,}600$ and $\overline{y} = \frac{\sum y}{n} = \frac{542}{10} = 54{,}200$

Regression coefficient,

$$b = \frac{n\sum xy - \sum x \sum y}{n\sum x^2 - (\sum x)^2}$$

$$= \frac{10(27\ 640) - (486)(542)}{10(25\ 122) - (486)^2}$$

$b = 0{,}865$

Regression coefficient,

$a = \overline{y} - b\overline{x}$

$= (54{,}200) - (0{,}865)(48{,}600)$

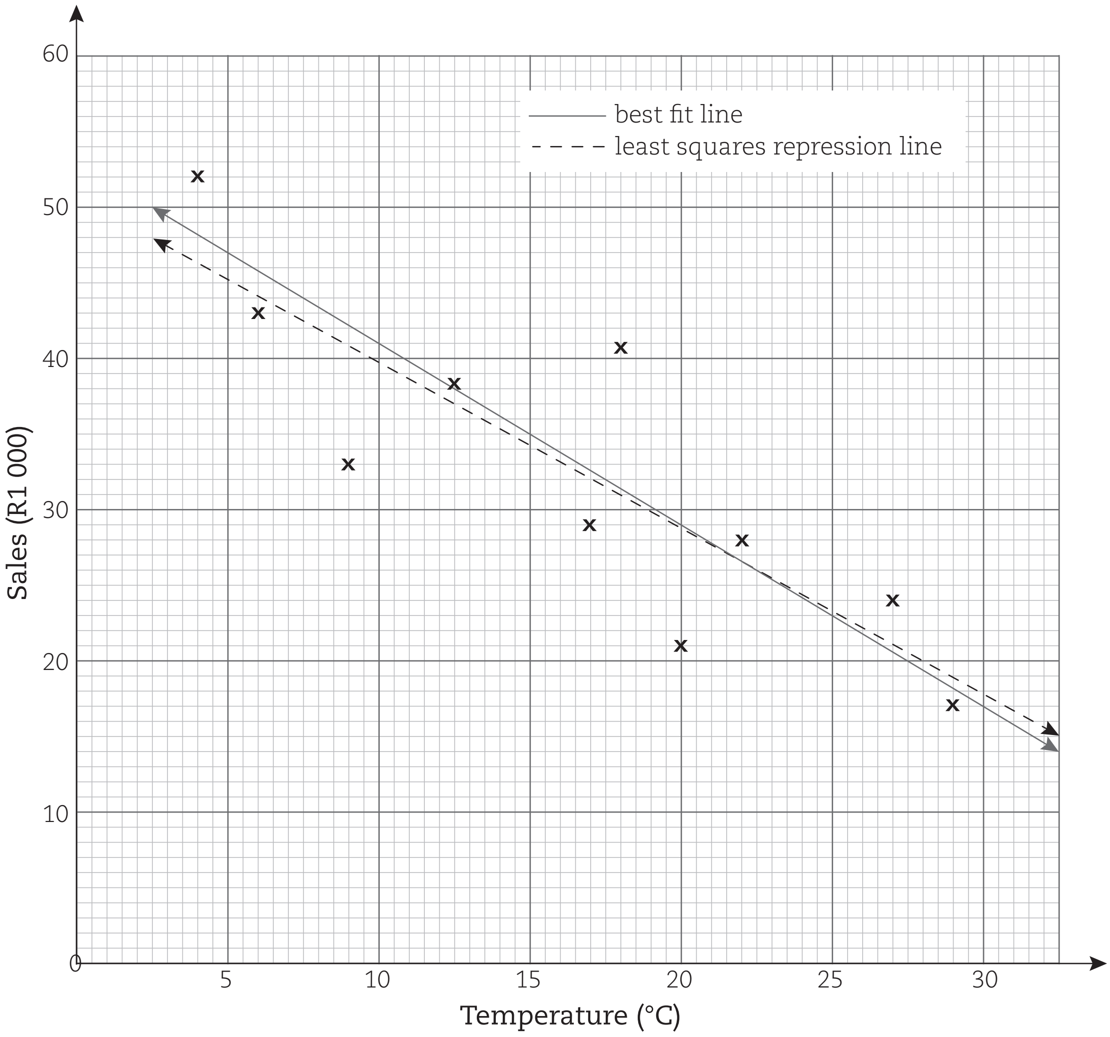
$a = 12{,}186$

Least squares regression line,

$12{,}186 = y - 0{,}865x$

$\therefore\ y = 0{,}865x + 12{,}186$

2. a)



b) The line of best fit has a negative weak association.

c)

x	x^2	y	xy
4	16	52	208
6	36	43	258
9	81	33	297
13	169	38	494
17	289	29	493
18	324	41	738
20	400	21	420
22	484	28	616
27	729	24	648
29	841	17	493
$\sum x = 165$	$\sum x^2 = 3\,369$	$\sum y = 326$	$\sum xy = 4\,665$

$\bar{x} = \frac{\sum x}{n} = \frac{165}{10} = 16{,}500$ and $\bar{y} = \frac{\sum y}{n} = \frac{326}{10} = 32{,}600$

Regression coefficient,

$$b = \frac{n\sum xy - \sum x \sum y}{n\sum x^2 - (\sum x)^2}$$

$$= \frac{10(4\,665) - (165)(326)}{10(3\,369) - (165)^2}$$

$b = -1{,}104$

Regression coefficient,

$a = \overline{y} - b\overline{x}$

$= (32{,}600) - (-1{,}104)(16{,}500)$

$a = 50{,}823$

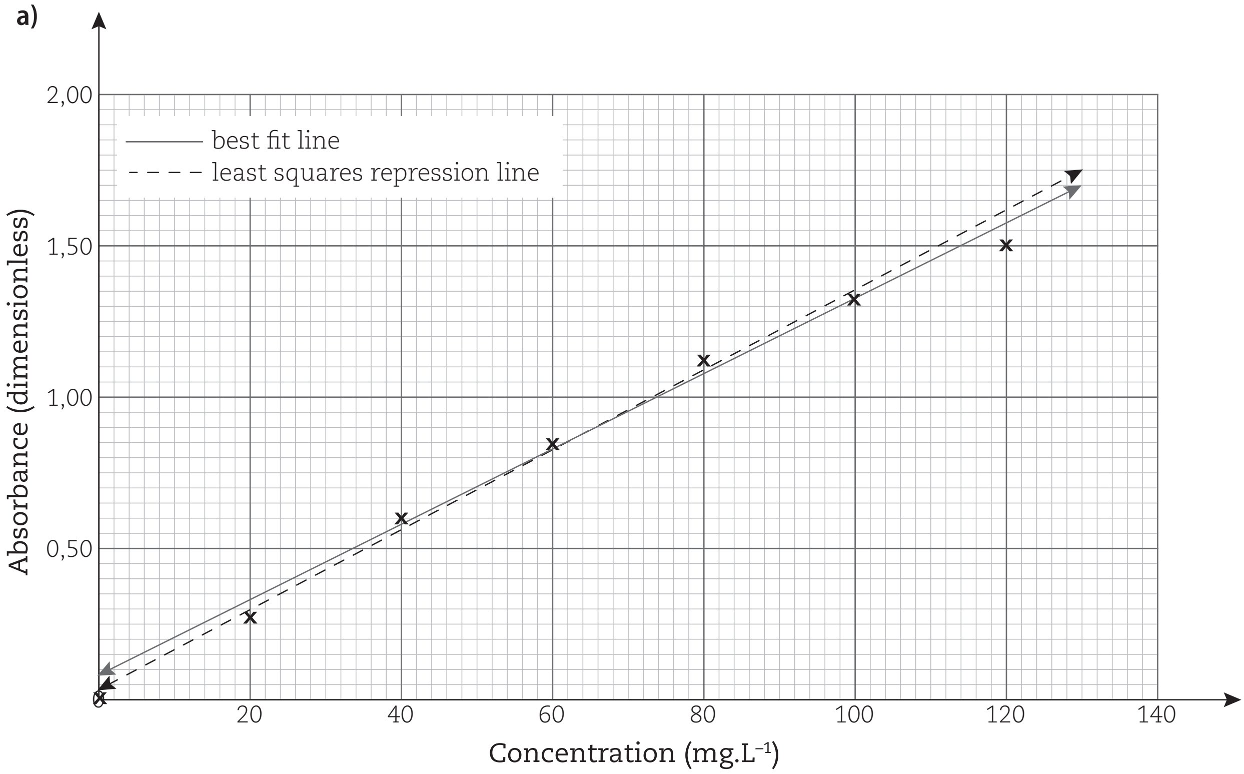
Least squares regression line,

$50{,}823 = y - (-1{,}104)x$

$50{,}823 = y + 1{,}104x$

$\therefore\ y = -1{,}104x + 50{,}823$

3. **a)**



b) The line of best fit has a positive strong association.

c)

x	x^2	y	xy
0	0	0,00	0,00
20	400	0,28	5,60
40	1600	0,60	24,00
60	3600	0,84	50,40
80	6400	1,13	90,40
100	10000	1,33	133,00
120	14400	1,51	181,20
$\sum x = 420$	$\sum x^2 = 36\ 400$	$\sum y = 5{,}69$	$\sum xy = 484{,}60$

$$\overline{x} = \frac{\sum x}{n} = \frac{420}{7} = 60 \text{ and } \overline{y} = \frac{\sum y}{n} = \frac{5{,}69}{7} = 0{,}813$$

Regression coefficient,

$$b = \frac{n\sum xy - \sum x \sum y}{n\sum x^2 - (\sum x)^2}$$

$$= \frac{7(484{,}60) - (420)(5{,}69)}{7(36\ 400) - (420)^2}$$

$b = 0{,}013$

Regression coefficient,

$a = \overline{y} - b\overline{x}$

$= (0{,}813) - (0{,}013)(60)$

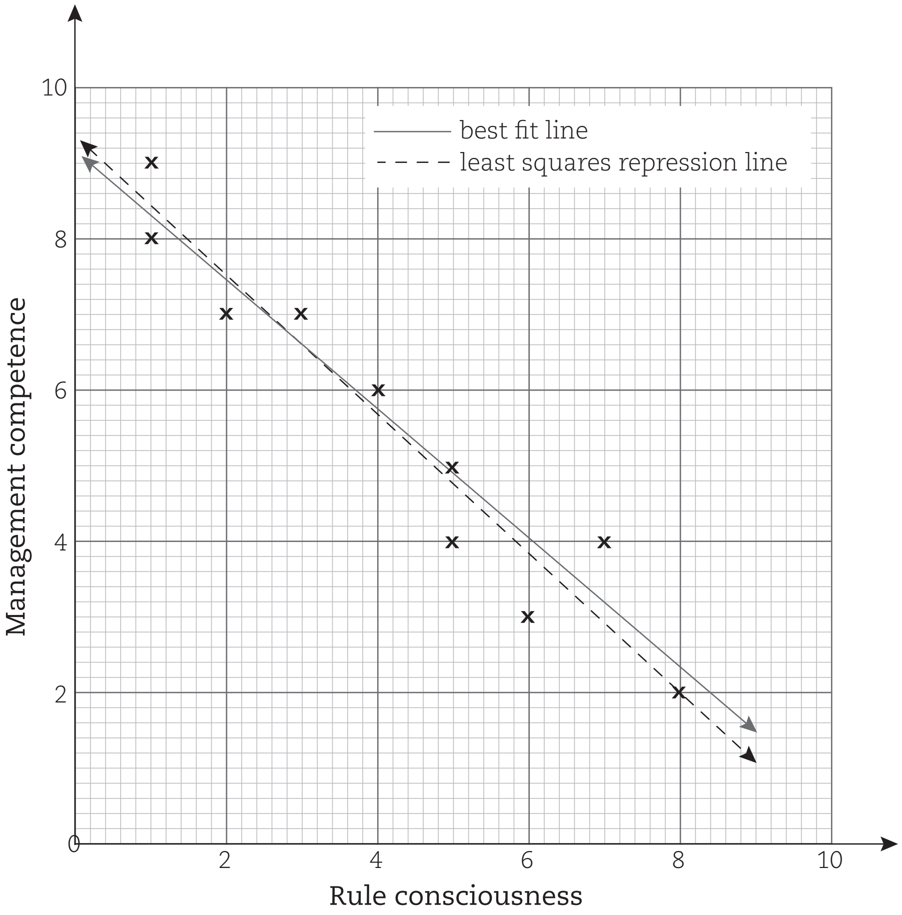
$a = 0{,}046$

Least squares regression line,

$0{,}046 = y - 0{,}013x$

$\therefore\ y = 0{,}013x + 0{,}046$

4. **a)**



b) The line of best fit has a negative strong association.

c)

x	x^2	y	xy
1	1	8	8
1	1	9	9
2	4	7	14
3	9	7	21
4	16	6	24
5	25	5	25
5	25	4	20
6	36	3	18
7	49	4	28
8	64	2	16
$\sum x = 42$	$\sum x^2 = 230$	$\sum y = 55$	$\sum xy = 183$

$\overline{x} = \frac{\sum x}{n} = \frac{42}{10} = 4{,}200$ and $\overline{y} = \frac{\sum y}{n} = \frac{55}{10} = 5{,}500$

Regression coefficient,

$$b = \frac{n\sum xy - \sum x \sum y}{n\sum x^2 - (\sum x)^2}$$

$$= \frac{10(183) - (42)(55)}{10(230) - (42)^2}$$

$$b = -0{,}896$$

Regression coefficient,

$$a = \overline{y} - b\overline{x}$$

$$= (5{,}500) - (-0{,}896)(4{,}200)$$

$$a = 9{,}261$$

Least squares regression line,

$$9{,}261 = y - (-0{,}896)x$$

$$9{,}261 = y + 0{,}896x$$

$$\therefore\ y = -0{,}896x + 9{,}261$$

Assessment activity 4.5

1. a) The sample space, S, is the set of all possible outcomes of an experiment.

b) (i) S = {H; T}

(ii) S = {1; 2; 3; 4; 5; 6}

(iii) S =

A	2	3	4	5	6	7	8	9	10	J	Q	K
♦;	♦;	♦;	♦;	♦;	♦;	♦;	♦;	♦;	♦;	♦;	♦;	♦
A	2	3	4	5	6	7	8	9	10	J	Q	K
♠;	♠;	♠;	♠;	♠;	♠;	♠;	♠;	♠;	♠;	♠;	♠;	♠
A	2	3	4	5	6	7	8	9	10	J	Q	K
♣;	♣;	♣;	♣;	♣;	♣;	♣;	♣;	♣;	♣;	♣;	♣;	♣
A	2	3	4	5	6	7	8	9	10	J	Q	K
♥;	♥;	♥;	♥;	♥;	♥;	♥;	♥;	♥;	♥;	♥;	♥;	♥

(iv) S = {HH; HT; TH; TT}

(v) S =

1; 1	1; 2	1; 3	1; 4	1; 5	1; 6
2; 1	2; 2	2; 3	2; 4	2; 5	2; 6
3; 1	3; 2	3; 3	3; 4	3; 5	3; 6
4; 1	4; 2	4; 3	4; 4	4; 5	4; 6
5; 1	5; 2	5; 3	5; 4	5; 5	5; 6
6; 1	6; 2	6; 3	6; 4	6; 5	6; 6

(vi) S = {H1; H2; H3; H4; H5; H6; T1; T2; T3; T4; T5; T6}

2. a) P(odd number) = $\frac{3}{6}$

P(odd number) = $\frac{1}{2}$

b) P(Two Heads) = P(Heads) × P(Heads)

$$= \frac{1}{2} \times \frac{1}{2}$$

P(Two Heads) = $\frac{1}{4}$

c) P(Jack or Black) = P(Jack) + P(Black) – P(Jack and Black)

$$= \frac{4}{52} + \frac{26}{52} - \frac{2}{52}$$

$$= \frac{28}{52}$$

P(Jack or Black) = $\frac{7}{13}$

d) P(Sum = 7) = $\frac{6}{36}$

P(Sum = 7) = $\frac{1}{6}$

e) P(5 or 2) = P(5) + P(2) – P(5 and 2)

$$= \frac{6}{36} + \frac{6}{36} - \frac{1}{36}$$

P(5 or 2) = $\frac{11}{36}$

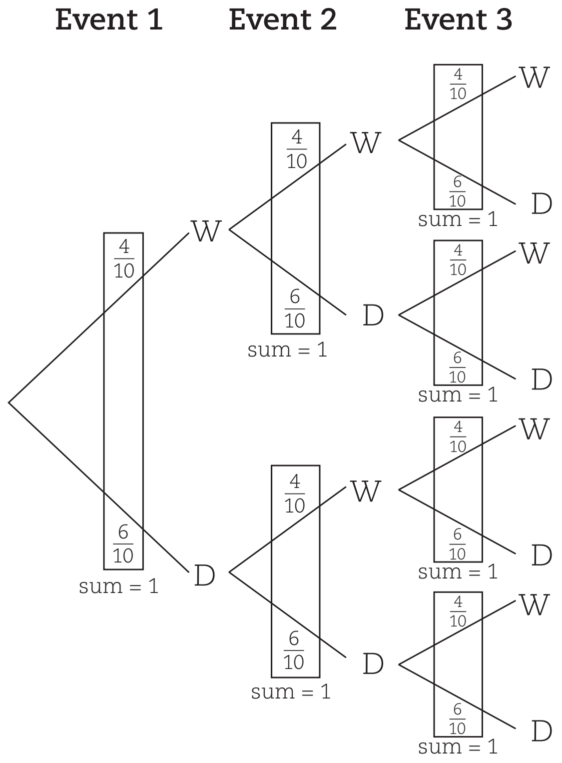
f) P(King of Diamonds twice) = P(King of Diamonds) × P(King of Diamonds)

$$= \frac{1}{52} \times \frac{1}{52}$$

P(King of Diamonds twice) = $\frac{1}{2\,704}$

3. (i) With replacement

a) & c)



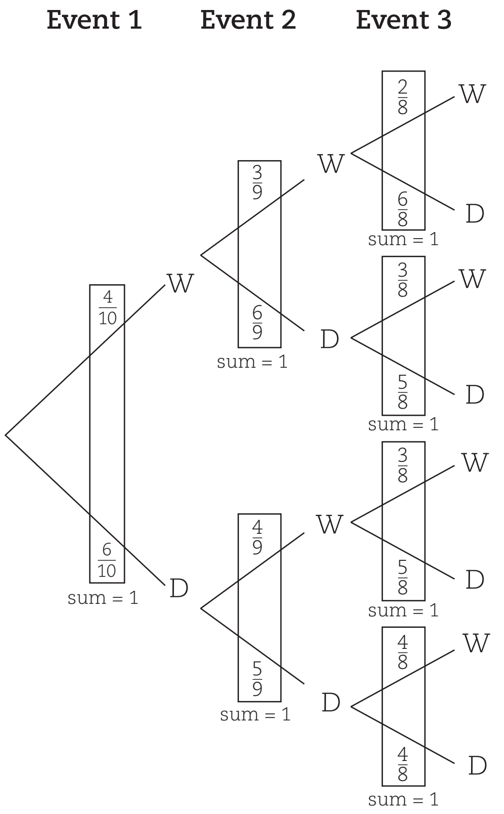
Outcome	Probability
WWW	$P(WWW) = \frac{4}{10} \times \frac{4}{10} \times \frac{4}{10} = \frac{64}{1\,000}$
WWD	$P(WWD) = \frac{4}{10} \times \frac{4}{10} \times \frac{6}{10} = \frac{96}{1\,000}$
WDW	$P(WDW) = \frac{4}{10} \times \frac{6}{10} \times \frac{4}{10} = \frac{96}{1\,000}$
WDD	$P(WDD) = \frac{4}{10} \times \frac{6}{10} \times \frac{6}{10} = \frac{144}{1\,000}$
DWW	$P(DWW) = \frac{6}{10} \times \frac{4}{10} \times \frac{4}{10} = \frac{96}{1\,000}$
DWD	$P(DWD) = \frac{6}{10} \times \frac{4}{10} \times \frac{6}{10} = \frac{144}{1\,000}$
DDW	$P(DDW) = \frac{6}{10} \times \frac{6}{10} \times \frac{4}{10} = \frac{144}{1\,000}$
DDD	$P(DDD) = \frac{6}{10} \times \frac{6}{10} \times \frac{6}{10} = \frac{216}{1\,000}$

P(S) = 1

b) S = {WWW; WWD; WDW; WDD; DWW; DWD; DDW; DDD}

(ii) Without replacement

a) & c)

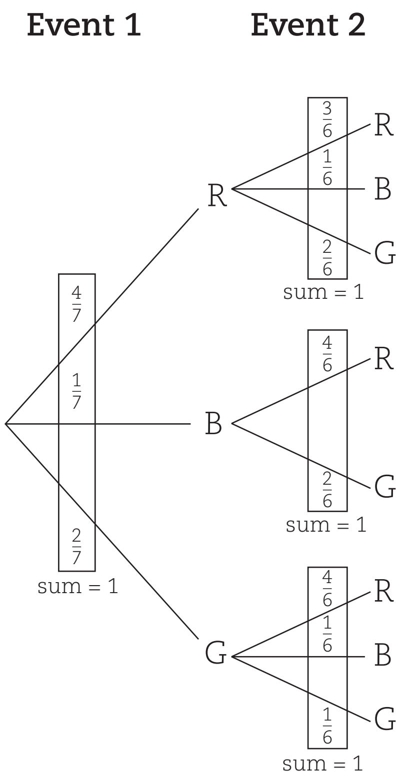


Outcome	Probability
WWW	$P(WWW) = \frac{4}{10} \times \frac{3}{9} \times \frac{2}{8} = \frac{24}{720}$
WWD	$P(WWD) = \frac{4}{10} \times \frac{3}{9} \times \frac{6}{8} = \frac{72}{720}$
WDW	$P(WDW) = \frac{4}{10} \times \frac{6}{9} \times \frac{3}{8} = \frac{72}{720}$
WDD	$P(WDD) = \frac{4}{10} \times \frac{6}{9} \times \frac{5}{8} = \frac{120}{720}$
DWW	$P(DWW) = \frac{6}{10} \times \frac{4}{9} \times \frac{3}{8} = \frac{72}{720}$
DWD	$P(DWD) = \frac{6}{10} \times \frac{4}{9} \times \frac{5}{8} = \frac{120}{720}$
DDW	$P(DDW) = \frac{6}{10} \times \frac{5}{9} \times \frac{4}{8} = \frac{120}{720}$
DDD	$P(DDD) = \frac{6}{10} \times \frac{5}{9} \times \frac{4}{8} = \frac{120}{720}$

P(S) = 1

b) S = {WWW; WWD; WDW; WDD; DWW; DWD; DDW; DDD}

4. a)



Outcome	Probability
RR	$P(RR) = \frac{4}{7} \times \frac{3}{6} = \frac{12}{42}$
RB	$P(RB) = \frac{4}{7} \times \frac{1}{6} = \frac{4}{42}$
RG	$P(RG) = \frac{4}{7} \times \frac{2}{6} = \frac{8}{42}$
BR	$P(BR) = \frac{1}{7} \times \frac{4}{6} = \frac{4}{42}$
BG	$P(BG) = \frac{1}{7} \times \frac{2}{6} = \frac{2}{42}$
GR	$P(GR) = \frac{2}{7} \times \frac{4}{6} = \frac{8}{42}$
GB	$P(GB) = \frac{2}{7} \times \frac{1}{6} = \frac{2}{42}$
GG	$P(GG) = \frac{2}{7} \times \frac{1}{6} = \frac{2}{42}$

P(S) = 1

b) S = {RR; RB; RG; BR; BG; GR; GB; GG}

c) **(i)** $P(RR) = \frac{12}{42}$

$P(BB) = 0$

$P(GG) = \frac{2}{42}$

(ii) $$\begin{aligned} \text{P(the same colour is drawn twice)} &= \text{P(RR or GG)} \\ &= P(RR) + P(GG) \\ &= \frac{12}{42} + \frac{2}{42} \end{aligned}$$

$$\text{P(the same colour is drawn twice)} = \frac{14}{42}$$

(iii) $$\begin{aligned} \text{P(at least one green ball is drawn)} &= \text{P(RG or BG or GR or GB or GG)} \\ &= P(RG) + P(BG) + P(GR) + P(GB) + P(GG) \\ &= \frac{8}{42} + \frac{2}{42} + \frac{8}{42} + \frac{2}{42} + \frac{2}{42} \end{aligned}$$

$$\text{P(at least one green ball is drawn)} = \frac{22}{42}$$

(iv) $$\begin{aligned} \text{P(at most one red ball is drawn)} &= \text{P(exactly one red ball or no red balls)} \\ &= \text{P(all outcomes except RR)} \\ &= 1 - P(RR) \\ &= 1 - \frac{12}{42} \end{aligned}$$

$$\text{P(at most one red ball is drawn)} = \frac{30}{42}$$

5. **a)**

Brand	**Usage**		**Total**
	Private	**Commercial**	
LG	**E = 80**	**F = 40**	120
Pioneer	90	20	**A = 110**
Samsung	110	**G = 60**	170
Total	**D = 280**	**C = 120**	**B = 400**

b)

Brand	**Usage**		**Total**
	Private	**Commercial**	
LG	0,200	0,100	0,300
Pioneer	0,225	0,050	0,275
Samsung	0,275	0,150	0,425
Total	**0,700**	**0,300**	**1,000**

c) $\text{P(private usage or Samsung)} = \frac{110}{400}$

6. **a)**

Drinks	**Gender**		**Total**
	Men	**Women**	
Coffee	45	35	80
Hot chocolate	40	60	100
Tea	20	50	70
Total	**105**	**145**	**250**

b) **(i)** P(hot chocolate drinker) = $\frac{100}{250}$

(ii) P(woman) = $\frac{145}{250}$

(iii) P(man or coffee drinker) = P(man) + P(coffee drinker) – P(man and coffee drinker)

$$= \frac{105}{250} + \frac{80}{250} - \frac{45}{250}$$

P(man or coffee drinker) = $\frac{140}{250}$

(iv) P(tea drinker and woman) = $\frac{50}{250}$

c)

Drinks	**Gender**		**Total**
	Men	**Women**	
Coffee	0,180	0,140	0,320
Hot chocolate	0,160	0,240	0,400
Tea	0,080	0,200	0,280
Total	**0,420**	**0,580**	**1,000**

Solutions for summative assessment: Chapter 4

Question 1

1.1

Ordered data
10
11
11
14
16
17
19
23
23
28
28
28
$x_{13} = \boxed{30} \longleftarrow Q_2 = x_{13} = 30$
30
33
33
34
35
36
39
40
42
47
48
48

$$P_{Q_2} = \frac{1}{2}(x + 1) = \frac{1}{2}(25 + 1) = 13$$

$$\bar{x} = \frac{\sum x_i}{n} = \frac{723}{25} = 28{,}920$$

1.1.1 Traditional method

x_i	$x_i - \overline{x}$	$(x_i - \overline{x})^2$
48	19,080	364,046
14	– 14,920	222,606
11	– 17,920	321,126
28	– 0,920	0,846
17	– 11,920	142,086
30	1,080	1,166
28	– 0,920	0,846
33	4,080	16,646
36	7,080	50,126
42	13,080	171,086
19	– 9,920	98,406
33	4,080	16,646
23	– 5,920	35,046
40	11,080	122,766
39	10,080	101,606
28	– 0,920	0,846
11	– 17,920	321,126
30	1,080	1,166
35	6,080	36,966
16	– 12,920	166,926
48	19,080	364,046
23	– 5,920	35,046
10	– 18,920	357,966
34	5,080	25,806
47	18,080	326,886
$\sum x_i = 723$	$\sum(x_i - \overline{x}) = 0{,}000$	$\sum(x_i - \overline{x})^2 = 3\ 301{,}840$

$$s^2 = \frac{\sum(x_i - \overline{x})^2}{n-1}$$

$$= \frac{3301{,}840}{25-1}$$

$$s^2 = 137{,}577$$

$$\therefore\ s = \sqrt{137{,}577}$$

$$s = 11{,}729$$

(8)

1.1.2 Shorthand method

x_i	x_i^2
48	2304
14	196
11	121
28	784
17	289
30	900
28	784
33	1089
36	1296
42	1764
19	361
33	1089
23	529
40	1600
39	1521
28	784
11	121
30	900
35	1225
16	256
48	2304
23	529
10	100
34	1156
47	2209
$\sum x_i = 723$	$\sum x_i^2 = 24\ 211$

$$s^2 = \frac{1}{n-1}[\sum x_i^2 - n\bar{x}^2]$$

$$= \frac{1}{25-1}[24211 - 25(28{,}920)^2]$$

$$s^2 = 137{,}577$$

$$\therefore s = \sqrt{137{,}577}$$

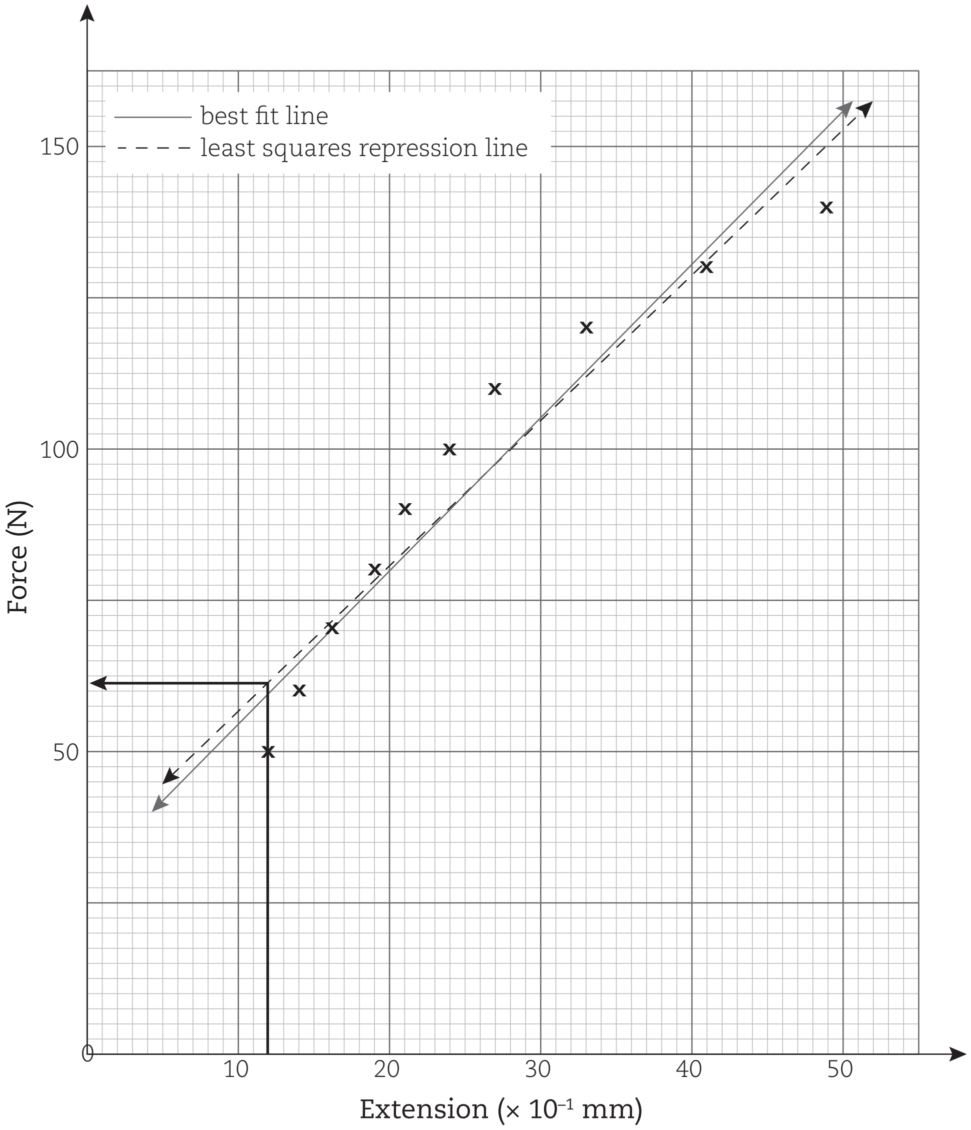
$$s = 11{,}729$$

(8)

1.2 Since $\bar{x} - Q_2 < 0$, therefore the data are skewed to the left, that is, negatively skewed. Since the data does not represent a normal distribution, therefore a 68% confidence interval cannot be computed. (4)

Question 2

2.1



(3)

2.2 The line of best fit has a positive strong association. (2)

2.3

x	x^2	y	xy
12	144	50	600
14	196	60	840
16	256	70	1120
19	361	80	1520
21	441	90	1890
24	576	100	2400
27	729	110	2970
33	1089	120	3960
41	1681	130	5330
49	2401	140	6860
$\sum x = 256$	$\sum x^2 = 7\ 874$	$\sum y = 950$	$\sum xy = 27\ 490$

$\bar{x} = \frac{\sum x}{n} = \frac{256}{10} = 25{,}600$ and $\bar{y} = \frac{\sum y}{n} = \frac{950}{10} = 95$

Regression coefficient,

$$b = \frac{n\sum xy - \sum x \sum y}{n\sum x^2 - (\sum x)^2}$$

$$= \frac{10(27\ 490) - (256)(950)}{10(7\ 874) - (256)^2}$$

$b = 2{,}401$

Regression coefficient,

$a = \overline{y} - b\overline{x}$

$= (95) - (2{,}401)(25{,}600)$

$a = 33{,}540$

Least squares regression line,

$33{,}540 = y - 2{,}401x$

$\therefore\ y = 2{,}401x + 33{,}540$ (7)

2.4 Least squares regression line,

$y = 2{,}401x + 33{,}540$

Force at 12 × 10^{-1} mm,

$y = 2{,}401x + 33{,}540$

$= 2{,}401(12) + 33{,}540$

$y = 62{,}349$

Therefore, force at 12 × 10^{-1} mm will be 62,349 N. (3)

Question 3

3.1.1 Sample space: all possible outcomes of a random experiment is called the sample space of the experiment. (1)

3.1.2 Event: an event is a subset of the sample space. (1)

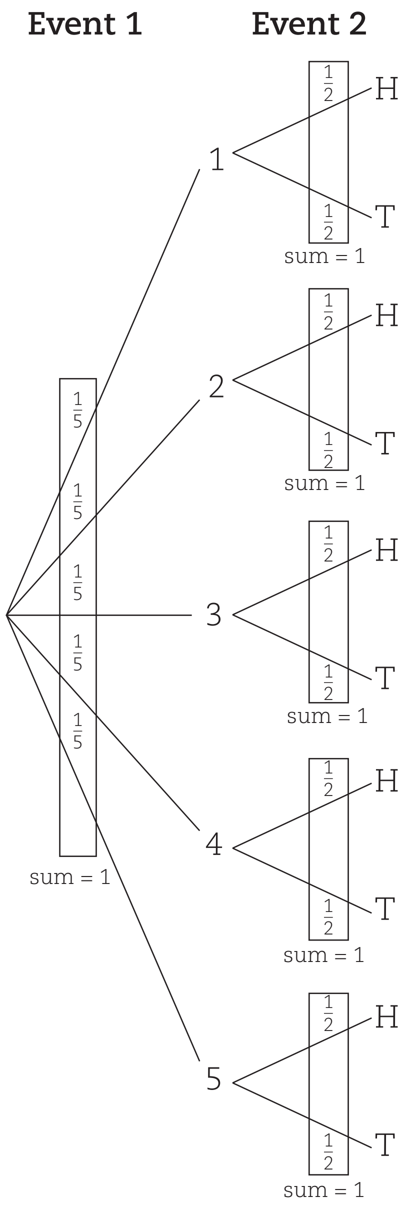
3.1.3 Complementary event: the complementary event are all those outcomes in the sample space that are not favourable. (1)

3.1.4 Mutual exclusive events: two events are mutually exclusive if the events cannot occur simultaneously. (1)

3.1.5 Mutual inclusive events: two events are mutually inclusive if the events occur simultaneously. (1)

3.2 The intersectional event is subtracted as to avoid double counting. (2)

3.3.1



Outcome	Probability
1H	$P(1H) = \frac{1}{5} \times \frac{1}{2} = \frac{1}{10}$
1T	$P(1T) = \frac{1}{5} \times \frac{1}{2} = \frac{1}{10}$
2H	$P(2H) = \frac{1}{5} \times \frac{1}{2} = \frac{1}{10}$
2T	$P(2T) = \frac{1}{5} \times \frac{1}{2} = \frac{1}{10}$
3H	$P(3H) = \frac{1}{5} \times \frac{1}{2} = \frac{1}{10}$
3T	$P(3T) = \frac{1}{5} \times \frac{1}{2} = \frac{1}{10}$
4H	$P(4H) = \frac{1}{5} \times \frac{1}{2} = \frac{1}{10}$
4T	$P(4T) = \frac{1}{5} \times \frac{1}{2} = \frac{1}{10}$
5H	$P(5H) = \frac{1}{5} \times \frac{1}{2} = \frac{1}{10}$
5T	$P(5T) = \frac{1}{5} \times \frac{1}{2} = \frac{1}{10}$

P(S) = 1 (5)

3.3.2 S = {1H ; 1T; 2H; 2T; 3H; 3T; 4H; 4T; 5H; 5T} (2)

3.3.3 **(i)** P(3 and tail) = P(3) × P(T)

$$= \frac{1}{5} \times \frac{1}{2}$$

P(3 and tail) = $\frac{1}{10}$ (1)

(ii) P(even number and head) = P(2H) + P(4H)

$$= \frac{1}{10} + \frac{1}{10}$$

P(even number and head) = $\frac{2}{10}$ (1)

(iii) P(5 or head) = P(5) + P(H) – P(5 and H)

$$= \frac{2}{10} + \frac{5}{10} - \frac{1}{10}$$

P(5 or head) = $\frac{6}{10}$ (1)

3.4.1

Sport code	Gender		Total
	Male	Female	
Diving	**A = 20**	25	45
Fencing	35	55	**C = 90**
Basketball	35	**B = 30**	65
Total	**90**	**110**	**200**

(3)

3.4.2 **(i)** P(female) = $\frac{110}{200}$ (1)

(ii) P(fencing) = $\frac{90}{200}$ (1)

(iii) P(male and diving) = $\frac{20}{200}$ (1)

(iv) P(female or basketball) = P(female) + P(basketball) – P(female or basketball)

$= \frac{110}{200} + \frac{65}{200} - \frac{30}{200}$

P(female or basketball) = $\frac{145}{200}$ (2)

Total [60]

Worked solutions • Chapter 5 Financial Mathematics

Assessment activity 5.1

1.
$$
\begin{aligned}
A &= P(1 + in)\\
\frac{A}{P} &= \frac{P(1 + in)}{P}\\
\frac{A}{P} &= 1 + in\\
\frac{A}{P} - 1 &= 1 + in - 1\\
\frac{A}{P} - 1 &= in\\
\frac{\frac{A}{P} - 1}{i} &= \frac{in}{i}\\
\frac{\frac{A}{P} - 1}{i} &= n\\
\therefore n &= \frac{\frac{A}{P} - 1}{i}\\
&= \frac{\frac{19\,500}{11\,500} - 1}{\frac{16,5}{100}}\\
n &= 4,216 \text{ years}
\end{aligned}
$$

2. Convert the interest rate of $9\frac{7}{12}$% per annum to $\frac{9\frac{7}{12}\%}{4} = \frac{115}{48}$% per quarter. Convert 3,75 years to quarters, that is, $(3,75 \times 4) = 15$ quarters.

$$
\begin{aligned}
A &= P(1 + i)^n\\
\frac{A}{(1 + i)^n} &= \frac{P(1 + i)^n}{(1 + i)^n}\\
\frac{A}{(1 + i)^n} &= P\\
\therefore P &= \frac{A}{(1 + i)^n}\\
&= \frac{37\,230}{(1 + \frac{115}{4\,800})^{15}}\\
P &= \text{R}26\ 101,10
\end{aligned}
$$

3. Convert 5 years and 2 months to years, that is, $\left(5 + \frac{2}{12}\right) = 5,167$ years.

$$
\begin{aligned}
A &= P(1 + in)\\
\frac{A}{P} &= \frac{P(1 + in)}{P}\\
\frac{A}{P} &= 1 + in\\
\frac{A}{P} - 1 &= 1 + in - 1\\
\frac{A}{P} - 1 &= in\\
\frac{\frac{A}{P} - 1}{n} &= \frac{in}{n}\\
\frac{\frac{A}{P} - 1}{n} &= i\\
\therefore i &= \frac{\frac{A}{P} - 1}{n}\\
&= \frac{\frac{32\,615}{15\,726} - 1}{5,167}\\
&= 0,208\\
i &= 20,786\% \text{ per annum}
\end{aligned}
$$

4. Convert the interest rate of 17,5% per annum to $\frac{17{,}5\%}{2}$ = 8,75% per semi-annual. Convert 7 years and 11 months to months, that is, (7 × 12 + 11) = 95 months. Then, convert 95 months to semi-annuals, that is $\left(\frac{95}{6}\right)$ = 15,833 semi-annuals.

$$\begin{aligned} A &= P(1 + i)^n \\ &= 47\,352\left(1 + \frac{8{,}75}{100}\right)^{15{,}833} \\ A &= R178\,704{,}31 \end{aligned}$$

5.

$$\begin{aligned} A &= P(1 + i)^n \\ \frac{A}{P} &= \frac{P(1 + i)^n}{P} \\ \frac{A}{P} &= (1 + i)^n \\ \sqrt[n]{\frac{A}{P}} &= \sqrt[n]{(1 + i)^n} \\ \sqrt[n]{\frac{A}{P}} &= 1 + i \\ \sqrt[n]{\frac{A}{P}} - 1 &= 1 + i - 1 \\ \sqrt[n]{\frac{A}{P}} - 1 &= i \\ \therefore\ i &= \sqrt[n]{\frac{A}{P}} - 1 \\ &= \sqrt[5]{\frac{800}{575}} - 1 \\ &= 0{,}068 \\ i &= 6{,}828\% \text{ per annum} \end{aligned}$$

6.

$$\begin{aligned} A &= P(1 + i)^n \\ &= 17\left(1 + \frac{6{,}5}{100}\right)^4 \\ A &= R21{,}87 \end{aligned}$$

7.

$$\begin{aligned} A &= P(1 + i)^n \\ \frac{A}{(1 + i)^n} &= \frac{P(1 + i)^n}{(1 + i)^n} \\ \frac{A}{(1 + i)^n} &= P \\ \therefore\ P &= \frac{A}{(1 + i)^n} \\ &= \frac{400}{(1 + \frac{10{,}5}{100})^{3{,}5}} \\ P &= R282{,}03 \end{aligned}$$

8. **a)** Principal loan amount = cash price – deposit

= cash price – 20% of cash price

$= 12\,500 - \frac{20}{100} \times 12\,500$

Principal loan amount = R10 000

b) Convert 48 months to years, that is, $\left(\frac{48}{12}\right)$ = 4 years.

$$\begin{aligned} A &= P(1 + in) \\ &= 10\,000\left(1 + \frac{20}{100}.4\right) \\ A &= R18\,000 \end{aligned}$$

The accumulated loan amount is R18 000.

c) Monthly repayment = $\frac{\text{accumulated loan amount}}{\text{number of monthly repayments}}$ + insurance premium

$= \frac{18\ 000}{48} + 58$

Monthly repayment = R433

d) Total amount paid = (monthly repayment × number of payments) + deposit

$= (\text{R}433 \times 48) + \frac{20}{100} \times \text{R}12\ 500$

Total amount paid = R23 284

Assessment activity 5.2

1. Albert Einstein is 69 years old and thus qualifies for the primary and secondary rebates, R11 440 and R6 390 respectively, which amounts to R17 830.

He pays a monthly medical aid contribution of R1 350. Since Albert Einstein is over the age of 65, he does not qualify for the medical scheme contribution tax credit, but he can consider his medical scheme contribution as a deduction of (R1 350 × 12) = R16 200 per annum.

Tax return for Albert Einstein:

Step 1: Annual pension received = R280 000

Step 2: Annual medical scheme contribution = R16 200

Step 3: Taxable income = gross income – deductions
= R280 000 – R16 200
Taxable income = R263 800

Step 4: From the individual rate table, Albert Einstein's taxable income falls in the category R250 001–R346 000, with the rates of tax R51 300 + 30% of the amount above R250 000.

Tax payable = R51 300 + 30% of (R263 800 – R250 000)
= R51 300 + 30% of R13 800
$= \text{R}51\ 300 + \frac{30}{100} \times \text{R}13\ 800$
= R51 300 + R4 140
Tax payable = R55 440

Step 5: Since Albert Einstein is over the age of 65 years, he qualifies for both the primary and secondary rebates.

Tax rebate = R11 440 + R6 390
Tax rebate = R17 830

Step 6: Tax due = tax payable – tax rebate
= R55 440 – R17 830
Tax due = R37 610

The annual tax due by Albert Einstein is R37 610.

2. Since Marie Curie is below the age of 65 years, she only qualifies for the primary rebate, R11 440. She also qualifies for a medical scheme contribution tax credit.

Tax return for Marie Curie:

Step 1: Gross income = monthly salary × 12
= R20 000 × 12
Gross income = R240 000

Step 2: Annual UIF contribution = 1% of R240 000

$= \frac{1}{100} \times$ R240 000

Annual UIF contribution = R2 400

Annual pension fund contribution = R1 200 × 12
Annual pension fund contribution = R14 400

Step 3: Taxable income = gross income – deductions
= R240 000 – (R2 400 + R14 400)
Taxable income = R223 200

Step 4: From the individual rate table, her taxable income falls in the category R160 001–R250 000, with the rates of tax R28 800 + 25% of the amount above R160 000.

Tax payable = R28 800 + 25% of (R223 200 – R160 000)
= R28 800 + 25% of R63 200
$=$ R28 800 + $\frac{25}{100} \times$ R63 200
= R28 800 + R15 800
Tax payable = R44 600

Step 5: Since Marie Curie is below the age of 65 years, she qualifies for the primary rebate only. She also qualifies for the medical scheme contribution tax credit for four individuals: the taxpayer and three dependants.

Tax rebate = R11 440

Medical scheme contribution tax credit = (R230 + R230 + R154 + R154) × 12
Medical scheme contribution tax credit = R9 216.

∴ Total tax rebate = R11 440 + R9 216
Total tax rebate = R20 656

Step 6: Tax due = tax payable – total tax rebate
= R44 600 – R20 656
Tax due = R23 944

The annual tax due by Marie Curie is R23 944.

3. Isaac Newton is 76 years old thus qualifies for the primary, secondary and tertiary rebates, R11 400, R6 390 and R2 130 respectively, which amounts to R19 960.

He received an annual medical aid tax certificate to the value of R25 000. Since Isaac Newton is over the age of 65, he does not qualify for the medical scheme contribution tax credit, but he can consider his medical scheme contribution as a deduction of R25 000.

Tax return for Isaac Newton:

Step 1: Annual pension received = R30 000 × 12
Annual pension received = R360 000

Step 2: Annual medical scheme contribution = R25 000

Step 3: Taxable income = gross income – deductions
= R360 000 – R25 000
Taxable income = R335 000

Step 4: From the individual tax table, Isaac Newton's taxable income falls in the category R250 001–R346 000, with the rates of tax R51 300 + 30% of the amount above R250 000.

Tax payable = R51 300 + 30% of (R335 000 – R250 000)
= R51 300 + 30% of R85 000
= R51 300 + $\frac{30}{100}$ × R85 000
= R51 300 + R25 500
Tax payable = R76 800

Step 5: Since Isaac Newton is over the age of 75 years, he qualifies for the primary, secondary and tertiary rebates.

Tax rebate = R11 440 + R6 390 + R2 130
Tax Rebate = R19 960

Step 6: Tax due = tax payable – tax rebate
= R76 800 – R19 960
Tax due = R56 840

The annual tax due by Isaac Newton is R56 840.

4. Pierre de Fermat is 40 years old, thus only qualifies for the primary rebate. He also qualifies for a medical scheme contribution tax credit.

Tax return for Pierre de Fermat:

Step 1: Gross income = monthly salary × 12 + annual bonus
= R30 000 × 12 + R50 000
Gross income = R410 000

Step 2: Since he is below the age of 65 years, his medical aid contribution cannot be considered as a deduction. The only deduction will therefore be the pension fund contribution.

Annual pension fund contribution = R40 000

Step 3: Taxable income = gross income – deduction
= R410 000 – R40 000
Taxable income = R370 000

Step 4: From the individual rate table, his taxable income falls in the category R346 001–R484 000, with the rates of tax R80 100 + 35% of the amount above R346 000.

Tax payable = R80 100 + 35% of (R370 000 – R346 000)
= R80 100 + 35% of R24 000
= R80 100 + $\frac{35}{100}$ × R24 000
= R80 100 + R8 400
Tax payable = R88 500

Step 5: Since the mathematician is below the age of 65 years, he qualifies for the primary rebate only. He also qualifies for the medical scheme contribution tax credit for the three individuals: the taxpayer and two dependants.

Tax rebate = R11 440

Medical scheme contribution tax credit = [R230 + R230 + R154] × 12
Medical scheme contribution tax credit = R7 368

∴ Total tax rebate = R11 440 + R7 368
Total tax rebate = R18 808

Step 6: Tax due = tax payable – total tax rebate

= R88 500 – R18 808

Tax due = R69 692

The annual tax due by the mathematician is R69 692.

Assessment activity 5.3

1.

$$A = P(1 - i)^n$$

$$\frac{A}{P} = \frac{P(1 - i)^n}{P}$$

$$\frac{A}{P} = (1 - i)^n$$

$$\sqrt[n]{\frac{A}{P}} = \sqrt[n]{(1 - i)^n}$$

$$\sqrt[n]{\frac{A}{P}} = 1 - i$$

$$\sqrt[n]{\frac{A}{P}} - 1 = 1 - i - 1$$

$$\sqrt[n]{\frac{A}{P}} - 1 = -i$$

$$1 - \sqrt[n]{\frac{A}{P}} = i$$

$$\therefore i = 1 - \sqrt[n]{\frac{A}{P}}$$

$$= 1 - \sqrt[19,25]{\frac{45\,750}{83\,000}}$$

$$= 0,030$$

$$i = 3,047\% \text{ per annum.}$$

2.

$$A = P(1 - in)$$

$$\frac{A}{P} = \frac{P(1 - in)}{P}$$

$$\frac{A}{P} = 1 - in$$

$$\frac{A}{P} - 1 = 1 - in - 1$$

$$\frac{A}{P} - 1 = -in$$

$$\frac{\frac{A}{P} - 1}{-i} = \frac{-in}{-i}$$

$$\frac{\frac{A}{P} - 1}{-i} = n$$

$$\therefore n = \frac{\frac{A}{P} - 1}{-i}$$

$$= \frac{\frac{11\,752}{33\,737} - 1}{-\frac{11}{75}}$$

$$n = 4,443 \text{ years}$$

3.

$$A = P(1 - i)^n$$

$$\frac{A}{(1 - i)^n} = \frac{P(1 - i)^n}{(1 - i)^n}$$

$$\frac{A}{(1 - i)^n} = P$$

$$\therefore P = \frac{A}{(1 - i)^n}$$

$$= \frac{43\,707}{(1 - \frac{67}{400})^{17}}$$

$$P = R986\,322,37$$

4.

$$A = P(1 - i.n)$$

$$= 65\,405\left(1 - \frac{13}{100}.7\right)$$

$$A = R5\,886,45$$

5. **a)**

$$A = P(1 - in)$$

$$= 40\,000\left(1 - \frac{9,5}{100}.6\right)$$

$$A = R17\,200$$

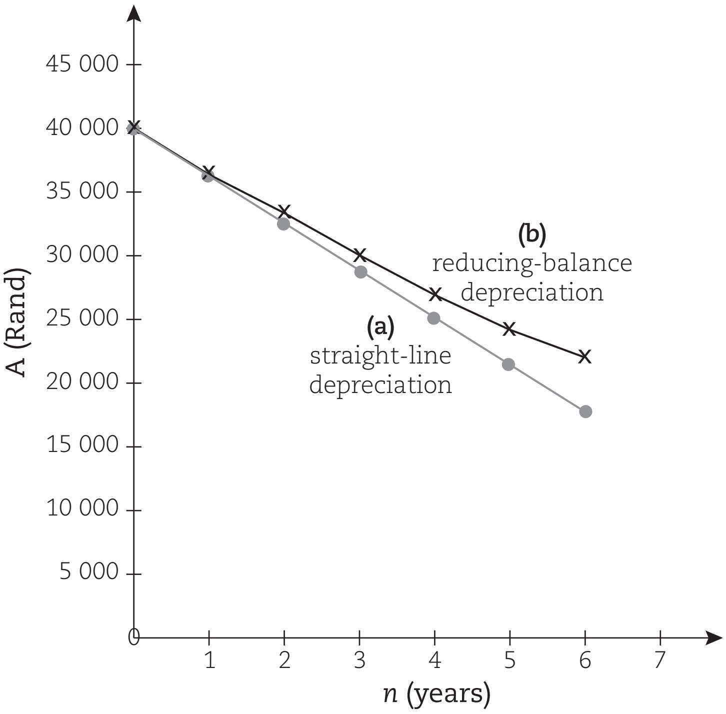
b)

$$A = P(1 - i)^n$$

$$= 40\,000\left(1 - \frac{9,5}{100}\right)^6$$

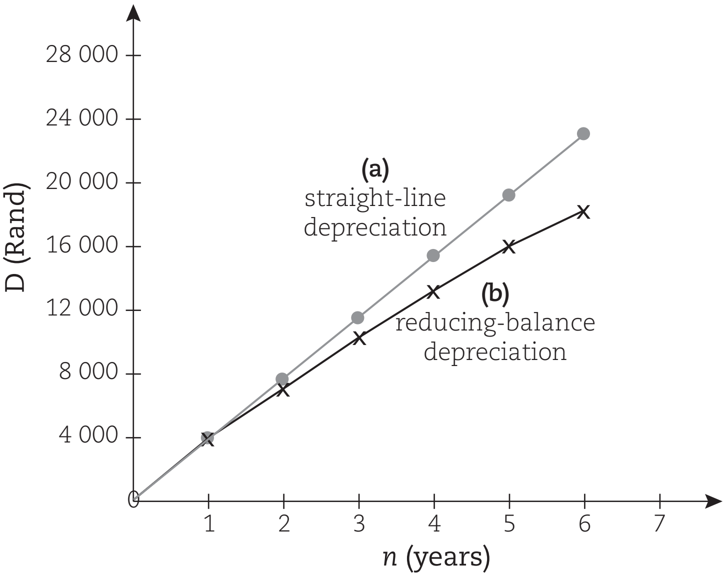
$$A = R21\,976,14$$

c)



n	(a)	(b)
0	40 000	40 000
1	36 200	36 200
2	32 400	32 761
3	28 600	29 648
4	24 800	26 832
5	21 000	24 283
6	17 200	21 976

d)



n	(a)	(b)
0	0	0
1	3 800	3 800
2	7 600	7 239
3	11 400	10 352
4	15 200	13 168
5	19 000	15 717
6	22 800	18 024

Solutions for summative assessment: Chapter 5

Question 1

1.1.1 Hire purchase: is a system of purchasing a product where the customer takes possession of the product on payment of a deposit (or no deposit in some instances) and completes the purchase by paying a series of regular instalments. (1)

1.1.2 Inflation: refers to an average percentage increase in the price of goods from year to year. (1)

1.1.3 Tax: a fee levied by government on a product, income or activity. (1)

1.1.4 Tax return: a declaration of personal income made annually to the tax authorities, and used as a basis for assessing an individual's liability for taxation. (1)

1.1.5 Taxable income: the difference between the gross income and deductions. (1)

1.1.6 Tax rate: a percentage of one's income that is payable in taxes. Tax rates vary according to income brackets. (1)

1.1.7 Tax rebate: a refund offered to taxpayers falling within a certain age category. (1)

1.1.8 Tax threshold: the level at which income is taxable. (1)

1.2.1 A tax deduction reduces the taxable income whereas the tax credit reduces the actual tax due. (2)

1.2.2 Straight-line depreciation represents a constant depreciation from year to year, whereas on a reducing-balance depreciation, the depreciation decreases from year to year. (2)

Question 2

a) Accumulated loan amount = (monthly repayment – insurance premium) × number of monthly repayments

= (R207,82 – R53) × 36

Accumulated loan amount = R5 573,52 (3)

b)

$$A = P(1 + in)$$

$$\frac{A}{(1 + in)} = \frac{P(1+in)}{(1+in)}$$

$$\frac{A}{(1 + in)} = P$$

$$\therefore P = \frac{A}{(1 + in)}$$

$$= \frac{5\,573{,}52}{(1 + \frac{20}{100}.3)}$$

$$P = R3\ 483{,}45$$

Principal loan amount = cash price – deposit

= cash price – 15% of cash price

$= \text{cash price} - \frac{15}{100} \times \text{cash price}$

$= \text{cash price}\left(1 - \frac{15}{100}\right)$

Principal loan amount = 0,85 × cash price

$\therefore \text{Cash price} = \frac{\text{Principal loan amount}}{0{,}85}$

$\text{Cash price} = \frac{3\,483{,}45}{0{,}85}$

Cash price = R4 098,18 (5)

Question 3

$$A = P(1 + i)^n$$

$$\frac{A}{P} = \frac{P(1 + i)^n}{P}$$

$$\frac{A}{P} = (1 + i)^n$$

$$\sqrt[n]{\frac{A}{P}} = \sqrt[n]{(1 + i)^n}$$

$$\sqrt[n]{\frac{A}{P}} = 1 + i$$

$$\sqrt[n]{\frac{A}{P}} - 1 = 1 + i - 1$$

$$\sqrt[n]{\frac{A}{P}} - 1 = i$$

$$\therefore i = \sqrt[n]{\frac{A}{P}} - 1$$

$$= \sqrt[6]{\frac{35\,000}{15\,000}} - 1$$

$$= 0{,}152$$

i = 15,167% per annum (3)

Question 4

The patent lawyer is 35 years old and thus qualifies for the primary rebate, R11 400. She also qualifies for a medical scheme contribution tax credit.

Tax return for the patent lawyer:

Step 1: Gross income = monthly gross income × 12
= R60 000 × 12
Gross income = R720 000

Step 2: Annual pension fund contribution = R5 000 × 12
Annual pension fund contribution = R60 000

Step 3: Taxable income = gross income – deductions
= R720 000 – R60 000
Taxable income = R660 000

Step 4: From the individual rate table, her taxable income falls in the category R617 001 and above, with the rates of tax R178 940 + 40% of the amount above R617 000.

Tax payable = R178 940 + 40% of (R660 000 – R617 000)

$$= \text{R}178\,940 + 40\% \text{ of R}43\,000$$

$$= \text{R}178\,940 + \frac{40}{100} \times \text{R}43\,000$$

$$= \text{R}178\,940 + \text{R}17\,200$$

Tax payable = R196 140

Step 5: Since the patent lawyer is below the age of 65 years, she qualifies for the primary rebate only. She also qualifies for the medical scheme contribution tax credit for five individuals: the taxpayer and four dependants.

Tax rebate = R11 440

Medical scheme contribution tax credit = [2 × R230 + 3 × R154) × 12
Medical scheme contribution tax credit = R11 064.

∴ Total tax rebate = R11 440 + R11 064
Total tax rebate = R22 504

Step 6: Tax due = tax payable – total tax rebate
= R196 140 – R22 504
Tax due = R173 636

The annual tax due by patent lawyer is R173 636. (7)

Question 5

a) $A = P(1 - in)$

$$= \text{R}60\,000\left(1 - \frac{9}{100}.10\right)$$

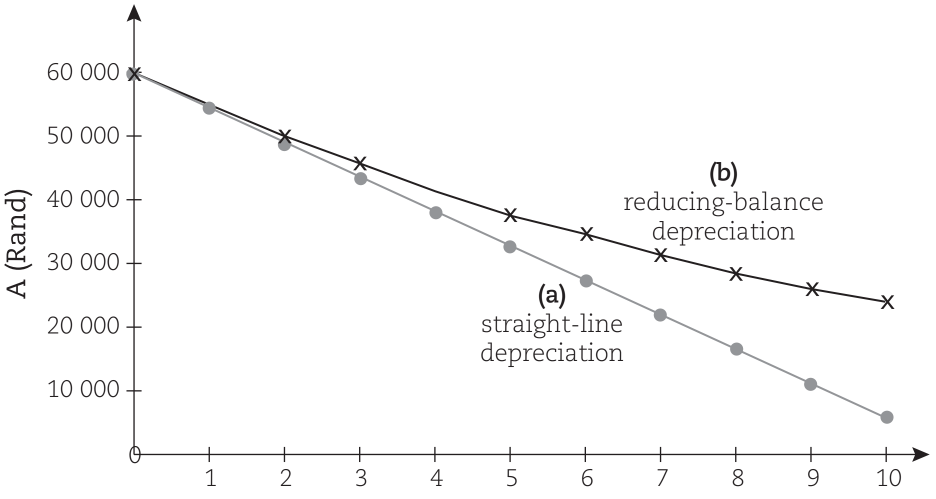
$A = \text{R}6\,000$ (2)

b) $A = P(1 - i)^n$

$$= \text{R}60\,000\left(1 - \frac{9}{100}\right)^{10}$$

$A = \text{R}23\,364{,}97$ (2)

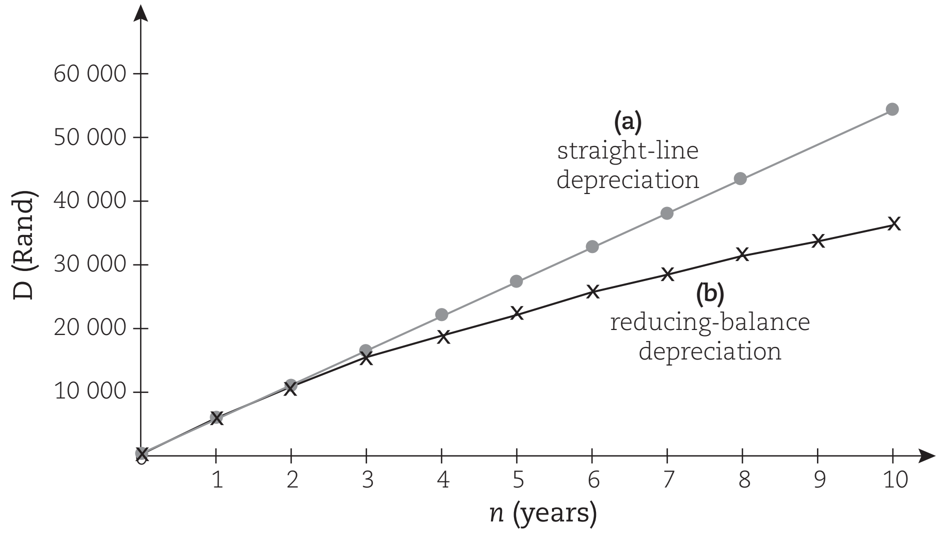
c)



n	(a)	(b)
0	60 000	60 000
1	54 600	54 600
2	49 200	49 686
3	43 800	45 214
4	38 400	41 144
5	33 000	37 441
6	27 600	34 072
7	22 200	31 005
8	16 800	28 215
9	11 400	25 675
10	6 000	23 364

(3)

d)



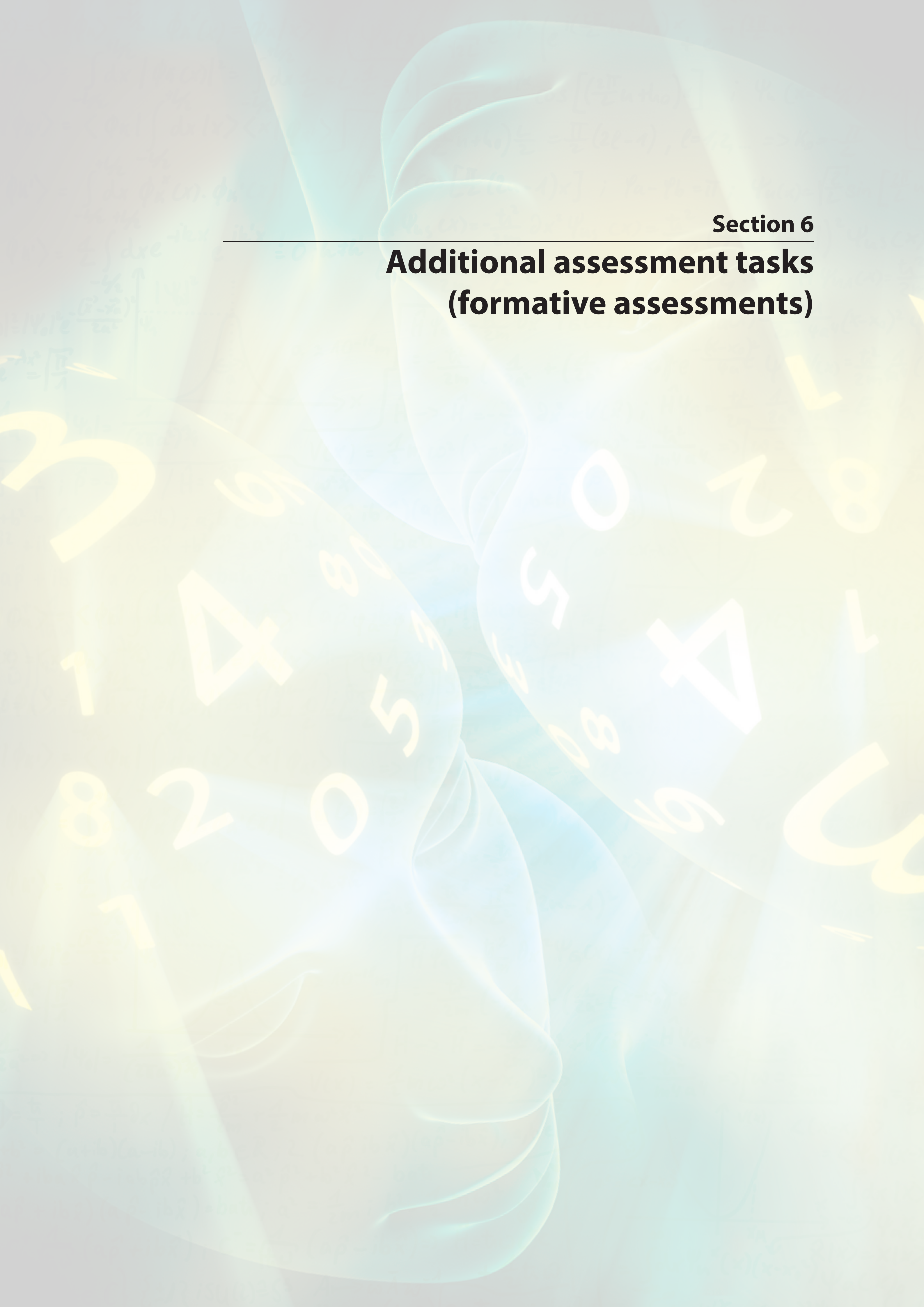
n	(a)	(b)
0	0	0
1	5 400	5 400
2	10 800	10 314
3	16 200	14 786
4	21 600	18 856
5	27 000	22 559
6	32 400	25 928
7	37 800	28 995
8	43 200	31 785
9	48 600	34 325
10	54 000	36 636

(3)

Total [40]

Section 6

Additional assessment tasks (formative assessments)



Topic 1

Formative assessment 1

Addition, subtraction, multiplication and division on complex numbers in standard form

Mark allocation: 15 **Time:** 30 minutes

1. Evaluate $(-i)^2 \times 6i^3$ where $i^2 = -1$ (3)

2. Simplify $\sqrt{-1}\,(\sqrt{-1} + \sqrt{25} - \sqrt{-49})$ without using a calculator. (3)

3. Simplify $(1 - 3i)(2 + i)^2$ to the form $a + bi$. (4)

4. Without using a calculator, write $\frac{2 + 4i}{3 - 5i}$ to the form $a + bi$. (5)

[15]

Formative assessment 2

Multiplication and division on complex numbers in polar form

Mark allocation: 10 **Time:** 20 minutes

Simplify each of the following expressions and give the answer in polar form. Do all the calculations in polar form.

1. $\frac{2-3i}{2 \text{ cis } -40^\circ}$ (5)

2. $(4+2i)(4 \text{ cis } 122^\circ)$ (5)

[10]

Formative assessment 3

De Moivre's theorem

Mark allocation: 10 **Time:** 20 minutes

1. Use De Moivre's theorem to evaluate the following. Leave the answer in polar form.

$\frac{4 \text{ cis } 45° \times (2 \text{ cis } 20°)^5}{3 \text{ cis } 80°}$ (2)

2. Use De Moivre's theorem to evaluate the following, leaving the answer in polar form.

$[2(\cos 30° + i \sin 30°)]^4$ (3)

3. Use De Moivre's theorem to evaluate the following. Leave the answer in polar form.

$\left(\frac{9 \text{ cis } 30°}{3 \text{ cis } 60°}\right)^4 \times \left(\frac{8 \text{ cis } 40°}{16 \text{ cis } 140°}\right)^{-2}$ (5)

[10]

Formative assessment 4

Identical complex numbers

Mark allocation: 15 **Time:** 30 minutes

Solve for x and y in the following complex identities.

1. $3x + 2yi - 2 = 4 - 5i$ (4)

2. $\frac{x - iy}{i^2} = (5 - 3i)^2$ (5)

3. $2x - yi = (i - 2) + \frac{3 + i}{3 - i}$ (6)

[15]

Formative assessment 5

Factorising and quadratic formula

Mark allocation: 8 **Time:** 15 minutes

Solve for *x* in the following equations.

1. $x^2 + 64 = 0$ (3)

2. $x^2 - 6x + 32 = 0$ (5)

[8]

Topic 2

Formative assessment 1

Remainder and factor theorem

Mark allocation: 10 **Time:** 10 minutes

1. Use the remainder theorem to determine the remainder of $\frac{x^3 - 8x^2 + x - 16}{3x - 9}$ (3)

2. Prove that $(x - 1)$ is a factor of $f(x) = 6x^3 - 11x^2 + x + 4$. (3)

3. Factorise $f(x) = x^3 - 7x^2 - 10x + 16$ completely by using the factor theorem. (4)

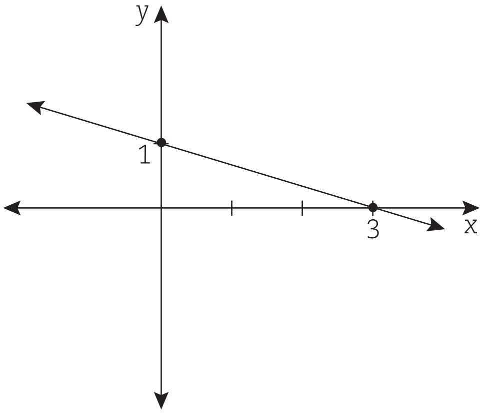
[10]

Formative assessment 2

Inverse graphs: Straight line and parabola

Mark allocation: 15 **Time:** 30 minutes

1. The diagram given below represents the graph $f(x) = -3x + 3$.



1.1 Determine the equation (in terms of y) of the inverse of the graph of $f(x)$ and express your answer as $f^{-1}(x) = \ldots$ (3)

1.2 Draw the graph of the inverse of $f(x)$. (3)

2. Given: $f(x) = 3x^2$

2.1 Determine the equation of the inverse of $f(x) = 3x^2$ algebraically. (2)

2.2 Sketch the graph of $f(x)$ and its inverse on the same set of axes. (5)

2.3 Write down the domain of $f(x)$. (1)

2.4 Write down the range of $f^{-1}(x)$. (1)

[15]

Formative assessment 3

Inverse graphs: Exponential graph

Mark allocation: 10 **Time:** 20 minutes

Given: $f(x) = 3x$

1. Write down the equation of the inverse of $f(x)$. (1)
2. Sketch the graph of $f(x) = 3x$ and the inverse on the same system of axes. (4)
3. Write down the equation of the horizontal asymptote of $f(x)$. (1)
4. Write down the equation of the vertical asymptote of $f^{-1}(x)$. (1)
5. Give the domain of $f(x)$. (1)
6. Give the domain of $f^{-1}(x)$. (2)

[10]

Formative assessment 4

Differentiation: First principles

Mark allocation: 15 **Time:** 30 minutes

Differentiate the following functions by use of first principles.

1. $f(x) = 3x + 7$ (3)

2. $y = -2x^2$ (4)

3. $f(x) = 5$ (3)

4. $f(x) = \frac{1}{x}$ (5)

[15]

Formative assessment 5

Differentiation rules

Mark allocation: 15 **Time:** 30 minutes

1 Differentiate with respect to x and **leave the answers with positive exponents and in surd form where applicable**.

$y = 4x^3 - \sqrt{3}\ \cos x + e^{-2x} - 4\ln x - x^{\frac{2}{3}} + 3$ (5)

2. Apply the product rule to differentiate the following in respect to x:
$f(x) = -4x^2(-6x + 3x^3)$ (3)

3. Given: $y = \frac{x^5}{e^{3x}}$
Differentiate by use of a quotient rule. (4)

4. Determine $f^{-1}(x)$ of $f(x) = (6x^2 - 3)^4$ (3)

[15]

Formative assessment 6

Differentiation: Rates of change

Mark allocation: 15 **Time:** 30 minutes

1. A body moves in a straight line from A to B where it comes to rest. After t seconds its distance from A, in metres, is given by:
$s = 102t + 14t^2 - t^3$.

Determine:

1.1 The velocity of the body after 3 seconds. (3)

1.2 The acceleration of the body after 3 seconds. (3)

2. Given: $f(x) = x^3 + x^2 - 6x$

2.1 Calculate, with the aid of differentiation, the coordinates of the maximum and the minimum turning points. (5)

2.2 Calculate the point of inflection (4)

[15]

Formative assessment 7

Integration

Mark allocation: 10 **Time:** 20 minutes

1. Integrate with respect to x:

$\int\left(\sin 2x + \sec^2 x - a + 6e^{-3x} + 3x^{\frac{1}{3}}\right) dx$ (6)

2. Determine:

$\int_0^1 (6x^2 - 3)\, dx$

(4)

[10]

Topic 3

Formative assessment 1

Use the Cartesian coordinate system to derive and apply the equation of a circle (any centre)

Mark allocation: 25 **Time:** 45 minutes

Show **all** the calculations and intermediary steps.
Simplify where possible.
All final answers must be approximated accurately to **three** decimal places.

1. Compute the equation of the circle, with the centre at the origin and:

a) passing through the point (7; –3) (2)

b) radius = 8 (2)

2. Compute the radius of the circle, with the centre at the origin:

a) $-x^2 - y^2 + 36 = 0$ (2)

b) $\frac{x^2}{3} + \frac{y^2}{3} = 27$ (2)

3. Compute the equation of the circle with:

a) centre (–2; 5) and radius = 7 (3)

b) centre (–1; –4) and passing through the point (3; 5) (3)

4. Compute the radius and centre of the circle:

a) $(x + 4)^2 + (y - 3)^2 = 144$ (3)

b) $x^2 - 6x + y^2 + 8y - 12 = 0$ (4)

5. Determine the numerical value(s) of n if $(-5; n)$ is a point on the circle $(x - 2)^2 + (y + 10)^2 = 490$. (4)

[25]

Formative assessment 2

Use the Cartesian coordinate system to derive and apply the equation of a tangent to a circle given a point on the circle

Mark allocation: 30 **Time:** 60 minutes

Show **all** the calculations and intermediary steps.
Simplify where possible.
All final answers must be approximated accurately to **three** decimal places.

1. Given, the circle $x^2 + y^2 - 17 = 2x - 8y$ and a point $(-2; -9)$ at the circumference of the circle.

- **a)** Determine the radius and the centre of the circle. (4)
- **b)** Calculate the gradient of the line between the centre of the circle and the point $(-2; -9)$. (2)
- **c)** Determine the equation of the tangent at the point $(-2; -9)$ to the circle. (3)

2.

- **a)** Describe in your own words the **condition of tangency**. (2)
- **b)** Illustrate by means of sketches:
  - **(i)** the tangents with the **same gradients** (m). (2)
  - **(ii)** the tangents with the **same y–intercepts** (c). (2)
- **c)** Determine the equation of the tangents:
  - **(i)** to the circle $x^2 + y^2 = 25$ and y–intercept, $c = \sqrt{125}$. (5)
  - **(ii)** to the circle $\frac{x^2}{7} + \frac{y^2}{7} = 7$ and has a gradient, $m = -2$. (5)
  - **(iii)** to the circle $x^2 + y^2 - 13 = 0$ and an angle of inclination, $\theta = 30°$. (5)

[30]

Formative assessment 3

Use geometry of straight lines and triangles to solve problems and justify relationships in geometric figures

Mark allocation: 25 **Time:** 45 minutes

1. Copy and complete the table below for different angles.

Type of angle	Description	Sketch
Acute angle		
Right angle		
Obtuse angle		
Straight angle		
Reflex angle		
Revolution		

(6)

2. Define and illustrate the following:

a) Complementary angles (2)

b) Supplementary angles (2)

3. Explain the meaning of **FUN** in geometry of lines. (3)

4. Copy and complete the table below for different triangles:

Type of triangle	Description	Sketch
Scalene triangle		
Isosceles triangle		
Equilateral triangle		
Acute-angled triangle		
Obtuse-angled triangle		
Right-angled triangle		

(6)

5. Fill in the missing words.

a) The sum of the … angles of a triangle is equal to …. (2)

b) The … angle of a triangle is equal to the sum of the two …. interior angles. (2)

6. Define the Theorem of Pythagoras. (2)

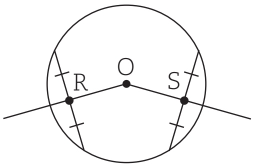
[25]

Formative assessment 4

State and apply the major theorems of circles

Divide the class into nine groups. Each group is assigned to an investigation. Prepare a poster and a 10 minute presentation for your class. Allow 5 minutes for questioning from your classmates. At the end of the presentations, the groups will vote for the best presentation.

Investigation 4.1 (Theorem 1)



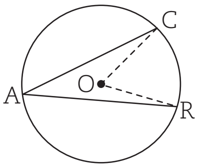
Step 1 Construct a large circle. Label the centre O.
Step 2 Construct two non-parallel chords that are not diameters.
Step 3 Bisect each of the chords.
Step 4 Draw two lines from the centre O, bisecting the two chords at R and S respectively.
Step 5 Measure the angles at R and S.

Compare your results with the results of others near you. State your observations as a conjecture.

Conclusion: If a line is drawn from the centre of a circle to the midpoint of a chord, that line is … to the chord.

Verify whether the converse of the above exists.

Investigation 4.2 (Theorem 2)



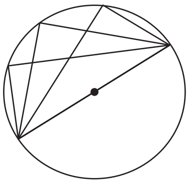
Step 1 Construct a large circle. Label the centre O.
Step 2 Construct an inscribed angle and its corresponding central angle.
Step 3 With the protractor, measure the central angle CÔR.
Step 4 Measure the inscribed angle CÂR. How does the measure of the inscribed angle CÂR compare with the central angle CÔR?

Compare your results with the results of others. State a conjecture.

Conclusion: The measure of a central angle in a circle is … the measure of the inscribed angle, subtended by the same arc.

Verify whether the converse of the above exists.

Investigation 4.3 (Theorem 3)



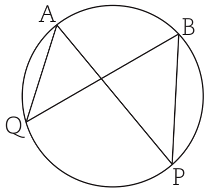
Step 1 Construct a large circle.
Step 2 Construct a diameter.
Step 3 Inscribe three angles in the same semicircle.
Step 4 Measure each angle with your protractor.

Compare your results with the results of others and make a conjecture.

Conclusion: Angles inscribed in a semicircle are ….

Verify whether the converse of the above exists.

Investigation 4.4 (Theorem 4)



Step 1 Construct a large circle.
Step 2 Select two points on the circle. Label them A and B.
Step 3 Select a point P on the major arc and construct inscribed $A\hat{P}B$.
Step 4 With the protractor, measure $A\hat{P}B$.
Step 5 Select another point Q on major arc APB and construct inscribed $A\hat{Q}B$.
Step 6 Measure. How does the measure of $A\hat{Q}B$ compare with the measure of $A\hat{P}B$?

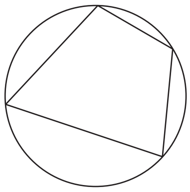
Repeat steps 1 to 6 with points P and Q selected on the minor arc AB. Compare the measure of $A\hat{Q}B$ with the measure of $A\hat{P}B$.

State the observations as a conjecture.

Conclusion: Inscribed angles that intercept the same arc are ….

Verify whether the converse of the above exists.

Investigation 4.5 (Theorem 5)



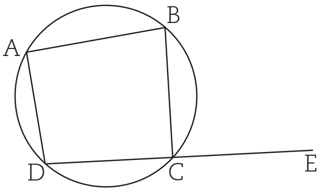
Step 1 Construct a large circle.
Step 2 Construct an inscribed quadrilateral.
Step 3 Measure each of the four inscribed angles. Write the measure in each angle.

There is a special relationship between some pairs of angles. Compare your observations with the observations of those near you. State your findings as your next conjecture.

Conclusion: The … angles of a quadrilateral inscribed in a circle are ….

Verify whether the converse of the above exists.

Investigation 4.6 (Theorem 6)

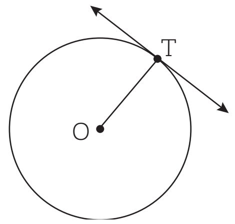


Step 1 Construct a large circle.
Step 2 Construct an inscribed quadrilateral and label it ABCD.
Step 3 Draw a line CE an extension of DC.
Step 4 With your protractor, measure $B\hat{A}D$.
Step 5 Measure $B\hat{C}E$. How does the measure of $B\hat{C}E$ compare with the measure of $B\hat{A}D$?

Compare your results with the results of others and make a conjecture.

Conclusion: An exterior angle of a cyclic quadrilateral is … to the interior opposite angle.

Verify whether the converse of the above exists.



Investigation 4.7 (Theorem 7)

Step 1 Construct a large circle. Label the centre O.

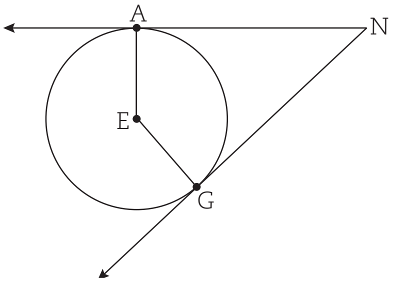
Step 2 Using your ruler, draw a line that appears to touch the circle at only one point. Label the point T. Construct the line OT.

Step 3 Use your protractor to measure the angles at T.

Compare your results with the results of others near you. State your observations as a conjecture.

Conclusion: A tangent to a circle is … to the radius drawn to the point of tangency.

Verify whether the converse of the above exists.



Investigation 4.8 (Theorem 8)

Step 1 Construct a circle. Label the centre E.

Step 2 Choose a point outside the circle and label it N.

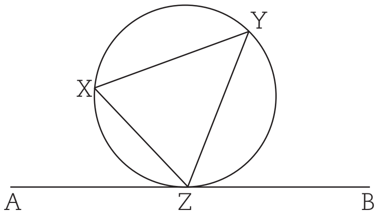
Step 3 Draw two lines through point N that appear to be tangent to the circle. Mark the points where these lines appear to touch the circle and label them A and G.

Step 4 Use your ruler to compare the lengths of segments NA and NG.

Compare your results with the results of others near you. State your observations as a conjecture.

Conclusion: Tangent segments to a circle from a point outside the circle are ….

Verify whether the converse of the above exists.



Investigation 4.9 (Theorem 9)

Step 1 Construct a large circle.

Step 2 Construct an inscribed triangle and label it XYZ.

Step 3 Draw a line AB through Z that appear to be tangent to the circle.

Step 4 With your protractor, measure $A\hat{Z}X$.

Step 5 Measure $X\hat{Y}Z$. How does the measure of $X\hat{Y}Z$ compare with measure of $A\hat{Z}X$?

Compare your results with the results of others and make a conjecture.

Conclusion: The angle between a tangent to a circle and a chord drawn from a point of contact, is … to an angle in the … segment.

Verify whether the converse of the above exists.

Formative assessment 5

Use the compound angle identities $\sin(\alpha \pm \beta) = \sin\alpha.\cos\beta \pm \sin\beta.\cos\alpha$ and $\cos(\alpha \pm \beta) = \cos\alpha.\cos\beta \mp \sin\alpha.\sin\beta$ to derive and apply the double angle identities, $\sin 2\alpha = 2\sin\alpha.\cos\alpha$, and $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$, $\cos 2\alpha = 2\cos^2\alpha - 1$ and $\cos 2\alpha = 1 - 2\sin^2\alpha$

Mark allocation: 25 **Time:** 45 minutes

Show **all** the calculations and intermediary steps.
Simplify where possible.

1. Expand using the appropriate compound angle identity for the following:

a) $\sin(\alpha - 20°)$ (1)
b) $\cos(63° + \beta)$ (1)
c) $\sin(\tau + 3\phi)$ (1)
d) $\cos(2\lambda - \omega)$ (1)

2. Use the appropriate compound angle identity to reduce the following:

a) $\sin\phi\cos 31° + \sin 31°.\cos\phi$ (1)
b) $\cos 47°.\cos\alpha - \sin 47°.\sin\alpha$ (1)
c) $\sin\mu.\cos 2\phi - \sin 2\phi.\cos\mu$ (1)
d) $\cos\frac{\rho}{2}.\cos\upsilon + \sin\frac{\rho}{2}.\sin\upsilon$ (1)

3. Evaluate without using a calculator: $\sin 105°$ (5)

4. Derive a formula for $\sin 2\alpha$ by using an appropriate compound angle identity. (2)

5.
a) Derive a formula for $\cos 2\alpha$ in terms of $\cos\alpha$ only. (3)
b) Hence, compute the value of $\cos 120°$. (3)
c) Derive a formula for $\cos(90° + \alpha)$ using a compound angle identity. (2)
d) Verify the solution of (b) by using the formula derived in (c). (2)

[25]

Formative assessment 6

Determine the specific solutions of trigonometric expressions using compound and double angle identities without a calculator

Mark allocation: 20 **Time:** 40 minutes

Show **all** the calculations and intermediary steps.
Simplify where possible.

1. Prove that: cos 53° – cos 7° = –sin 23° (5)

2. Simplify: $\frac{1 - \tan^2 x}{1 + \tan^2 x}$ (5)

3. Given, $\sin A = -\frac{8}{10}$ with $A \in [0°; 270°]$ and $\tan B = -\frac{5}{12}$ with $B \in [180°; 360°]$.

Compute, with the aid of sketches, the numerical value of the following (the magnitude of A and B may not be calculated):

a) tan (180° – A) (2)
b) sin (A + B) (2)
c) sin 2B (2)
d) cos (A – B) (2)
e) cos 2A (2)

[20]

Formative assessment 7

Determine the specific solutions of trigonometric equations by using knowledge of compound angles and identities

Mark allocation: 20 **Time:** 40 minutes

Show **all** the calculations and intermediary steps.
Simplify where possible.
All final answers must be approximated accurately to **three** decimal places.

1. Determine the value(s) of $x \in [0°; 90°]$ without using a calculator:

a) $\sin 3x = \cos x.\cos 30° - \sin x.\sin 30°$ (5)

b) $\cos 3x.\cos 15° + \sin 3x.\sin 15° = -\cos 60°$ (5)

2. Compute the value(s) of θ with the aid of a calculator:

a) $3 \sin \theta + 1 = 2 \cos 2\theta$ with $\theta \in [0°; 360°]$ (5)

b) $\frac{\cos 2\theta}{\cos^2 \theta} + 2 = 0$ with $\theta \in [0°; 360°]$ (5)

[20]

Formative assessment 8

Solve problems from a given diagram in two and three dimensions by applying the sine and cosine rule

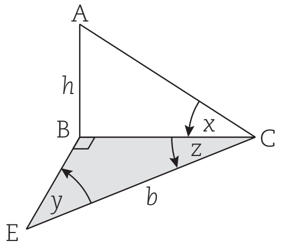
Mark allocation: 30 **Time:** 60 minutes

Show **all** the calculations and intermediary steps.
Simplify where possible.
All final answers must be approximated accurately to **three** decimal places.

1. Solve ΔABC in which BC = 15 cm, $\hat{B} = 40°$ and $\hat{C} = 103°$. (10)

2. Solve ΔDEF in which DE = 104 mm, EF = 140 mm and $\hat{E} = 20°$. (10)

3. In the diagram below, AB is a tower that stands on a horizontal plane BEC. The angle of elevation of A from C is x.



If it is given that $A\hat{B}C = 90°$, $B\hat{C}E = z$, $B\hat{E}C = y$, EC = b and AB = h:

a) Prove that $h = \frac{b \sin y . \tan x}{\sin(y + z)}$. (7)

b) Find the value of h correct to three decimal places if $b = 650$ m, $x = 14{,}9°$, $y = 41{,}8°$ and $z = 66{,}7°$. (3)

[30]

Topic 4

Formative assessment 1

Calculate variance and standard deviation manually for small sets of data only, and interpret the meaning of variance and standard deviation for small sets of data only

Mark allocation: 20 **Time:** 40 minutes

Show **all** the calculations and intermediary steps.
Simplify where possible.
All final answers must be approximated accurately to **three** decimal places.

1. Compute the sample mean, sample variance and standard deviation of scores in an Indoor Cricket League, using the traditional method.

13	8	7	6	16
9	15	6	14	20

Determine whether the data has a normal distribution and, if so, compute the confidence interval within which 95% of the scores would be expected to occur.

Note: Traditional method, $s^2 = \frac{\Sigma(x_i - \bar{x})^2}{n-1}$ and $s = \sqrt{\frac{\Sigma(x_i - \bar{x})^2}{n-1}}$ (10)

2. Find the sample mean, sample variance and standard deviation of the following test scores out of thirty using the shorthand method.

13	15	21	3	6
2	30	26	27	5

Determine whether the data has a normal distribution and, if so, compute the confidence interval within which 68% of the test scores would be expected to occur.

Note: Shorthand method, $s^2 = \frac{1}{n-1}[\Sigma x_i^2 - n\bar{x}^2]$ and $s = \sqrt{\frac{1}{n-1}[\Sigma x_i^2 - n\bar{x}^2]}$ (10)

[20]

Formative assessment 2

Represent bivariate numerical data as a scatter plot, and identify intuitively whether a linear, quadratic or exponential function would best fit the data

Mark allocation: 20 **Time:** 40 minutes

Draw a scatter plot for the data sets below and comment on the relationship between the variables.

1.

Time (s)	0	2	4	6	8	10
Velocity (m.s^{-1})	0	11	18	32	39	48

(4)

2.

Time (s)	0	10	20	30	40	50
Displacement (*m*)	0	1 500	2 000	1 500	0	–2 500

(4)

3.

Time (days)	0	20	40	60	80	100
Mass (%)	100	74	54	40	29	22

(4)

4.

Time (years)	0	1	2	3	4	5
Accrued (rands)	25 000	22 500	20 000	17 500	15 000	12 500

(4)

5.

Time (years)	0	2	4	6	8	10
Population	100	114	130	147	168	191

(4)

[20]

Formative assessment 3

Draw the intuitive line of best fit, and use least squares regression method to determine a function which best fits a given set of bivariate data, and use the regression line to predict the outcome of the given problem

Mark allocation: 20 **Time:** 40 minutes

Show **all** the calculations and intermediary steps.
Simplify where possible.
All final answers must be approximated accurately to **three** decimal places.

The following data was recorded during a tensile test on a mild steel test piece of circular cross-section.

Extension (× 10^{-2} mm)	6	11	17	22	27	32	37	42
Load (kN)	20	40	60	80	100	120	140	160

1. Draw a scatter diagram to illustrate the data. (2)
2. Draw a line of best fit on the scatter diagram and describe the association between extension and load. (3)
3. Compute the equation of the least square regression line and sketch the line on the scatter diagram. (11)
4. Verify from the graph whether the least square regression line cuts the means of the extension and load. (2)
5. With the aid of the equation of the regression line, compute the load at 24 × 10^{-2} mm. (2)

[20]

Formative assessment 4

Explain and distinguish between the probability terminology, and make predictions based on validated experimental or theoretical probabilities

Mark allocation: 30 **Time:** 60 minutes

Show **all** the calculations and intermediary steps.
Simplify where possible.
All final answers must be approximated accurately to **three** decimal places.

1. Define the following probability terminology:

a)	Sample space	(2)
b)	Event	(2)
c)	Probability	(2)
d)	Dependent events	(2)
e)	Independent events	(2)
f)	Mutually exclusive events	(2)
g)	Mutually inclusive events	(2)
h)	Complementary event	(2)

2. Two balls are drawn from a bag containing 5 red, 6 blue and 4 green balls, without replacing the first ball. What is the probability of drawing two red balls? (3)

3. A coin is flipped and a pentagonal spinner with sectors 1, 2, 3, 4 and 5 is spun. What is the probability the coin will reveal heads and the pentagonal spinner will show a 3? (3)

4. A hexagon is drawn from a bag containing 8 purple, 6 yellow and 6 orange hexagons. What is the probability that it is a purple or an orange hexagon? (3)

5. What is the probability of drawing a king or a black card from a deck of cards? (3)

6. What is the probability of not drawing a club, diamond or spade from a deck of cards? (2)

[30]

Formative assessment 5

Draw tree diagrams, Venn diagrams and complete contingency two–way tables to solve probability problems, and interpret and clearly communicate results of the experiments correctly in terms of real context

Mark allocation: 25 **Time:** 45 minutes

Show **all** the calculations and intermediary steps.
Simplify where possible.
All final answers must be approximated accurately to **three** decimal places.

1. A bag contains 9 black balls, 6 red balls and 5 yellow balls. Two balls are chosen from the bag, that is, the first ball is chosen from the bag and the colour recorded. The ball is then replaced and a second ball is chosen.

a) Construct a tree diagram to show that the sum of the branches is equal to 1. (5)

b) Record the sample space. (2)

c) Compute the following probabilities:

(i) P(BB); P(RR); P(YY) (3)

(ii) P(BB or RR); P(RR or YY); P(BB or YY) (3)

(iii) P(at least one yellow ball is drawn);
P(two different colour balls) (2)

2. Illustrate each of the following terms with a Venn diagram:

a) Sample space (1)

b) Event (1)

c) Mutual exclusive events (1)

d) Mutual inclusive events (1)

e) Complementary event (1)

3. A survey conducted at an office block yielded the following contingency table.

	Gender		
Sport code	**Female**	**Male**	**Total**
Soccer	40	**B**	75
Rugby	35	20	**C**
Cricket	**A**	90	120
Total	**105**	**145**	**250**

a) Complete the contingency table by assigning values to A, B and C. (3)

b) Convert the contingency table into a joint probability table. (2)

[25]

Topic 5

Formative assessment 1

Use simple and compound growth formulae A = P(1 + in) and A = P(1 + i)^n to solve problems, including interest, hire purchase and inflation

Mark allocation: 20 **Time:** 40 minutes

Show **all** the calculations and intermediary steps.
Simplify where possible.
All final answers must be approximated accurately to **three** decimal places.

1. Define the following financial terminology:
  a) Hire purchase (2)
  b) Inflation (2)

2. Make n subject of the formula: $A = P(1 + in)$ (3)

3. Make i subject of the formula: $A = P(1 + i)^n$ (3)

4. Zachary wants to purchase a laptop on a hire purchase agreement. The laptop's cash price is R5 000. He wants to pay it off over 24 months at an interest of 20% per annum. The supplier's terms and conditions require that Zachary pays a deposit of 10% and an insurance premium of R25 monthly.
  a) Compute the principal amount. (2)
  b) Calculate the accumulated loan amount. (2)
  c) Determine Zachary's monthly repayment. (2)
  d) Compute the total amount paid for the laptop. (2)

5. An electronic tracking device presently costs R2 000. Determine the average inflation rate, if it cost R1 200 four years ago. (2)

[20]

Formative assessment 2

Understand, use and interpret tax tables

Mark allocation: 20 **Time:** 40 minutes

Show **all** the calculations and intermediary steps.
Simplify where possible.
All final answers must be approximated accurately to **three** decimal places.

1. Define the following tax terminology:

a) Tax (2)
b) Tax return (2)
c) Tax rate (2)
d) Tax rebate (2)
e) Tax threshold (2)
f) Tax credit (2)

2. The individual rate table obtained from SARS is given below.

Statutory rates applicable to individuals are:	
Taxable income (R)	**Rates of tax (R)**
0–160 000	18% of each R1
160 001–250 000	28 800 + 25% of the amount above R160 000
250 001–346 000	51 300 + 30% of the amount above R250 000
346 001–484 000	80 100 + 35% of the amount above R346 000
484 001–617 000	128 400 + 38% of the amount above R484 000
617 001 and above	178 940 + 40% of the amount above R617 000

Tax rebates applicable to individuals are:

• Primary rebate	R11 440
• Secondary rebate (for persons 65 years and older)	R6 390
• Tertiary rebate (for persons 75 years and older)	R2 130

Tax thresholds applicable to individuals (excluding the allowable medical scheme fees tax credit) are:

• Persons under 65 years	R63 556
• Persons 65 years and older	R99 056
• Persons 75 years and older	R110 889

South African Revenue Service (SARS) 2012/2013

Dr Harper, an economist, is 54 years old and has two dependants.

The following monthly earning and deductions have been provided:

- Monthly salary: R50 000,00
- Medical aid contribution: R2 550,00
- Pension fund contribution: R5 000,00

In addition, he received an annual bonus of R180 000,00.

Use the individual rate table to compute his annual tax due. (8)

[20]

Formative assessment 3

Use the simple and compound decay formulae A = P(1 – in) and A = P(1 – i)^n to solve problems (straight-line depreciation and depreciation on a reducing balance)

Mark allocation: 20 **Time:** 40 minutes

Show **all** the calculations and intermediary steps.
Simplify where possible.
All final answers must be approximated accurately to **three** decimal places.

1. Explain the difference between straight-line depreciation and reducing-balance depreciation. (2)

2. Make i subject of the formula for both decay models:

a) $A = P(1 - in)$ (3)

b) $A = P(1 - i)^n$ (3)

3. Calculate the period for straight-line depreciation for a depreciation rate of 17,5% per annum, with an accrued amount of R13 500 and a principal amount of R45 000. (4)

4. Calculate the principal amount for reducing–balance depreciation for a depreciation rate of $11\frac{1}{3}\%$ per annum for 5 years, with an accrued amount of R15 725. (2)

5. Daniel, an electronics engineer, bought a fault-finding instrument for R45 000. The fault finding instrument is depreciated at 12% per annum on a reducing-balance basis.

a) Determine the value of the instrument at the end of 4 years. (2)

b) Compute the total depreciation after 4 years. (2)

c) Draw a graph of the value of the fault finding instrument versus time for the 4 years. (2)

[20]

Topic 1 • Solutions

Formative assessment 1

1.

$$\begin{aligned}&(-i)^2 \times 6i^3\\ &= i^2 \times 6i^3\\ &= (-1) \times 6(i^2)i\\ &= (-1) \times 6(-1)i\\ &= -1 - 6i\end{aligned}$$

2.

$$\begin{aligned}&\sqrt{-1}\,(\sqrt{-1} + \sqrt{25} - \sqrt{-49})\\ &= i\,(i + 5 - \sqrt{49}.\sqrt{-1})\\ &= i\,(i + 5 - 7i)\\ &= i\,(5 - 6i)\\ &= 5i - 6i^2\\ &= 5i - 6(-1)\\ &= 6 + 5i\end{aligned}$$

3.

$$\begin{aligned}&(1 - 3i)(2 + i)^2\\ &= (1 - 3i)(2 + i)\,(2 + i)\\ &= (1 - 3i)(4 + 4i + i^2)\\ &= (1 - 3i)(4 + 4i + (-1))\\ &= (1 - 3i)(3 + 4i)\\ &= 3 + 4i + 9i - 12i^2\\ &= 3 - 5i + 12\\ &= 16 - 5i\end{aligned}$$

4.

$$\begin{aligned}&\frac{2 + 4i}{3 - 5i}\\ &= \frac{2 + 4i}{3 - 5i} \times \frac{3 + 5i}{3 + 5i}\\ &= \frac{6 + 10i + 12i + 20i^2}{9 - 25i^2}\\ &= \frac{6 + 22i + 20(-1)}{9 - 25(-1)}\\ &= \frac{6 + 22i - 20}{9 + 25}\\ &= \frac{-14 + 22i}{34}\\ &= -\frac{14}{34} + \frac{22}{34}i\\ &= -\frac{7}{17} + \frac{11}{17}i\end{aligned}$$

Formative assessment 2

1.

$$\begin{aligned}&\frac{2 - 3i}{2\lfloor\underline{-40^\circ}}\\ &= \frac{\sqrt{13}\lfloor\underline{359{,}974^\circ}}{2\lfloor\underline{-40^\circ}}\\ &= \frac{\sqrt{13}}{2}\lfloor\underline{359{,}974^\circ + 40^\circ}\\ &= 1{,}803\lfloor\underline{399{,}974^\circ}\\ &= 1{,}803\lfloor\underline{39{,}974^\circ}\end{aligned}$$

$$\begin{aligned}r(\text{mod}) &= \sqrt{(2)^2 + (-3)^2}\\ &= \sqrt{13}\\ \theta(\text{arg}) &= 360^\circ - \tan^{-1}\frac{3}{2}\\ &= 303{,}69^\circ\end{aligned}$$

2.

$$\begin{aligned}&(4 + 2i)(4 \text{ cis } 122^\circ)\\ &= (\sqrt{20}\lfloor\underline{26{,}565^\circ})\,(4\lfloor\underline{122^\circ})\\ &= (\sqrt{20})(4)\lfloor\underline{26{,}565^\circ + 122^\circ}\\ &= 17{,}889\lfloor\underline{148{,}565^\circ}\end{aligned}$$

$$\begin{aligned}r(\text{mod}) &= \sqrt{(4)^2 + (2)^2}\\ &= \sqrt{20}\\ \theta(\text{arg}) &= \tan^{-1}\frac{2}{4}\\ &= 26{,}565^\circ\end{aligned}$$

Formative assessment 3

1. $$\frac{4 \text{ cis } 45^\circ \times (2 \text{ cis } 20^\circ)^5}{3 \text{ cis } 80^\circ}$$

$$= \frac{4\lfloor\underline{45^\circ}.2^5\lfloor\underline{20^\circ \times 5}}{3\lfloor\underline{80^\circ}}$$

$$= \frac{4 \times 32\lfloor\underline{45^\circ + 100^\circ}}{3\lfloor\underline{80^\circ}}$$

$$= \frac{4 \times 32}{3}\lfloor\underline{145^\circ - 80^\circ}$$

$$= 42{,}667\lfloor\underline{65^\circ}$$

2. $$[2(\cos 30^\circ + i \sin 30^\circ)]^4$$

$$= [2\lfloor\underline{30^\circ}\,]^4$$

$$= 16\lfloor\underline{30^\circ \times 4}$$

$$= 16\lfloor\underline{120^\circ}$$

3. $$\left(\frac{9 \text{ cis } 30^\circ}{3 \text{ cis } 60^\circ}\right)^4 \times \left(\frac{8 \text{ cis } 40^\circ}{16 \text{ cis } 140^\circ}\right)^{-2}$$

$$= (3\lfloor\underline{30^\circ - 60^\circ})^4 \times \left(\frac{16 \text{ cis } 140^\circ}{8 \text{ cis } 40^\circ}\right)^2$$

$$= (3\lfloor\underline{-30^\circ})^4 \times (2\lfloor\underline{100^\circ})^2$$

$$= 3^4\lfloor\underline{-30^\circ \times 4} \times 2^2\lfloor\underline{100^\circ \times 2}$$

$$= 81\lfloor\underline{-120^\circ} \times 4\lfloor\underline{200^\circ}$$

$$= 324\lfloor\underline{80^\circ}$$

Formative assessment 4

1. $3x + 2yi - 2 = 4 - 5i$

$\therefore 3x - 2 = 4; \quad 2y = -5$

$3x = 6; \quad y = -\frac{5}{2}$

$x = 2$

2. $$\frac{x - iy}{i^2} = (5 - 3i)^2$$

$$\frac{x - iy}{(-1)} = (5 - 3i)(5 - 3i)$$

$-x + iy = 25 - 30i + 9i^2$

$-x + iy = 25 - 30i + 9(-1)$

$-x + iy = 16 - 30i$

$\therefore -x = 16 \qquad y = -30$

$x = -16$

3. $$2x - yi = (i - 2) + \frac{3 + i}{3 - i}$$

$$= (i - 2) + \left(\frac{3 + i}{3 - i} \times \frac{3 + i}{3 + i}\right)$$

$$= (i - 2) + \left(\frac{9 + 6i + i^2}{9 - i^2}\right)$$

$$= (i - 2) + \left(\frac{8 + 6i}{10}\right)$$

$$2x - yi = i - 2 + \frac{8}{10} + \frac{6}{10}i$$

$2x - yi = -1{,}2 + 1{,}6i$

$\therefore 2x = -1{,}2; \qquad -y = 1{,}6$

$x = -0{,}6 \qquad y = 1{,}6$

Formative assessment 5

1. $x^2 + 64 = 0$

$$x^2 = -64$$
$$x = \pm \sqrt{-64}$$
$$x = \pm \sqrt{8}\,i$$

2. $x^2 - 6x + 32 = 0$

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(32)}}{2(1)}$$
$$= \frac{6 \pm \sqrt{36 - 128}}{2}$$
$$= \frac{6 \pm \sqrt{-92}}{2}$$
$$= \frac{6 \pm \sqrt{92}\,i}{2}$$
$$= \frac{6}{2} \pm \frac{\sqrt{92}}{2}\,i$$
$$= 3 \pm 4{,}796i$$

Topic 2 • Solutions

Formative assessment 1

1. $3x - 9 = 0$

$$3x = 9$$
$$x = 3$$

$$\therefore f(3) = (3)^3 - 8(3)^2 + (3) - 16$$
$$= 27 - 72 + 3 - 16$$
$$\therefore \text{rem} = -58$$

2. $f(x) = 6x^3 - 11x^2 + x + 4$

$$f(1) = 6(1)^3 - 11(1)^2 + (1) + 4$$
$$= 6 - 11 + 1 + 4$$
$$= 0$$
$$\therefore x - 1 \text{ is a factor}$$

3. $f(x) = x^3 - 7x^2 - 10x + 16$

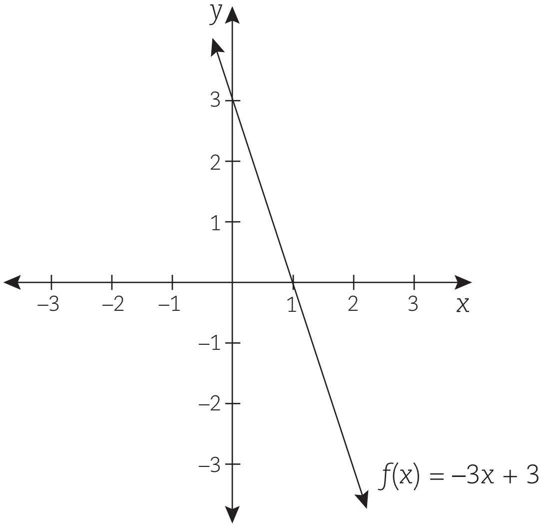
$$f(1) = (1)^3 - 7(1)^2 - 10(1) + 16$$
$$= 0$$
$$\therefore (x - 1)(x^2 - 6x - 16)$$
$$-x^2$$
$$-6x^2$$
$$= (x - 1)(x - 8)(x + 2)$$

Formative assessment 2

1.1

$$y = -3x + 3$$
$$f(x) = -3x + 3$$
$$f^{-1}: x = -3y + 3$$
$$3y = -x + 3$$
$$y = -\frac{1}{3}x + 1$$

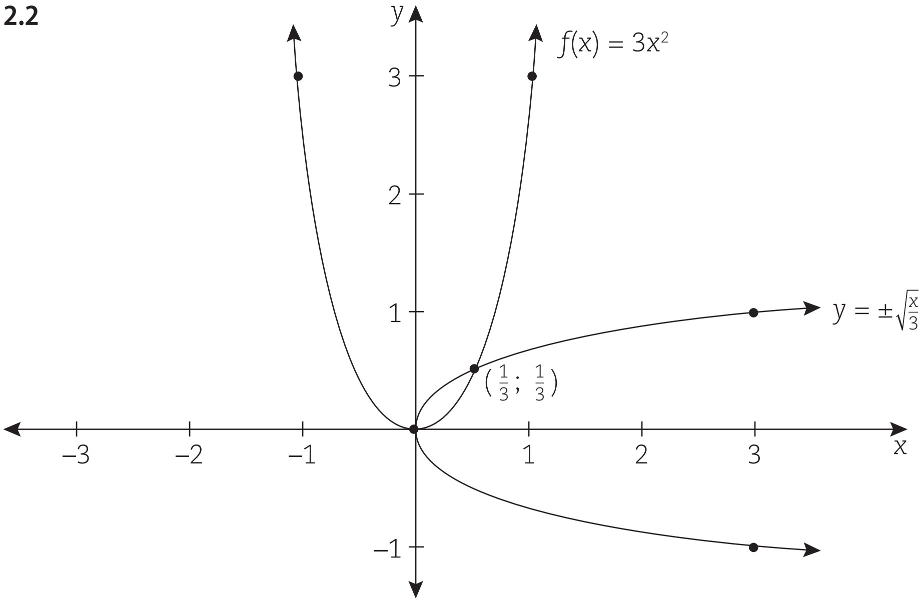
1.2 $f^{-1}(x) = -\frac{1}{3}x + 1$



2.1 $f(x) = 3x^2$

$$\therefore f^{-1}: x = 3y^2$$
$$y^2 = \frac{x}{3}$$
$$y = \pm\sqrt{\frac{x}{3}}$$

2.2



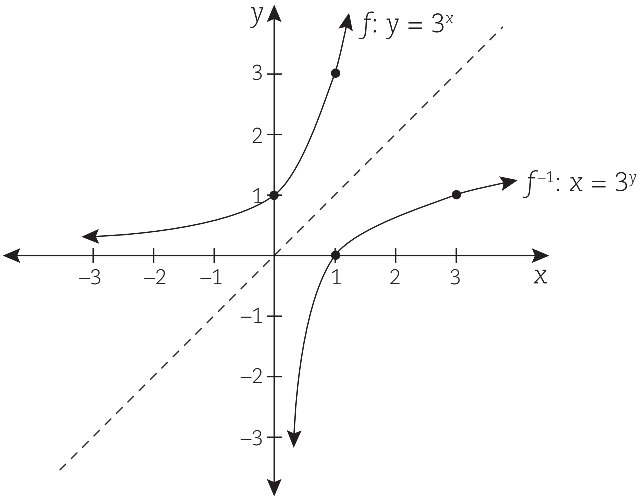
2.3 $\{x : x \in \mathbb{R}\}$

2.4 $\{y : y \in \mathbb{R}\}$

Formative assessment 3

1. $x = 3y$

2.



3. $y = 0$

4. $x = 0$

5. $\{x : x \in \mathbb{R}\}$

6. $\{x : x > 0; x \in \mathbb{R}\}$

Formative assessment 4

1.

$$
\begin{aligned}
f(x) &= 3x + 7 \\
f(x+h) &= 3(x+h) + 7 \\
&= 3x + 3h + 7
\end{aligned}
$$

$$\lim_{h\to 0} \frac{3x + 3h + 7 - 3x - 7}{h} =$$

$$= \lim_{h\to 0} \frac{3h}{h}$$

$$= \lim_{h\to 0} 3$$

$$= 3$$

2.

$$
\begin{aligned}
f(x) &= -2x^2 \\
f(x+h) &= -2(x+h)^2 \\
&= -2(x+h)(x+h) \\
&= -2(x^2 + 2xh + h^2) \\
&= -2x^2 - 4xh - 2h^2
\end{aligned}
$$

$$\lim_{h\to 0} \frac{-2x^2 - 4xh - 2h^2 + 2x^2}{h}$$

$$= \lim_{h\to 0} \frac{h(-4x - 2h)}{h}$$

$$= \lim_{h\to 0} -4x - 2h$$

$$= -4x$$

3.

$$f(x) = 5x^0$$

$$f(x+h) = 5(x+h)^0$$

$$\lim_{h\to 0} \frac{5(x+h)^0 - 5x^0}{h}$$

$$= \lim_{h\to 0} \frac{0}{h}$$

$$= \lim_{h\to 0} 0$$

$$= 0$$

4.

$$f(x) = \frac{1}{x}$$

$$f(x+h) = \frac{1}{x+h}$$

$$\lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$

$$= \lim_{h\to 0} \frac{x-(x+h)}{x(x+h)} \times \frac{1}{h}$$

$$= \lim_{h\to 0} \frac{x-x-h}{x(x+h)} \times \frac{1}{h}$$

$$= \lim_{h\to 0} \frac{1}{x(x+h)}$$

$$= -\frac{1}{x^2}$$

Formative assessment 5

1.

$$y = 4x^3 - \sqrt{3}\cos x + e^{-2x} - 4\ln x - x^{\frac{2}{3}} + 3$$

$$\frac{dy}{dx} = 12x^2 + \sqrt{3}\sin x - e^{-2x} - \frac{4}{x} - \frac{2}{3}x^{-\frac{1}{3}}$$

$$= 12x^2 + \sqrt{3}\sin x - e^{-2x} - \frac{4}{x} - \frac{2}{3\sqrt[3]{x}}$$

2.

$$f(x) = -4x^2(-6x + 3x^3)$$

$$f'(x) = -4x^2(-6 + 9x^2) + (-8x)(-6x + 3x^3)$$

$$= 24x^2 - 36x^4 + 48x^2 - 24x^4$$

$$= 72x^2 - 60x^4$$

3.

$$y = \frac{x^5}{e^{3x}}$$

$$\frac{dy}{dx} = \frac{e^{3x}(5x^4) - x^5(3e^{3x})}{(e^{3x})^2}$$

$$= \frac{e^{3x}(5x^4 - 3x^5)}{(e^{3x})^2}$$

$$= \frac{5x^4 - 3x^5}{e^{3x}}$$

4.

$$f(x) = (6x^2 - 3)^4$$

$$y = u^4 \qquad u = 6x^2 - 3$$

$$\frac{dy}{du} = 4u^3 \qquad \frac{du}{dx} = 12x$$

$$\frac{dy}{dx} = 4u^3.12x$$

$$= 4(6x^2 - 3)^3\ 12x$$

$$= 48x(6x^2 - 3)^3$$

Formative assessment 6

1. $s = 102t + 14t^2 - t^3$

1.1 $\frac{ds}{dt} = 102 + 28t - 3t^2$
$= 102 + 28(3) - 3(3)^2$
$= 159$ m/s

1.2 $\frac{d^2s}{dt^2} = 28 - 6t$
$= 28 - 6(3)$
$= 10$ m/s^2

2. **2.1** $f(x) = x^3 + x^2 - 6x$

TP: $\frac{dy}{dx} = 0$

$f'(x) = 3x^2 + 2x - 6$
$0 = 3x^2 + 2x - 6$

$x = 1{,}120$ and $x = -1{,}786$
$y = (1{,}12)^3 + (1{,}12)^2 - 6(1{,}12)$
$= -4{,}061$
TP: (1,120; −4,061) and (−1,786; 8,209)

2.2 $\frac{d^2y}{dx^2} = 6x + 2$
$0 = 6x + 2$
$\therefore\ 6x = -2$
$x = -\frac{1}{3}$
$y = x^3 + x^2 - 6x$
$= \left(-\frac{1}{3}\right)^3 + \left(-\frac{1}{3}\right)^2 - 6\left(-\frac{1}{3}\right)$
$= 2{,}074$

Point of inflection: $(-\frac{1}{3}\,; -2{,}074)$

Formative assessment 7

1. $\int\left(\sin 2x + \sec^2 x - a + 6e^{-3x} + 3x^{\frac{1}{3}}\right) dx$

$= -\frac{\cos 2x}{2} + \tan x - ax + \frac{6e^{-3x}}{-3} + \frac{3x^{\frac{4}{3}}}{\frac{4}{3}} + c$

$= -\frac{1}{2}\cos 2x + \tan x - ax - \frac{2}{e^{3x}} + \frac{9}{4}\sqrt[3]{x^4} + c$

2. $\int_0^1 (6x^2 - 3)\, dx$

$= \left[\frac{6x^3}{3} - 3x\right]_0^1$

$= [2x^3 - 3x]_0^1$

$= [2(1)^3 - 3(1)] - [2(0)^3 - 3(0)]$
$= 2 - 3$
$= -1$

Topic 3 • Solutions

Formative assessment 1

1. **a)**

$$x^2 + y^2 = r^2$$
$$(7)^2 + (-3)^2 = r^2$$
$$58 = r^2$$
$$\therefore x^2 + y^2 = 58$$

b)

$$x^2 + y^2 = r^2$$
$$x^2 + y^2 = (8)^2$$
$$\therefore x^2 + y^2 = 64$$

2. **a)**

$$-x^2 - y^2 + 36 = 0$$
$$-x^2 - y^2 = -36 \quad : \times -1$$
$$x^2 + y^2 = 36$$
$$r^2 = 36$$
$$\therefore r = 6 \text{ units}$$

b)

$$\frac{x^2}{3} + \frac{y^2}{3} = 27 \quad : \times 3$$
$$x^2 + y^2 = 81$$
$$r^2 = 81$$
$$\therefore r = 9 \text{ units}$$

3. **a)** centre (–2; 5) and radius = 7

$$(x - h)^2 + (y - k)^2 = r^2$$
$$(x - (-2))^2 + (y - (5))^2 = (7)^2$$
$$\therefore (x + 2)^2 + (y - 5)^2 = 49$$

b) centre (–1; –4) and point (3; 5)

$$(x - h)^2 + (y - k)^2 = r^2$$
$$((3) - (-1))^2 + ((5) - (-4))^2 = r^2$$
$$16 + 81 = r^2$$
$$97 = r^2$$
$$(x - (-1))^2 + (y - (-4))^2 = 97$$
$$\therefore (x + 1)^2 + (y + 4)^2 = 97$$

4. **a)** $(x + 4)^2 + (y - 3)^2 = 144$

$r^2 = 144$

$\therefore r = 12$ units

$x + 4 = 0$ and $y - 3 = 0$

$x = -4$ $y = 3$

$\therefore$ centre = $(h; k) = (-4; 3)$

b)

$$x^2 - 6x + y^2 + 8y - 12 = 0$$
$$(x^2 - 6x) + (y^2 + 8y) = 12$$
$$\left(x^2 - 6x + \left(\frac{1}{2}.-6\right)^2\right) + \left(y^2 + 8y + \left(\frac{1}{2}.8\right)^2\right) = 12 + \left(\frac{1}{2}.-6\right)^2 + \left(\frac{1}{2}.8\right)^2$$
$$(x^2 - 6x + 9) + (y^2 + 8y + 16) = 12 + 9 + 16$$
$$(x - 3)^2 + (y + 4)^2 = 37$$

$r^2 = 37$

$\therefore r = \sqrt{37}$ units

$x - 3 = 0$ and $y + 4 = 0$

$x = 3$ $y = -4$

$\therefore$ centre = $(h; k) = (3; -4)$

5.

$$\begin{aligned}
(x-2)^2 + (y+10)^2 &= 490\\
((-5)-2)^2 + ((n)+10)^2 &= 490\\
(-7)^2 + (n+10)^2 &= 490\\
49 + (n+10)^2 &= 490\\
(n+10)^2 &= 490 - 49\\
(n+10)^2 &= 441\\
n + 10 &= \pm\ 21
\end{aligned}$$

$\therefore n + 10 = 21$ and $n + 10 = -21$

$n = 21 - 10$ $\qquad$ $n = -21 - 10$

$n = 11$ $\qquad$ $n = -31$

Formative assessment 2

1. a)

$$\begin{aligned}
x^2 + y^2 - 17 &= 2x - 8y\\
(x^2 - 2x) + (y^2 + 8y) &= 17\\
\left(x^2 - 2x + \left(\tfrac{1}{2}.-2\right)^2\right) + \left(y^2 + 8y + \left(\tfrac{1}{2}.8\right)^2\right) &= 17 + \left(\tfrac{1}{2}.-2\right)^2 + \left(\tfrac{1}{2}.8\right)^2\\
(x-1)^2 + (y+4)^2 &= 17 + 1 + 16\\
(x-1)^2 + (y+4)^2 &= 34
\end{aligned}$$

$$\begin{aligned}
r^2 &= 34\\
\sqrt{r^2} &= \sqrt{34}\\
\therefore r &= \sqrt{34}
\end{aligned}$$

$x - 1 = 0$ and $y + 4 = 0$

$x = 1$ $\qquad$ $y = -4$

$\therefore$ centre $= (h; k) = (1; -4)$

b) $P(x_1; y_1) = (-2; -9)$ and $P(x_2; y_2) = (1; -4)$

$$\begin{aligned}
m_{OP} &= \frac{y_2 - y_1}{x_2 - x_1}\\
&= \frac{(-4) - (-9)}{(1) - (-2)}\\
m_{OP} &= \frac{5}{3}
\end{aligned}$$

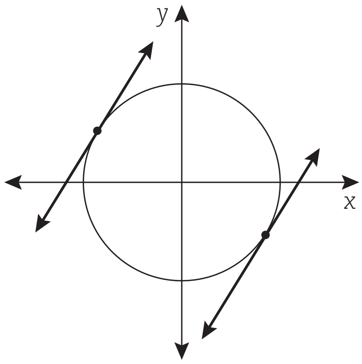
c) $m_{OP} \times m_{tan} = -1$

$$\begin{aligned}
m_{tan} &= \frac{-1}{m_{OP}}\\
&= \frac{-1}{\frac{5}{3}}\\
m_{tan} &= -\frac{3}{5}
\end{aligned}$$

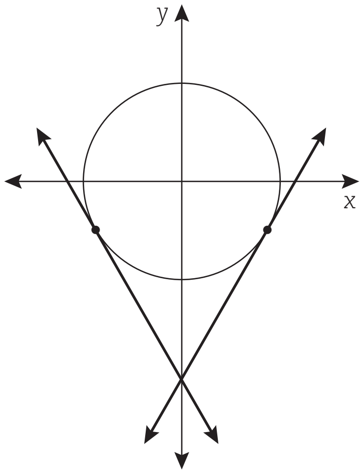
$$\begin{aligned}
y - y_1 &= m(x - x_1)\\
y - (-9) &= -\frac{3}{5}(x - (-2))\\
y + 9 &= -\frac{3}{5}x - \frac{6}{5}\\
y &= -\frac{3}{5}x - \frac{51}{5}
\end{aligned}$$

2. a) The condition of tangency refers to the relationship between the y-intercept (c), the radius (r) and the gradient (m), that is, $c^2 = r^2\,(m^2 + 1)$.

b) (i) The tangents with the same gradients,



(ii) The tangents with the same y-intercepts,



c) (i)

$$x^2 + y^2 = 25$$
$$\therefore\ r^2 = 25$$
$$\sqrt{r^2} = \sqrt{25}$$
$$r = 5 \text{ units}$$

$$c = \sqrt{125}$$

$$c^2 = r^2\,(m^2 + 1)$$
$$(\sqrt{125})^2 = (5)^2\,(m^2 + 1)$$
$$125 = 25\,(m^2 + 1)$$
$$5 = m^2 + 1$$
$$5 - 1 = m^2$$
$$4 = m^2$$
$$\sqrt{4} = \sqrt{m^2}$$
$$\pm 2 = m$$

$$y = \pm 2x + \sqrt{125}$$

$y = +\,2x + \sqrt{125}$ and $y = -2x + \sqrt{125}$

$y = +\,2x + 5\sqrt{5}$ $\quad$ $y = -2x + 5\sqrt{5}$

(ii) $\frac{x^2}{7} + \frac{y^2}{7} = 7 \qquad : \times 7$

$$x^2 + y^2 = 49$$
$$\therefore r^2 = 49$$
$$\sqrt{r^2} = \sqrt{49}$$
$$r = 7 \text{ units}$$

$$m = -2$$

$$c^2 = r^2(m^2 + 1)$$
$$= (7)^2\ ((-2)^2 + 1)$$
$$c^2 = 245$$
$$\sqrt{c^2} = \sqrt{245}$$
$$c = \pm\sqrt{245}$$

$$y = -2x \pm \sqrt{245}$$

$$y = -2x + \sqrt{245} \text{ and } y = -2x - \sqrt{245}$$
$$y = -2x + 7\sqrt{5} \qquad y = -2x - 7\sqrt{5}$$

(iii) $x^2 + y^2 - 13 = 0$

$$x^2 + y^2 = 13$$
$$\therefore r^2 = 13$$
$$\sqrt{r^2} = \sqrt{13}$$
$$r = \sqrt{13} \text{ units}$$

$$\theta = 30°$$
$$\therefore m = \tan\theta$$
$$= \tan 30°$$
$$m = \frac{1}{\sqrt{3}}$$

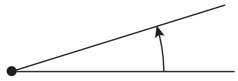
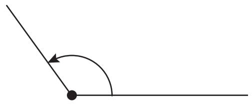
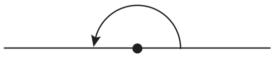
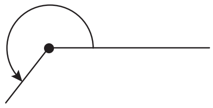
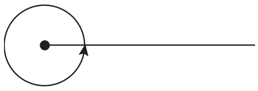
$$c^2 = r^2\ (m^2 + 1)$$
$$= \left(\sqrt{13}\right)^2 \left(\left(\frac{1}{\sqrt{3}}\right)^2 + 1\right)$$
$$c^2 = \frac{52}{3}$$
$$\sqrt{c^2} = \sqrt{\frac{52}{3}}$$
$$c = \pm\sqrt{\frac{52}{3}}$$

$$y = \frac{1}{\sqrt{3}}x \pm \sqrt{\frac{52}{3}}$$

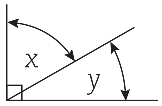
$$y = \frac{1}{\sqrt{3}}x + \sqrt{\frac{52}{3}} \text{ and } y = \frac{1}{\sqrt{3}}x - \sqrt{\frac{52}{3}}$$
$$y = \frac{1}{\sqrt{3}}x + \frac{2\sqrt{13}}{\sqrt{3}} \qquad y = \frac{1}{\sqrt{3}}x - \frac{2\sqrt{13}}{\sqrt{3}}$$

Formative assessment 3

1.

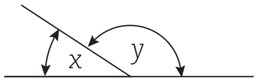
Type of angle	Description	Sketch
Acute angle	An angle less than 90°	
Right angle	An angle equal to 90°	
Obtuse angle	An angle greater than 90°, but smaller than 180°	
Straight angle	An angle equal to 180°	
Reflex angle	An angle greater than 180°	
Revolution	An angle equal to 360°	

2. **a)** Complementary angles: sum of angles equals 90°.



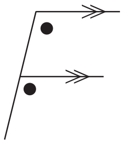
$\hat{x} + \hat{y} = 90°$

b) Supplementary angles: sum of angles equals 180°.

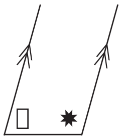


$\hat{x} + \hat{y} = 180°$

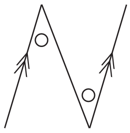
3.



Corresponding angles

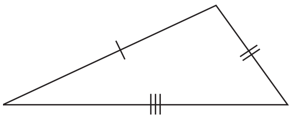
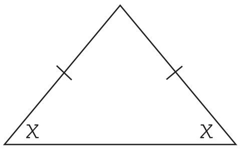
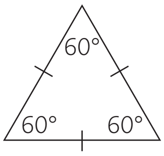
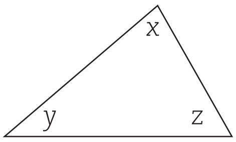
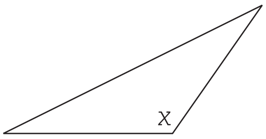
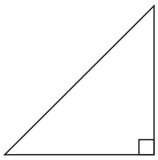


Consecutive interior angles



Alternate interior angles

4.

Type of triangle	Description	Sketch
Scalene triangle	Sides are all different lengths and all three angles different	
Isosceles triangle	Two equal sides and the angles opposite the sides are also equal	
Equilateral triangle	All three sides are equal and each angle measures 60°	
Acute-angled triangle	All interior angles are less than 90°	
Obtuse-angled triangle	One interior angle is more than 90°	
Right-angled triangle	One interior angle is a right angle, that is, 90°	

5. **a)** interior; 180°
   b) exterior; opposite

6. Theorem of Pythagoras states that in a right-angled triangle the square of the hypotenuse is equal to the sum of the squares of the other two sides.

Formative assessment 4

Investigation 4.1
perpendicular

Investigation 4.2
twice

Investigation 4.3
complementary

Investigation 4.4
equal

Investigation 4.5
opposite; supplementary

Investigation 4.6
equal

Investigation 4.7
perpendicular

Investigation 4.8
equal

Investigation 4.9
equal; alternate

Formative assessment 5

1. **a)** $\sin(\alpha - 20°)$
   $= \sin \alpha.\cos 20° - \sin 20°.\cos \alpha$

   b) $\cos (63° + \beta)$
   $= \cos 63°.\cos \beta - \sin 63°.\sin \beta$

   c) $\sin (\tau + 3\phi)$
   $= \sin \tau.\cos 3\phi + \sin 3\phi.\cos \tau$

   d) $\cos (2\lambda - \omega)$
   $= \cos 2\ \lambda.\cos \omega + \sin 2\ \lambda.\sin \omega$

2. **a)** $\sin\phi.\cos 31° + \sin 31°.\cos\phi$
$= \sin(\phi + 31°)$

b) $\cos 47°.\cos\alpha - \sin 47°.\sin\alpha$
$= \cos(47° + \alpha)$

c) $\sin\mu.\cos 2\phi - \sin 2\phi.\cos\mu$
$= \sin(\mu - 2\phi)$

d) $\cos\frac{\rho}{2}.\cos\upsilon + \sin\frac{\rho}{2}.\sin\upsilon$
$= \cos\left(\frac{\rho}{2} - \upsilon\right)$

3. $\sin 105°$
$= \sin(45° + 60°)$
$= \sin 45°.\cos 60° + \sin 60°.\cos 45°$
$= \left(\frac{1}{\sqrt{2}}\right).\left(\frac{1}{2}\right) + \left(\frac{\sqrt{3}}{2}\right).\left(\frac{1}{\sqrt{2}}\right)$
$= \frac{1}{2\sqrt{2}} + \frac{\sqrt{3}}{2\sqrt{2}}$
$= \frac{1+\sqrt{3}}{2\sqrt{2}}$
$= \frac{1+\sqrt{3}}{2\sqrt{2}} + \frac{\sqrt{2}}{\sqrt{2}}$
$= \frac{\sqrt{2}+\sqrt{6}}{4}$

4. $\sin 2\alpha = \sin(\alpha + \alpha)$
$= \sin\alpha.\cos\alpha + \sin\alpha.\cos\alpha$
$\sin 2\alpha = 2\sin\alpha.\cos\alpha$

5. **a)** $\cos 2\alpha = \cos(\alpha + \alpha)$
$= \cos\alpha.\cos\alpha - \sin\alpha.\sin\alpha$
$= \cos^2\alpha - \sin^2\alpha$
$= \cos^2\alpha - (1 - \cos^2\alpha)$
$\cos 2\alpha = 2\cos^2\alpha - 1$

b) $\cos 120° = \cos[2(60°)]$
$= 2\cos^2 60° - 1$
$= 2\left(\frac{1}{2}\right)^2 - 1$
$\cos 120° = -\frac{1}{2}$

c) $\cos(90° + \alpha) = \cos 90°.\cos\alpha - \sin 90°.\sin\alpha$
$= 0.\cos\alpha - 1.\sin\alpha$
$\cos(90° + \alpha) = -\sin\alpha$

d) $\cos 120° = \cos(90° + 30°)$
$= -\sin 30°$
$\cos 120° = -\frac{1}{2}$

Formative assessment 6

1. $\cos 53° - \cos 7° = -\sin 23°$

L.H.S. $\cos 53° - \cos 7°$
$= \cos(23° + 30°) - \cos(30° - 23°)$
$= (\cos 23°.\cos 30° - \sin 23°.\sin 30°) - (\cos 30°.\cos 23° + \sin 30°.\sin 23°)$
$= \cos 23°.\cos 30° - \sin 23°.\sin 30° - \cos 30°.\cos 23° - \sin 30°.\sin 23°$
$= -2\sin 30°.\sin 23°$
$= -2\left(\frac{1}{2}\right).\sin 23°$
$= -\sin 23°$

$\therefore$ L.H.S. = R.H.S.

2. $$\frac{1-\tan^2 x}{1+\tan^2 x}$$

$$= \frac{1-\frac{\sin^2 x}{\cos^2 x}}{1+\frac{\sin^2 x}{\cos^2 x}}$$

$$= \frac{\frac{\cos^2 x-\sin^2 x}{\cos^2 x}}{\frac{\cos^2 x+\sin^2 x}{\cos^2 x}}$$

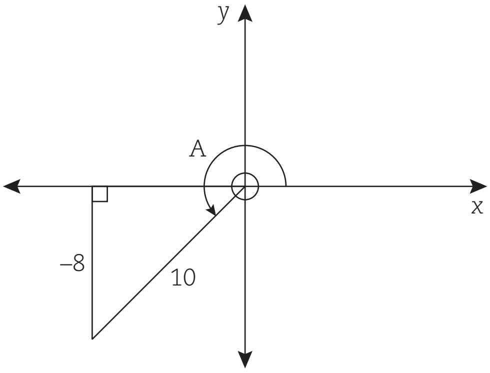
$$= \frac{\cos^2 x-\sin^2 x}{\cos^2 x} \times \frac{\cos^2 x}{\cos^2 x+\sin^2 x}$$

$$= \frac{\cos^2 x-\sin^2 x}{\cos^2 x+\sin^2 x}$$

$$= \frac{\cos 2x}{1}$$

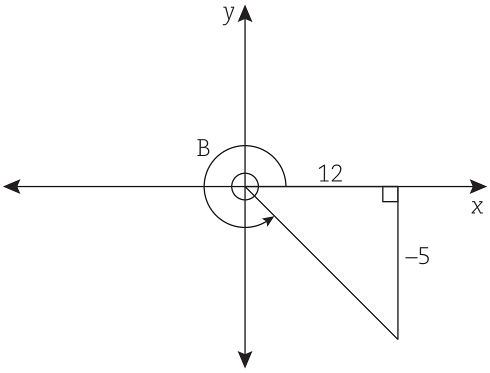
$$= \cos 2x$$

3. $\sin A = -\frac{8}{10}$, $A \in [0°; 270°]$



$$x^2 + y^2 = r^2$$
$$x^2 + (-8)^2 = (10)^2$$
$$x^2 + 64 = 100$$
$$x^2 = 100 - 64$$
$$x^2 = 36$$
$$x = -6$$

$\tan B = -\frac{5}{12}$, $B \in [180°; 360°]$



$$x^2 + y^2 = r^2$$
$$(12)^2 + (-5)^2 = r^2$$
$$144 + 25 = r^2$$
$$169 = r^2$$
$$r = 13$$

a) $\tan(180° - A)$
$= -\tan A$
$= -\left(\frac{-8}{-6}\right)$
$= -\frac{4}{3}$

b) $\sin(A + B)$
$= \sin A.\cos B + \sin B.\cos A$
$= \left(-\frac{8}{10}\right).\left(\frac{12}{13}\right) + \left(-\frac{5}{13}\right).\left(-\frac{6}{10}\right)$
$= -\frac{96}{130} + \frac{30}{130}$
$= -\frac{33}{65}$

c) $\sin 2B$
$= 2\sin B.\cos B$
$= 2\left(-\frac{5}{13}\right).\left(\frac{12}{13}\right)$
$= -\frac{120}{169}$

d) $\cos(A - B)$
$= \cos A.\cos B + \sin A.\sin B$
$= \left(-\frac{6}{10}\right).\left(\frac{12}{13}\right) + \left(-\frac{8}{10}\right).\left(-\frac{5}{13}\right)$
$= -\frac{72}{130} + \frac{40}{130}$
$= -\frac{16}{65}$

e) $\cos 2A$
$= \cos^2 A - \sin^2 A$
$= \left(-\frac{6}{10}\right)^2 - \left(-\frac{8}{10}\right)^2$
$= \frac{36}{100} - \frac{64}{100}$
$= -\frac{7}{25}$

Formative assessment 7

1. **a)** $\sin 3x = \cos x.\cos 30° - \sin x.\sin 30°$
$\sin 3x = \cos(x + 30°)$
$\sin 3x = \sin[90° - (x + 30°)]$

$\therefore\ 3x = 90° - (x + 30°)$
$3x = 90° - x - 30°$
$4x = 60°$
$x = 15°$
OR
$\therefore\ 3x = 180° - (90° - x - 30°)$
$3x = 180° - 90° + x + 30°$
$2x = 120°$
$x = 60°$

$\sin 3x = \cos x.\cos 30° - \sin x.\sin 30°$
$\sin 3x = \cos(x + 30°)$
$\sin 3x = \sin[90° + (x + 30°)]$

$\therefore\ 3x = 90° + (x + 30°)$
$3x = 90° + x + 30°$
$2x = 120°$
$x = 60°$
OR
$3x = 180° - (90° + x + 30°)$
$3x = 180° - 90° - x - 30°$
$4x = 60°$
$x = 15°$

But $x \in [0°; 90°]$,
$x = 15°$ and $x = 60°$

b) $\cos 3x.\cos 15° + \sin 3x.\sin 15° = -\cos 60°$

$$\cos(3x - 15°) = -\cos 60°$$

$$\therefore 3x - 15° = 180° - 60°$$
$$3x = 135°$$
$$x = 45°$$

OR

$$\therefore 3x - 15° = 180° + 60°$$
$$3x = 255°$$
$$x = 85°$$

But $x \in [0°; 90°]$,
$x = 45°$ and $x = 85°$

2. **a)**

$$3\sin\theta + 1 = 2\cos 2\theta$$
$$3\sin\theta + 1 = 2(1 - 2\sin^2\theta)$$
$$3\sin\theta + 1 = 2 - 4\sin^2\theta$$
$$4\sin^2\theta + 3\sin\theta - 1 = 0$$
$$(4\sin\theta - 1)(\sin\theta + 1) = 0$$

$$4\sin\theta - 1 = 0$$
$$4\sin\theta = 1$$
$$\sin\theta = \oplus\,\frac{1}{4}$$

sin +; 1st

$$\theta = \theta_{ref}$$
$$\theta = \sin^{-1}\left(\frac{1}{4}\right)$$
$$\theta = 14{,}478°$$

sin +; 2nd

$$\theta = 180° - \theta_{ref}$$
$$\theta = 180° - \sin^{-1}\left(\frac{1}{4}\right)$$
$$\theta = 165{,}523°$$

$$\sin\theta + 1 = 0$$
$$\sin\theta = \ominus\,1$$

sin –; 3rd

$$\theta = 180° + \theta_{ref}$$
$$\theta = 180° + \sin^{-1}(1)$$
$$\theta = 270°$$

sin –; 4th

$$\theta = 360° - \theta_{ref}$$
$$\theta = 360° - \sin^{-1}(1)$$
$$\theta = 270°$$

Since all the angles are within the restriction $\theta \in [0°; 360°]$, the solutions will be $\theta = 14{,}478°$, $\theta = 165{,}523°$ and $\theta = 270°$.

b)

$$\frac{\cos 2\theta}{\cos^2\theta} + 2 = 0$$
$$\cos 2\theta + 2\cos^2\theta = 0$$
$$2\cos^2\theta - 1 + 2\cos^2\theta = 0$$
$$4\cos^2\theta - 1 = 0$$
$$4\cos^2\theta = 1$$
$$\cos^2\theta = \frac{1}{4}$$
$$\cos\theta = \pm\,\frac{1}{2}$$

$$\cos\theta = \oplus\,\frac{1}{2}$$

cos +; 1st

$$\theta = \theta_{ref}$$
$$\theta = \cos^{-1}\left(\frac{1}{2}\right)$$
$$\theta = 60°$$

cos +; 4th

$$\theta = 360° - \theta_{ref}$$
$$\theta = 360° - \cos^{-1}\left(\frac{1}{2}\right)$$
$$\theta = 300°$$

$$\cos\theta = \ominus\,\frac{1}{2}$$

cos –; 2nd

$$\theta = 180° - \theta_{ref}$$
$$\theta = 180° - \cos^{-1}\left(\frac{1}{2}\right)$$
$$\theta = 120°$$

cos –; 3rd

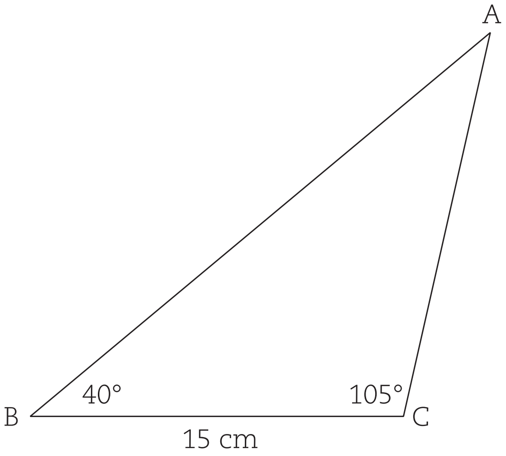
$$\theta = 180° + \theta_{ref}$$
$$\theta = 180° + \cos^{-1}\left(\frac{1}{2}\right)$$
$$\theta = 240°$$

Since all the angles are within the restriction $\theta \in [0°; 360°]$, the solutions will be $\theta = 60°$, $\theta = 120°$, $\theta = 240°$ and $\theta = 300°$.

Formative assessment 8

1.

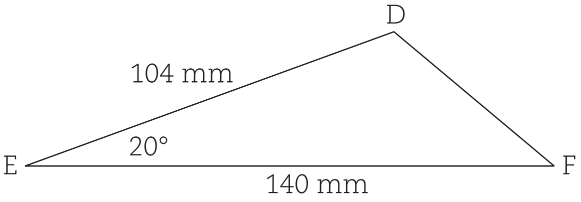
$$\hat{A} + \hat{B} + \hat{C} = 180°$$
$$\therefore \hat{A} = 180° - (\hat{B} + \hat{C})$$
$$= 180° - (40° + 103°)$$
$$\hat{A} = 37°$$



$$\frac{AB}{\sin \hat{C}} = \frac{BC}{\sin \hat{A}}$$
$$\therefore AB = \frac{BC \sin \hat{C}}{\sin \hat{A}}$$
$$= \frac{15 \sin 103°}{\sin 37°}$$
$$AB = 24{,}286 \text{ cm}$$

$$\frac{AC}{\sin \hat{B}} = \frac{BC}{\sin \hat{A}}$$
$$\therefore AC = \frac{BC \sin \hat{B}}{\sin \hat{A}}$$
$$= \frac{15 \sin 40°}{\sin 37°}$$
$$AC = 16{,}021 \text{ cm}$$

2.

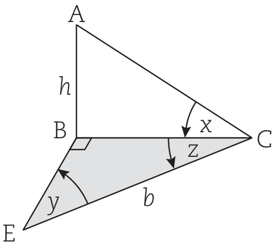


$$DF^2 = DE^2 + EF^2 - 2\,(DE)(EF) \cos \hat{E}$$
$$= (104)^2 + (140)^2 - 2(104)(140) \cos 20°$$
$$DF^2 = 3052{,}151$$
$$\therefore DF = 55{,}246 \text{ mm}$$

$$\frac{\sin \hat{D}}{EF} = \frac{\sin \hat{E}}{DF}$$
$$\sin \hat{D} = \frac{EF \sin \hat{E}}{DF}$$
$$\therefore \hat{D} = \sin^{-1}\left[\frac{EF \sin \hat{E}}{DF}\right]$$
$$= \sin^{-1}\left[\frac{140 \sin 20°}{55{,}246}\right]$$
$$\hat{D} = 60{,}079°$$

$$\hat{D} + \hat{E} + \hat{F} = 180°$$
$$\therefore \hat{F} = 180° - (\hat{D} + \hat{E})$$
$$= 180° - (60{,}079° + 20°)$$
$$\hat{F} = 99{,}921°$$

3.



a) In ΔBCE, $E\hat{B}C = 180° - (y + z)$

$$\frac{BC}{\sin B\hat{E}C} = \frac{EC}{\sin E\hat{B}C}$$

$$\frac{BC}{\sin y} = \frac{b}{\sin[180° - (y + z)]}$$

$$BC = \frac{b \sin y}{\sin(y + z)}$$

In ΔABC,

$$\tan x = \frac{AB}{BC}$$

$$\therefore AB = BC \tan x$$

$$h = \frac{b \sin y}{\sin(y + z)}.\tan x$$

$$h = \frac{b \sin y.\tan x}{\sin(y + z)}$$

b) $h = \frac{b \sin y.\tan x}{\sin(y + z)}$

$$= \frac{650 \sin 41,8°.\tan 14,9°}{\sin(41,8° + 66,7°)}$$

h = 121,560 m

Topic 4 • Solutions

Formative assessment 1

1.

	Ordered Data
	6
	6
	7
	8
x_5 =	⑨
x_6 =	⑬
	14
	15
	16
	20

$$P_{Q_2} = \frac{1}{2}\ (n + 1) = \frac{1}{2}\,(10 + 1) = 5{,}5 \qquad \overline{x} = \frac{\Sigma\, x_i}{n} = \frac{114}{10} = 11{,}400$$

$$Q_2 = \frac{x_5 + x_6}{2} = \frac{9 + 13}{2} = 11$$

x_i	$x_i - \overline{x}$	$(x_i - \overline{x})^2$
13	1.600	2.560
8	–3.400	11.560
7	–4.400	19.360
6	–5.400	29.160
16	4.600	21.160
9	–2.400	5.760
15	3.600	12.960
6	–5.400	29.160
14	2.600	6.760
20	8.600	73.960
$\Sigma x_i = 114$	$\Sigma(x_i - \overline{x}) = 0{,}000$	$\Sigma(x_i - \overline{x})^2 = 212{,}400$

$$s^2 = \frac{\Sigma(x_i - \overline{x})^2}{n - 1}$$

$$= \frac{212,400}{10 - 1}$$

$$s^2 = 23{,}600$$

$$\therefore\ s = \sqrt{23{,}600}$$

$$s = 4{,}858$$

Since $\overline{x} - Q_2 > 0$, therefore the data are skewed to the right, that is, positively skewed.
Since the data does not represent a normal distribution, therefore a 95% confidence interval cannot be computed.

2.

	Ordered Data
	2
	3
	5
	6
x_5 =	⑬
x_6 =	⑮
	21
	26
	27
	30

$$P_{Q_2} = \frac{1}{2}\,(n + 1) = \frac{1}{2}\,(10 + 1) = 5{,}5 \qquad \overline{x} = \frac{\Sigma\, x_i}{n} = \frac{148}{10} = 14{,}800$$

$$Q_2 = \frac{x_5 + x_6}{2} = \frac{13 + 15}{2} = 14$$

x_i	x_i^2
13	169
15	225
21	441
3	9
6	36
2	4
30	900
26	676
27	729
5	25
$\sum x_i = 148$	$\sum x_i^2 = 3214$

$$s^2 = \frac{1}{n-1}\left[\sum x_i^2 - n\bar{x}^2\right]$$

$$= \frac{1}{10-1}\left[3214 - 10(14{,}800)^2\right]$$

$$s^2 = 113{,}733$$

$$\therefore s = \sqrt{113{,}733}$$

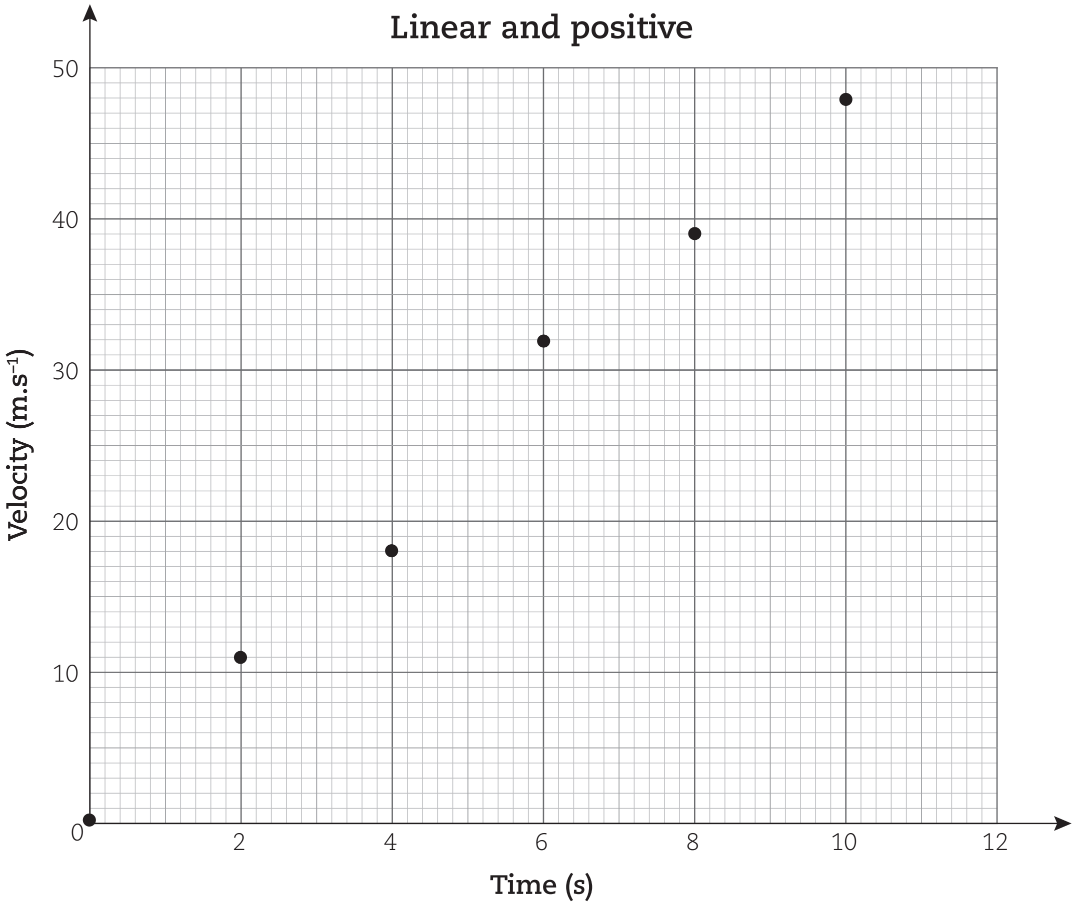
$$s = 10{,}655$$

Since $\bar{x} - Q_2 > 0$, therefore the data are skewed to the right, that is, positively skewed.

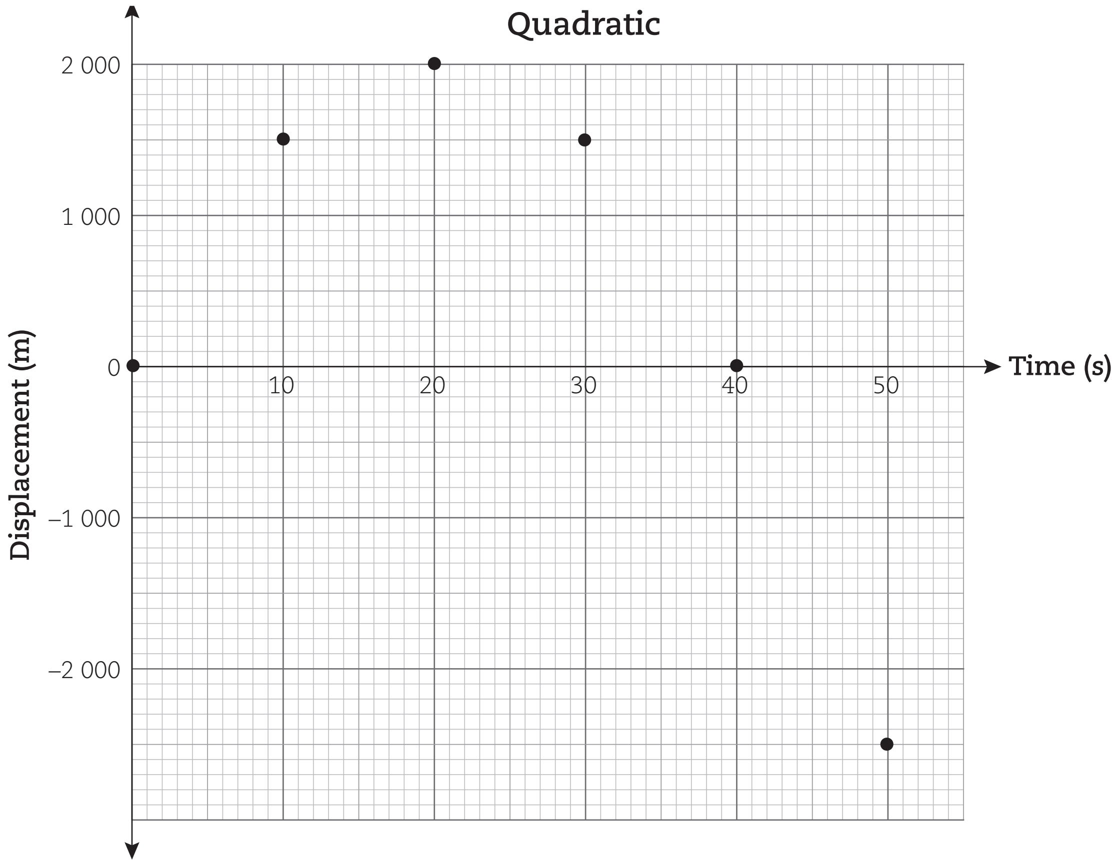
Since the data does not represent a normal distribution, therefore a 68% confidence interval cannot be computed.

Formative assessment 2

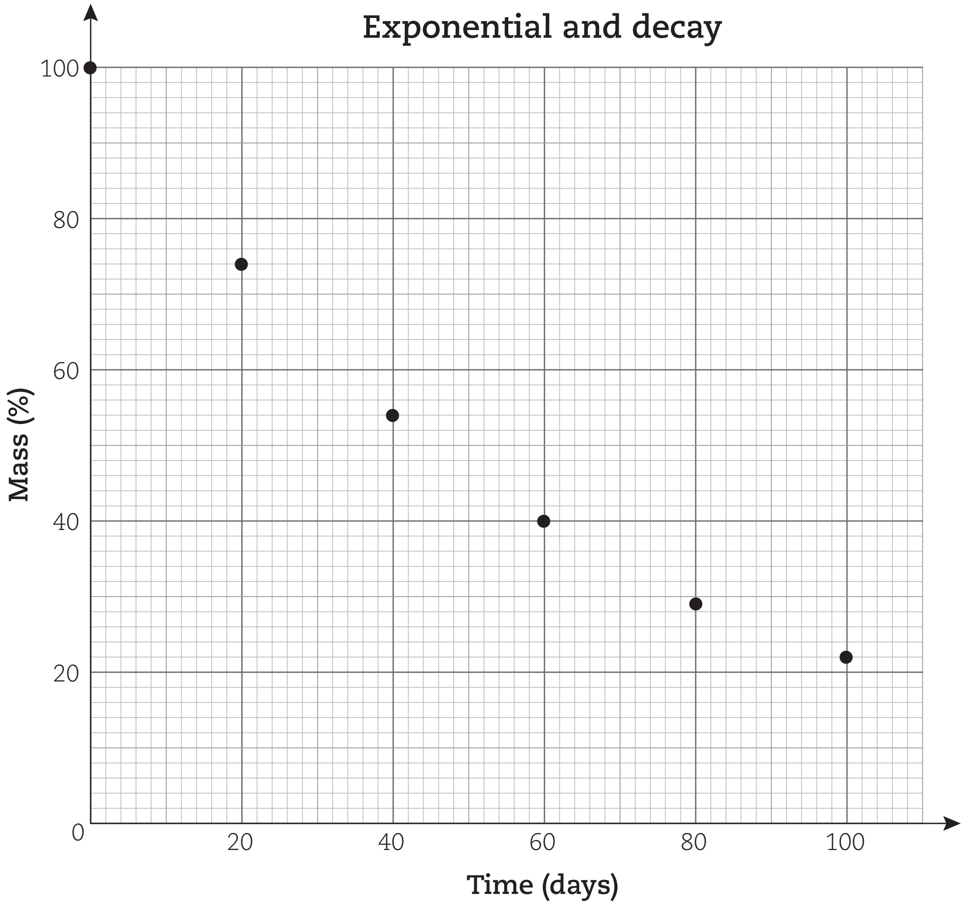
1.



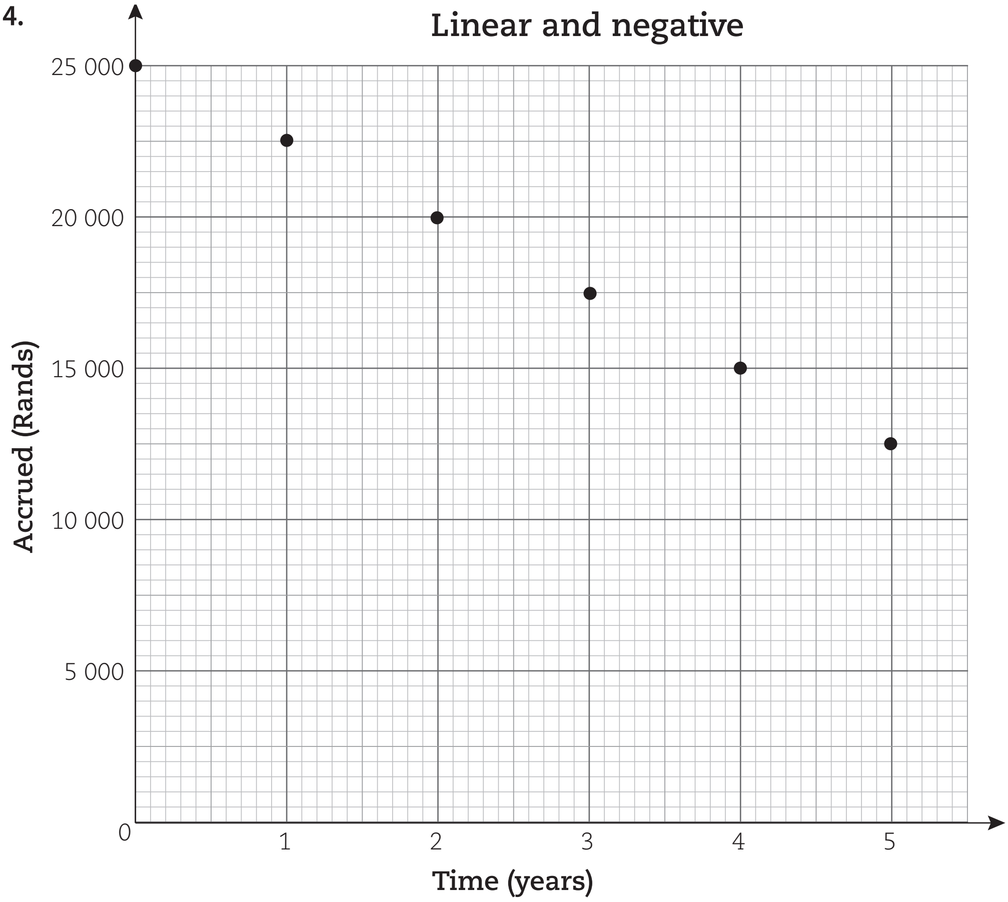
2.


Quadratic
2 000
1 000
0
–1 000
–2 000
Displacement (m)
10
20
30
40
50
Time (s)

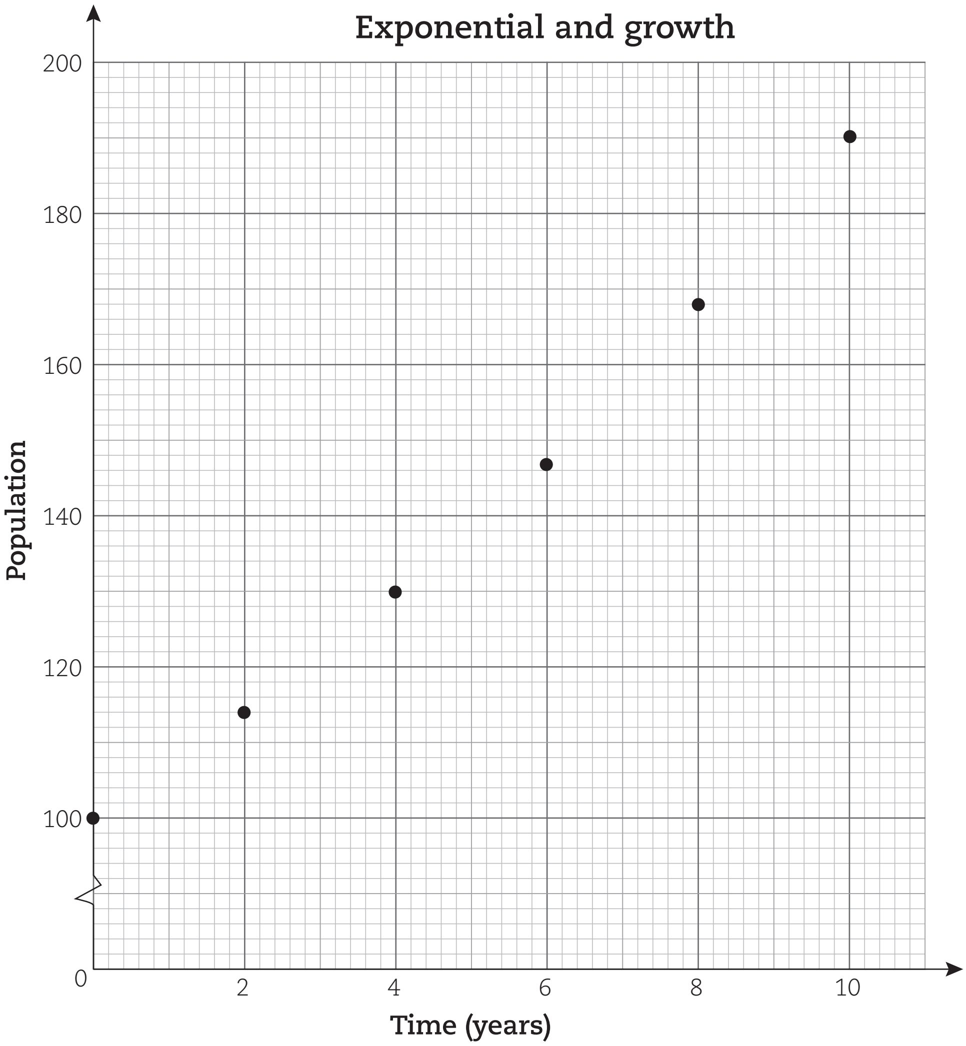
3.


Exponential and decay
100
80
60
40
20
0
Mass (%)
20
40
60
80
100
Time (days)

4.

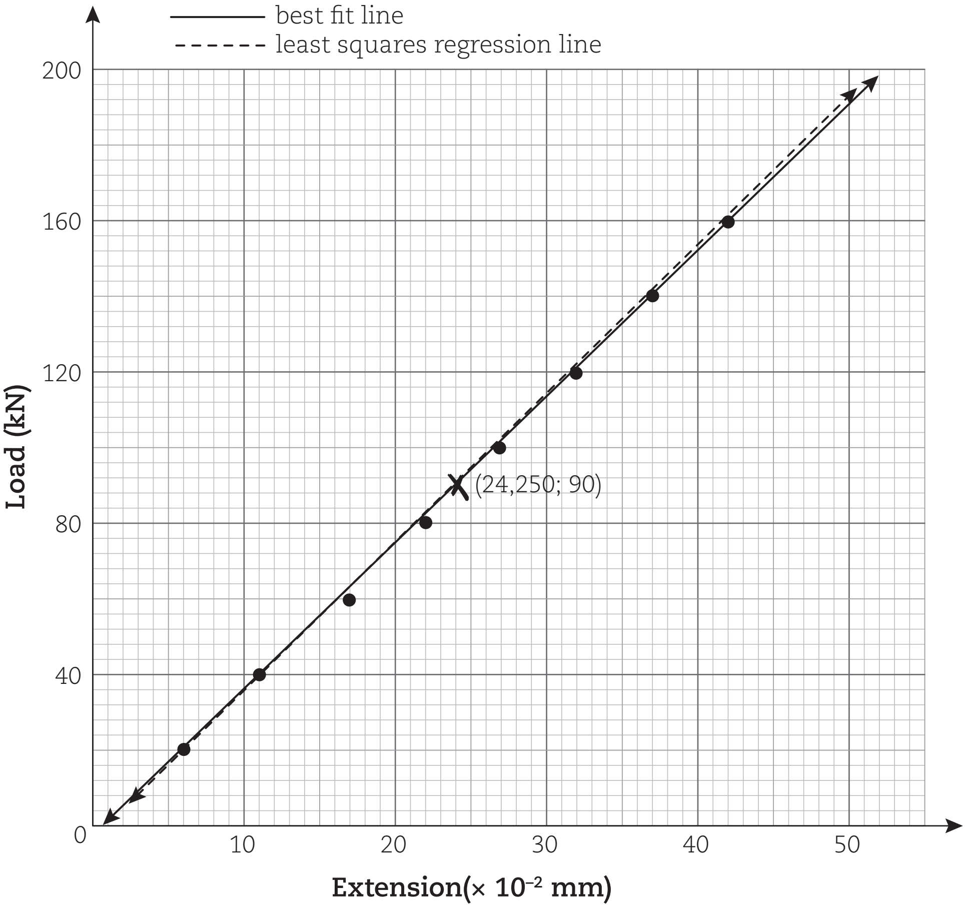

Linear and negative
Accrued (Rands)
25 000
20 000
15 000
10 000
5 000
0
1
2
3
4
5
Time (years)

5.


Exponential and growth
Population
200
180
160
140
120
100
0
2
4
6
8
10
Time (years)

Formative assessment 3

1.



2. The line of best fit has a positive strong association.

3.

x	x^2	y	xy
6	36	20	120
11	121	40	440
17	289	60	1020
22	484	80	1760
27	729	100	2700
32	1024	120	3840
37	1369	140	5180
42	1764	160	6720
$\sum x = 194$	$\sum x^2 = 5816$	$\sum y = 720$	$\sum xy = 21780$

$\overline{x} = \frac{\sum x}{n} = \frac{194}{8} = 24{,}250$ and $\overline{y} = \frac{\sum y}{n} = \frac{720}{8} = 90{,}000$

Regression coefficient,

$$b = \frac{n\sum xy - \sum x \sum y}{n\sum x^2 - (\sum x)^2}$$

$$= \frac{8(21780) - (194)(720)}{8(5816) - (194)^2}$$

$b = 3{,}887$

Regression coefficient,

$a = \overline{y} - b\overline{x}$

$= (90{,}000) - (3{,}887)(24{,}250)$

$a = -4{,}251$

Least squares regression line,

$-4{,}251 = y - 3{,}887x$

$\therefore\ y = 3{,}887x - 4{,}251$

4. The least squares regression line cuts the means of the extension and load as indicated on the graph.

5. Least squares regression line,
$y = 3{,}887x - 4{,}251$

Force at 24×10^{-2} mm,
$y = 3{,}887x - 4{,}251$
$= 3{,}887(24) - 4{,}251$
$y = 89{,}028$
Therefore, force at 24×10^{-2} mm will be 89,028 kN

Formative assessment 4

1. **a)** Sample space: all possible outcomes of a random experiment is called the sample space of the experiment.
- **b)** Event: an event is a subset of the sample space.
- **c)** Probability: if an experiment can produce N mutually exclusive and equally likely outcomes of which *n* outcomes are favourable to occurrence of event A.
- **d)** Dependent events: two events, A and B, are statistically dependent if the one event affects the outcome of the other event.
- **e)** Independent events: two events, A and B, are statistically independent if there is no influence of one event on the other event.
- **f)** Mutually exclusive events: two events are mutually exclusive if the events cannot occur simultaneously.
- **g)** Mutually inclusive events: two events are mutually inclusive if the events occur simultaneously.
- **h)** Complementary event: the complementary events are all those outcomes in the sample space that are not favourable.

2. $P(RR) = P(R) \times P(R)$

$= \frac{5}{15} \times \frac{4}{14}$

$P(RR) = \frac{2}{21}$

3. $P(\text{heads and } 3) = P(\text{heads}) \times P(3)$

$= \frac{1}{2} \times \frac{1}{5}$

$P(\text{heads and } 3) = \frac{1}{10}$

4. $$\begin{aligned}P(\text{purple or orange hexagon}) &= P(\text{purple hexagon}) + P(\text{orange hexagon})\\ &= \frac{8}{20} + \frac{6}{20}\\ P(\text{purple or orange hexagon}) &= \frac{7}{10}\end{aligned}$$

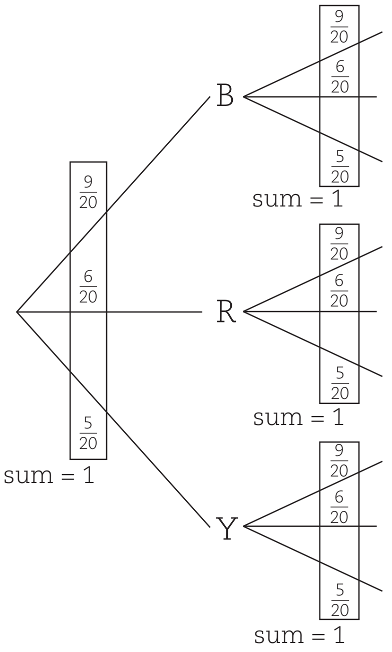
5. $$\begin{aligned}P(\text{king or black}) &= P(\text{king}) + P(\text{black}) - P(\text{king and black})\\ &= \frac{4}{52} + \frac{26}{52} - \frac{2}{52}\\ P(\text{king or black}) &= \frac{7}{13}\end{aligned}$$

6. $$\begin{aligned}P(\text{club or diamond or spade}) &= P(\text{club}) + P(\text{diamond}) + P(\text{spade})\\ &= \frac{13}{52} + \frac{13}{52} + \frac{13}{52}\\ P(\text{club or diamond or spade}) &= \frac{3}{4}\end{aligned}$$

$$\begin{aligned}\therefore P(\overline{\text{club or diamond or spade}}) &= 1 - P(\text{club or diamond or spade})\\ &= 1 - \frac{3}{4}\\ P(\overline{\text{club or diamond or spade}}) &= \frac{1}{4}\end{aligned}$$

Formative assessment 5

1. a) **Event 1** **Event 2** **Outcome** **Probability**



BB $P(BB) = \frac{9}{20} \times \frac{9}{20} = \frac{81}{400}$

BR $P(BR) = \frac{9}{20} \times \frac{6}{20} = \frac{54}{400}$

BY $P(BY) = \frac{9}{20} \times \frac{5}{20} = \frac{45}{400}$

RB $P(RB) = \frac{6}{20} \times \frac{9}{20} = \frac{54}{400}$

RR $P(RR) = \frac{6}{20} \times \frac{6}{20} = \frac{36}{400}$

RY $P(RY) = \frac{6}{20} \times \frac{5}{20} = \frac{30}{400}$

YB $P(YB) = \frac{5}{20} \times \frac{9}{20} = \frac{45}{400}$

YR $P(YR) = \frac{5}{20} \times \frac{6}{20} = \frac{30}{400}$

YY $P(YY) = \frac{5}{20} \times \frac{5}{20} = \frac{25}{400}$

$P(S) = 1$

b) S = {BB; BR; BY; RB; RR; RY; YB; YR; YY}

c) **(i)** $P(BB) = \frac{81}{400}$

$P(RR) = \frac{36}{400}$

$P(YY) = \frac{25}{400}$

(ii) $$\begin{aligned}P(BB \text{ or } RR) &= P(BB) + P(RR)\\ &= \frac{81}{400} + \frac{36}{400}\\ P(BB \text{ or } RR) &= \frac{117}{400}\end{aligned}$$

$$P(RR \text{ or } YY) = P(RR) + P(YY)$$
$$= \frac{36}{400} + \frac{25}{400}$$
$$P(RR \text{ or } YY) = \frac{61}{400}$$

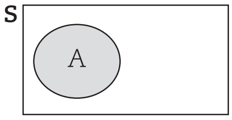
$$P(BB \text{ or } YY) = P(BB) + P(YY)$$
$$= \frac{81}{400} + \frac{25}{400}$$
$$P(BB \text{ or } YY) = \frac{106}{400}$$

(iii) $$P(\text{at least one yellow ball is drawn}) = P(BY \text{ or } RY \text{ or } YB \text{ or } YR \text{ or } YY)$$
$$= P(BY) + P(RY) + P(YB) + P(YR) + P(YY)$$
$$= \frac{45}{400} + \frac{30}{400} + \frac{45}{400} + \frac{30}{400} + \frac{25}{400}$$
$$P(\text{at least one yellow ball is drawn}) = \frac{175}{400}$$

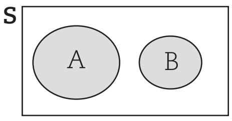
$$P(\text{two different colour balls}) = 1 - [P(BB) + P(RR) + P(YY)]$$
$$= 1 - \left[\frac{81}{400} + \frac{36}{400} + \frac{25}{400}\right]$$
$$P(\text{two different colour balls}) = \frac{258}{400}$$

2. **a)**

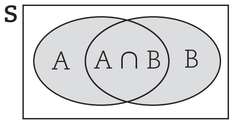
b)



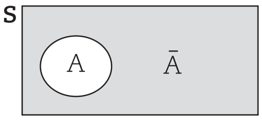
c)



d)



e)



3. **a)**

Sport code	Gender		Total
	Female	Male	
Soccer	40	**B = 35**	75
Rugby	35	20	**C = 55**
Cricket	**A = 30**	90	120
Total	**105**	**145**	**250**

b)

Sport code	Gender		Total
	Female	Male	
Soccer	0,160	0,140	0,300
Rugby	0,140	0,080	0,220
Cricket	0,120	0,360	0,480
Total	**0,420**	**0,580**	**1,000**

Topic 5 • Solutions

Formative assessment 1

1. **a)** Hire purchase: is a system of purchasing a product where the customer takes possession of the product on payment of a deposit (or no deposit in some instances) and completes the purchase by paying a series of regular instalments.

b) Inflation: refers to an average percentage increase in the price of goods from year to year.

2.

$$A = P(1 + in)$$
$$\frac{A}{P} = \frac{P(1 + in)}{P}$$
$$\frac{A}{P} = 1 + in$$
$$\frac{A}{P} - 1 = 1 + in - 1$$
$$\frac{A}{P} - 1 = in$$
$$\frac{\frac{A}{P} - 1}{i} = \frac{in}{i}$$
$$\frac{\frac{A}{P} - 1}{i} = n$$

3.

$$A = P(1 + i)^n$$
$$\frac{A}{P} = \frac{P(1 + i)^n}{P}$$
$$\frac{A}{P} = (1 + i)^n$$
$$\sqrt[n]{\frac{A}{P}} = \sqrt[n]{(1 + i)^n}$$
$$\sqrt[n]{\frac{A}{P}} = 1 + i$$
$$\sqrt[n]{\frac{A}{P}} - 1 = 1 + i - 1$$
$$\sqrt[n]{\frac{A}{P}} - 1 = i$$

4. **a)** Principal loan amount = cash price – deposit

= cash price – 10% of cash price

$= 5\ 000 - \frac{10}{100} \times R5\ 000$

Principal loan amount = R4 500

b) Convert 24 months to years, that is, $\left(\frac{24}{12}\right) = 2$ years

$A = P(1 + in)$

$= 4\ 500\ (1 + \frac{20}{100}.2)$

$A = R6\ 300$

The accumulated loan amount is R6 300.

c) Monthly repayment = $\frac{\text{accumulated loan amount}}{\text{number of monthly repayments}}$ + insurance premium

$= \frac{6\ 300}{24} + 25$

Monthly repayment = R287,50

d) Total amount paid = (monthly repayment × number of payment) + deposit

$= (R287{,}50 \times 24) + \frac{10}{100} \times R5\ 000$

Total amount paid = R7 400.

5.

$$
\begin{aligned}
A &= P(1+i)^n \\
\frac{A}{P} &= \frac{P(1+i)^n}{P} \\
\frac{A}{P} &= (1+i)^n \\
\sqrt[n]{\frac{A}{P}} &= \sqrt[n]{(1+i)^n} \\
\sqrt[n]{\frac{A}{P}} &= 1+i \\
\sqrt[n]{\frac{A}{P}} - 1 &= 1+i-1 \\
\sqrt[n]{\frac{A}{P}} - 1 &= i \\
\therefore i &= \sqrt[n]{\frac{A}{P}} - 1 \\
&= \sqrt[4]{\frac{2\ 000}{1\ 200}} - 1 \\
&= 0{,}136 \\
i &= 13{,}622\% \text{ per annum}
\end{aligned}
$$

Formative assessment 2

1. **a)** Tax: a fee levied by government on a product, income or activity.

b) Tax return: a declaration of personal income made annually to the tax authorities, and used as a basis for assessing an individual liability for taxation.

c) Tax rate: a percentage of one's income that is payable in taxes. Tax rates vary according to income brackets.

d) Tax rebate: a refund offered to taxpayers falling within a certain age category.

e) Tax threshold: the level at which income is taxable.

f) Tax credit: an item that reduces your actual tax, and differs from tax deduction that reduces only your taxable income.

2. Dr Harper is 54 years old, and thus only qualifies for the primary rebate. He also qualifies for a medical scheme contribution tax credit.

Tax return for Dr Harper:

Step 1 Gross income = monthly salary × 12 + annual bonus
= R50 000 × 12 + R180 000
Gross income = R780 000

Step 2 Annual pension fund contribution = R5 000 × 12
Annual pension fund contribution = R60 000

Step 3 Taxable income = gross income – deductions
= R780 000 – R60 000
Taxable income = R720 000

Step 4 From the individual rate table, his taxable income falls in the category R617 001 and above, with the rate of tax R178 940 + 40% of the amount above R617 000.

Tax payable = R178 940 + 40% of (R720 000 – R617 000)
= R178 940 + 40% of R103 000
= R178 940 + $\frac{40}{100}$ × R103 000
= R178 940 + R41 200
Tax payable = R220 140

Step 5: Since the economist is below the age of 65 years, he qualifies for the primary rebate only. He also qualifies for the medical scheme contribution tax credit for the three individuals: the taxpayer and two dependants.

Tax rebate = R11 440

Medical scheme contribution tax credit = [2 × R230 + R154] × 12
Medical scheme contribution tax credit = R7 368

∴ Total tax rebate = R11 440 + R7 368
Total tax rebate = R18 808

Step 6 Tax due = tax payable – total tax rebate
= R220 140 – R18 808
Tax due = R201 332

The annual tax due by the economist is R201 332.

Formative assessment 3

1. Straight-line depreciation represents a constant depreciation from year to year, whereas on a reducing–balance depreciation, the depreciation decreases from year to year.

2. **a)**

$$A = P(1 - in)$$
$$\frac{A}{P} = \frac{P(1 - in)}{P}$$
$$\frac{A}{P} = 1 - in$$
$$\frac{A}{P} - 1 = 1 - in - 1$$
$$\frac{A}{P} - 1 = -in$$
$$\frac{\frac{A}{P} - 1}{-n} = \frac{-in}{-n}$$
$$\frac{\frac{A}{P} - 1}{-n} = i$$

b)

$$A = P(1 - i)^n$$
$$\frac{A}{P} = \frac{(1 - i)^n}{P}$$
$$\frac{A}{P} = (1 + i)^n$$
$$\sqrt[n]{\frac{A}{P}} = \sqrt[n]{(1 - i)^n}$$
$$\sqrt[n]{\frac{A}{P}} = 1 - i$$
$$\sqrt[n]{\frac{A}{P}} - 1 = 1 - i - 1$$
$$\sqrt[n]{\frac{A}{P}} - 1 = -i$$
$$1 - \sqrt[n]{\frac{A}{P}} = i$$

3.

$$\begin{aligned}
A &= P(1 - in)\\
\frac{A}{P} &= \frac{(1 - in)}{P}\\
\frac{A}{P} &= 1 - in\\
\frac{A}{P} - 1 &= 1 - in - 1\\
\frac{A}{P} - 1 &= - in\\
\frac{\frac{A}{P} - 1}{-i} &= \frac{-in}{-i}\\
\frac{\frac{A}{P} - 1}{-i} &= n\\
\therefore n &= \frac{\frac{A}{P} - 1}{-i}\\
&= \frac{\frac{13\ 500}{45\ 000} - 1}{-\frac{17,5}{100}}\\
n &= 4 \text{ years}
\end{aligned}$$

4.

$$\begin{aligned}
A &= P(1 - i)^n\\
\frac{A}{(1 - i)^n} &= \frac{P(1 - i)^n}{(1 - i)^n}\\
\frac{A}{(1 - i)^n} &= P\\
\therefore P &= \frac{A}{(1 - i)^n}\\
&= \frac{15\ 725}{(1 - \frac{34}{300})^5}\\
P &= R28\ 693{,}85
\end{aligned}$$

5. **a)**

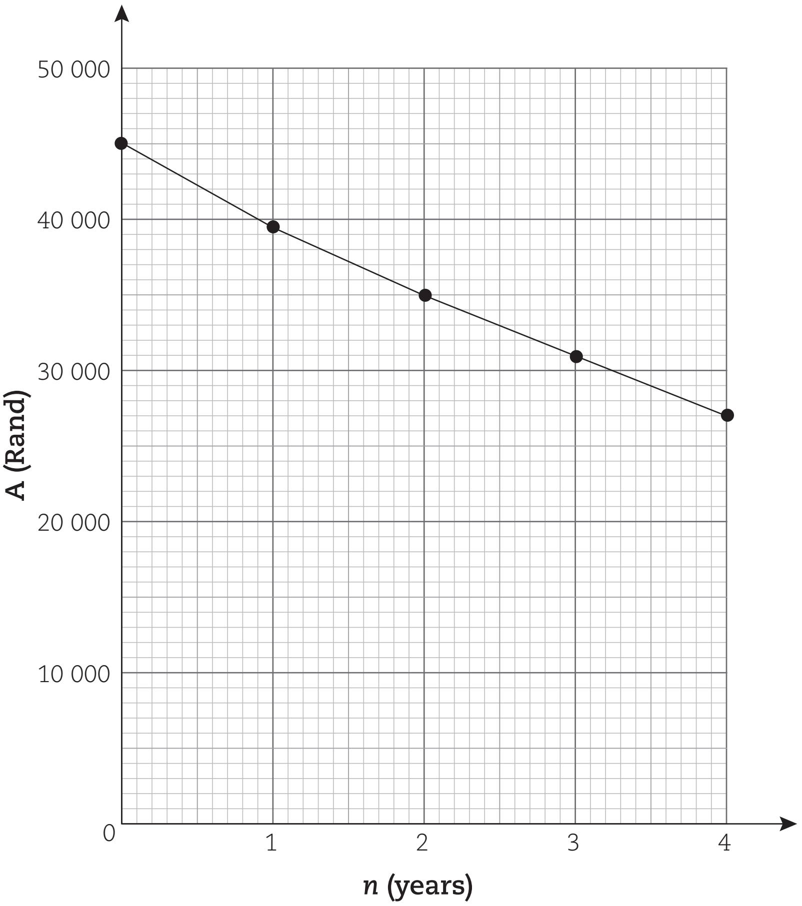
$$\begin{aligned}
A &= P(1 - i)^n\\
&= 45\ 000\left(1 - \frac{12}{100}\right)^4\\
A &= R26\ 986{,}29
\end{aligned}$$

b) Total depreciation after 4 years = P – A

= R45 000 – R26 986,29

Total depreciation after 4 years = R18 013,71

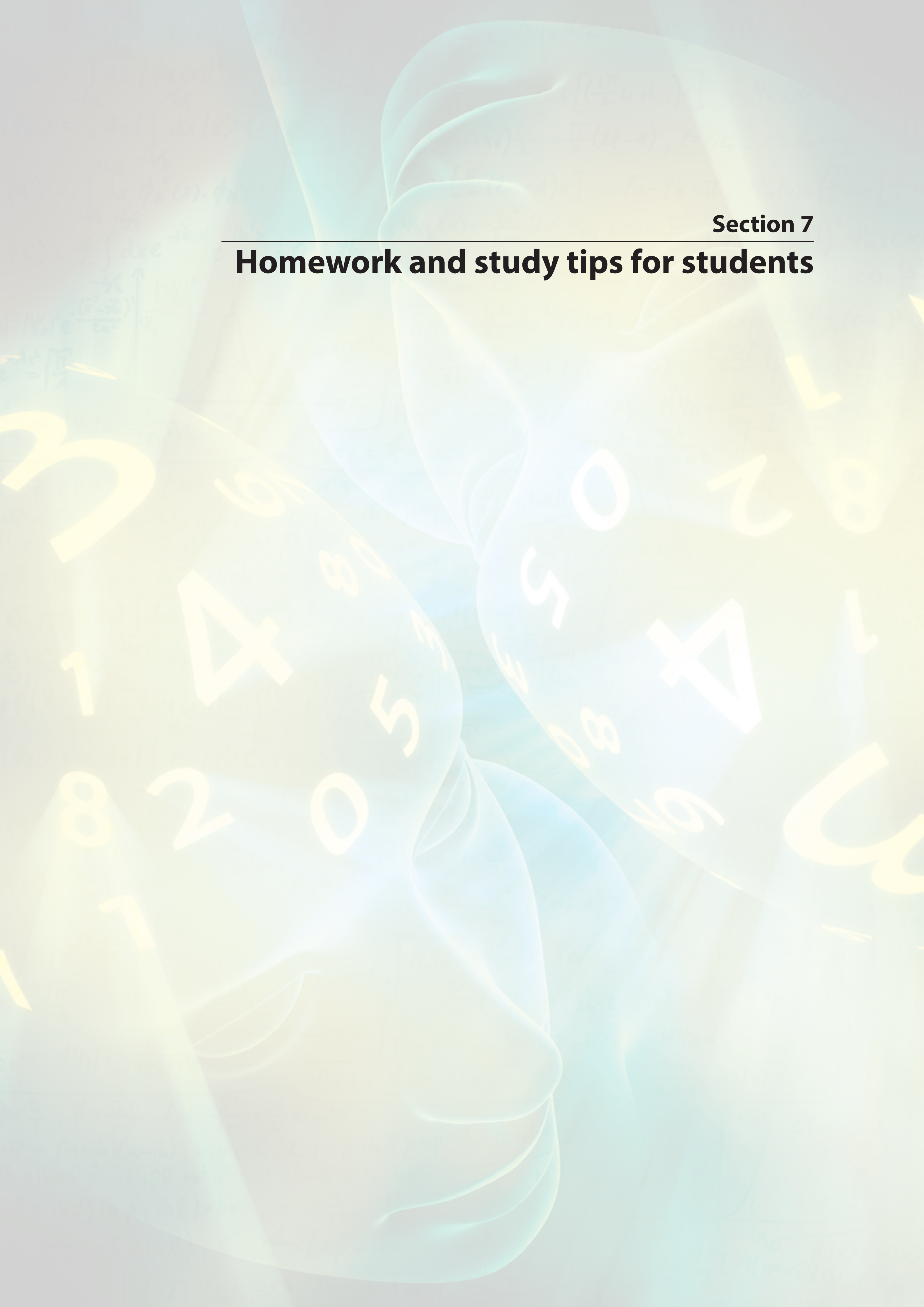
c)



n	A
0	45 000
1	39 600
2	34 848
3	30 666
4	26 986

Section 7

Homework and study tips for students



Homework tips for learners

This section contains some general tips on making the most of your homework.

- **Do the activity or assignment promptly.**
- **Be organised.**
- **Review.** Review any examples that the lecturer worked out, to make sure you understand all the ideas from the section completed in class. (Work through the examples in your text book.)
- **Read/follow the instructions carefully.**
- **Be neat.**
- **Show all your workings out.**
- **Check your work at the end. Mark your answers**. (The answers are at the end of each chapter.)
- **Mark the wrong answers** clearly and **ask** your lecturer for assistance.
- **Do your Mathematics homework in a single book**, such as a hardcover book.
- Clearly indicate **the number of the exercise** you are doing.
- Do your work in **pencil**, with mistakes cleanly erased, not crossed or scratched out. If you work in ink, use Tippex to correct mistakes.
- **Write legibly** (suitably large and suitably dark).
- **Write neatly across the page**, with each succeeding problem **below** the preceding one.
- Do not work in multiple columns down the page (like a newspaper); your page should contain **only one column**.
- Use **enough space** for each problem, with at least one blank line between problems.
- **Show all your work.** This means showing your steps. Show everything in between the question and the answer. For your work to be complete, you need to **explain your reasoning**.
- For tables and graphs, **use a ruler to draw the straight lines**, and clearly label the axes, and the points of interest. **Use a consistent scale** on the axes. Also, make your table or graph large enough to be clear. If you can fit more than three or four graphs on one side of a sheet of paper, then you're drawing them too small.
- **Do not perform magic**. Plus/minus signs, "= 0", radicals and denominators should not disappear in the middle of your calculations, and then reappear at the end. Each step should be complete.
- **Write your final answer at the end** of your work, and mark it clearly, by for example, underlining it. Label your answer appropriately; if the question asks for measured units, make sure to put appropriate units next to the answer. **If the question is a word problem, the answer should be in words.**

Note
Summative and **formative** assessment tasks should be done in **blue** or **black** ink.

Remember
Do *all* the homework problems, not just some of them!

Learn from your errors

Learning from your mistakes can only help you.

- **Review homework**. When you check your homework look for errors that you made.
- **Review assessments/exams completed**. Do your corrections!
- **Understand the error made**. When you find an error in your homework or exams try to understand what the error is and just what you did wrong. To help you to avoid making it again, look for something about the error that you can remember.
- **Get help**. If you’t can find the error and/or don’t understand why it was an error then get help. Ask the lecturer or a classmate who got the problem correct.
- **Rushed errors**. If you find yourself continually making silly arithmetic errors then slow down when you are working the problems. Most of these types of errors happen because students rush and don’t pay attention to what they are doing.
- **Repeated errors**. If you find yourself continually making errors on one particular type of problem then you probably don’t have a really good grasp of the concept behind that specific type of problem. Find more examples and really try to understand just what you are doing wrong or don’t understand.
- **Keep a list of errors made**. Put errors that you keep making in a “list of errors”. Write down the correct method/solution next to each error.

Problem-solving

Here are some tips to help you actually work the problems.

- **Read the problem**. Get an idea of what you're being asked to do.
- **Read the problem again**. Now that you know what you're being asked to do, read the problem again. Make notes of what you are given and what you need to find and make sure that you understand just what you're being asked to do.
- **Write down what you are asked to find.**
- **Write down what you know**. Write down all the information you've been given.
- **Draw a diagram**. If appropriate draw a diagram and label what you know and what you need to find.
- **Decide on a plan**. Try to figure out what you're going to need to work the problem. Identify formulas that may help you. See if there are any intermediate steps/answers that will be needed in order to arrive at the final answer.
- **Do a similar problem**. If you can't figure out how to work the problem find a similar problem (refer to examples and explanations in text book) that is simpler. Work this problem then go back and compare what you did in the simpler problem to the problem you're asked to do.
- **Work the plan**. Once you've got the plan, work it out to get the answer.
- **Check your solution**. Is your answer in proper form? Does your answer make sense? If possible, check your answer.
- **Always go over the problem again**. Once you're satisfied that you've got the correct answer go back over the problem. Identify concepts/ methods/ formulas that were used for the problem. Try to understand why these concepts/methods/formulas were used for this problem. Look for identifying characteristics that will help you identify this kind of problem in the future.

Note-taking tips for Mathematics

Here are a couple of tips for taking notes in class.

- ***Always* listen in class**. Do not just write down what you see on the board. No lecturer is going to write down every word they say and sometimes the important ideas won't get written down. You must follow the lesson.
- **Write down explanatory remarks** that the lecturer makes. These often won't get written down by the lecturer, but can tell you how to work a particular kind of problem or why the lecturer used one formula/method over another for a given problem.
- **Note the important formulas/concepts**. If a lecturer emphasizes a particular formula or concept then make a note of it because this probably means that it's important and important formulas are much more likely to show up in an exam.
- **Ask questions to your lecturer**. If you are unclear on something ask questions.
- **Note the specific topics or parts of topics that you don't understand.**
- **Review/edit your notes**. As soon you can after class, go back over your notes.
- **Look for any errors and/or omissions**. Fill in any information you didn't have time to write down in class.
- **Review all the work done regularly**. At regular intervals sit down and revise your notes so that you can learn the information. Remember, that this information will probably be required at a later stage so it's best to learn it as soon as possible.

Getting help

Getting help when you are in trouble is *very* important. Here are a couple of things that you can do to get help.

- **Get help as soon as you need it**. Do not wait until the last minute to get help. As soon as you start running into problems it is time to get help. Remember that maths is cumulative. If you don't get help immediately you will be making it more difficult to understand future lessons.
- **Ask questions in class.**
- **Form a study group or get a study buddy**. Different people will see things differently and may see another way to work a problem.
- If your college has a free maths tutoring laboratory, make use of it!
- **Get a private tutor for extra Maths lessons**. You can always hire a private tutor for some help. In almost every college you will find people who are willing to tutor you for a fee.
- **Ask 'good' and specific questions**. Saying "I don't understand this section" or "I don't get it." is not the best way to seek help. It just does not imply what you're having trouble with and so will probably not get your questions answered. Be specific with your questions. What exactly is it about this section don't you understand?
- **Have the attempted problems with you**. Bring the attempts that you've made on the problem to class or to the tutor. This will help the person helping you to understand just where you're having difficulties.

Tips on studying for exams

- **Start on Day One**. Do a little each day, or at the very least start studying 3–4 days before the exam. ***Never*** start studying the night before the exam. **Remember to get a good night's sleep!**
- **Make a list of important concepts and formulas**. Learn the proper notations.
- **Rework homework problems**. Don't just read over the homework problems. Actually rework them. Repeat the problems that you found difficult to solve.
- **Rework textbook examples and notes**. Cover up the solutions to the examples in the text book and try to rework them.
- **Look for identifying specific characteristics in problems.**
- **Take a practice exam**. Find some problems and take a practice test. Use the summative assessments at the end of each chapter. Give yourself a time limit and don't use your notes or textbook.

Taking an exam

Writing an assessment or exam is probably one of the most important things that you will do in a Mathematics class. So it is important to do the best you can. Here are some ideas to help you while you take an exam.

- **Relax!** This is the first step to successfully taking an exam.
- **Be smart.** This means, be smart on how you take the exam. You should go through the exam paper three times. **First**, work all the problems that you KNOW you can do. **Second**, work all the problems that you *think* you can do, but are not too sure about. **Third**, work the remaining problems. This way you will get all the points that you know you can get.
- **Be time-efficient**. Watch the clock. Don’t spend valuable minutes trying to get the points for only one problem. It could help to work out the amount of time to spend on each problem based on the marks allocated to that problem.
- **If you are stuck, move on**. If you find that you’re stuck on a problem, move on to a different problem and come back later to those that you were stuck on.
- **Show all your work**. Make it as easy as possible for the lecturer to see how much you do know. Try to write a well-reasoned solution. If your answer is incorrect, the lecturer will be able to assign partial credit based on the work and understanding you do show.
- **Never leave a problem blank**. Even if you don’t know how to finish the problem, write down as much as you do know. Always attempt all questions.
- **Read the problem**. Read all the questions carefully and completely before you answer the question.
- **Does your answer make sense?**
- **Recheck your work**. If time allows, recheck every problem.

☺ **Just do the best you can!**